Mean Field Games
and
Systemic Risk

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Financial Mathematics Tutorials

IPAM-UCLA

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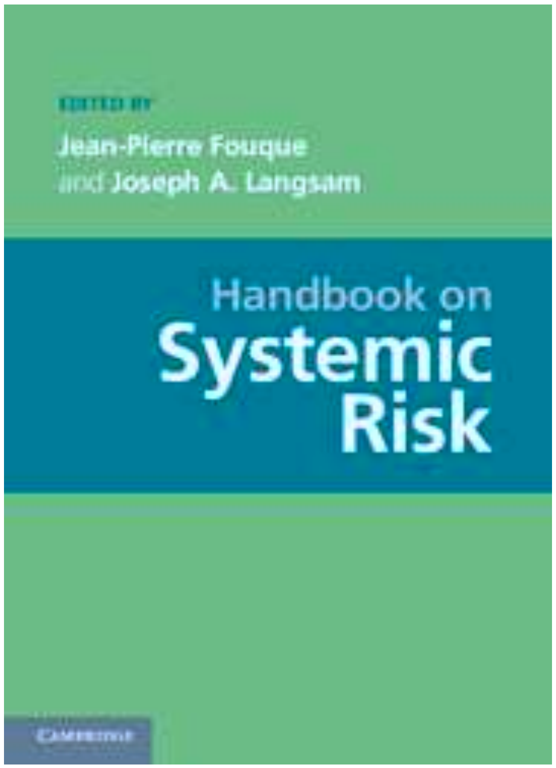
2010: Dodd-Frank Bill includes the creation of the

Office of Financial Research (OFR)



Editors: J.-P. Fouque and J. Langsam

Cambridge University Press (2013)

Coupled Diffusions: Liquidity Risk

$X_t^{(i)}, i = 1, \ldots, N$ denote log-monetary reserves of N banks

$$dX_t^{(i)} = b_t^{(i)} dt + \sigma_t^{(i)} dW_t^{(i)} \qquad i = 1, \ldots, N,$$

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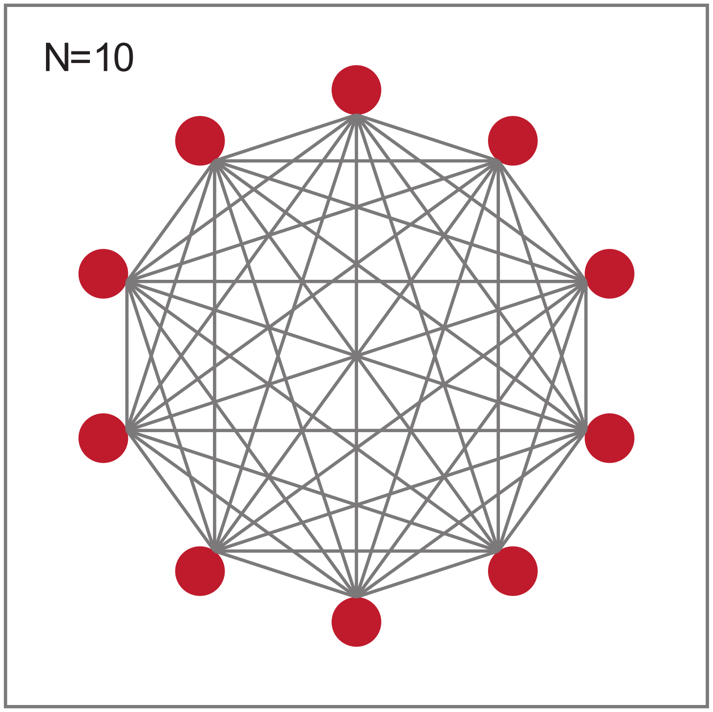
Assume **independent Brownian motions** $W_t^{(i)}, i = 1, \dots, N$
and **identical constant volatilities** $\sigma_t^{(i)} = \sigma$
Model **borrowing and lending** through the drifts:

$$dX_t^{(i)} = \frac{a}{N} \sum_{j=1}^{N} (X_t^{(j)} - X_t^{(i)})\, dt + \sigma dW_t^{(i)}\,, \quad i = 1, \dots, N.$$

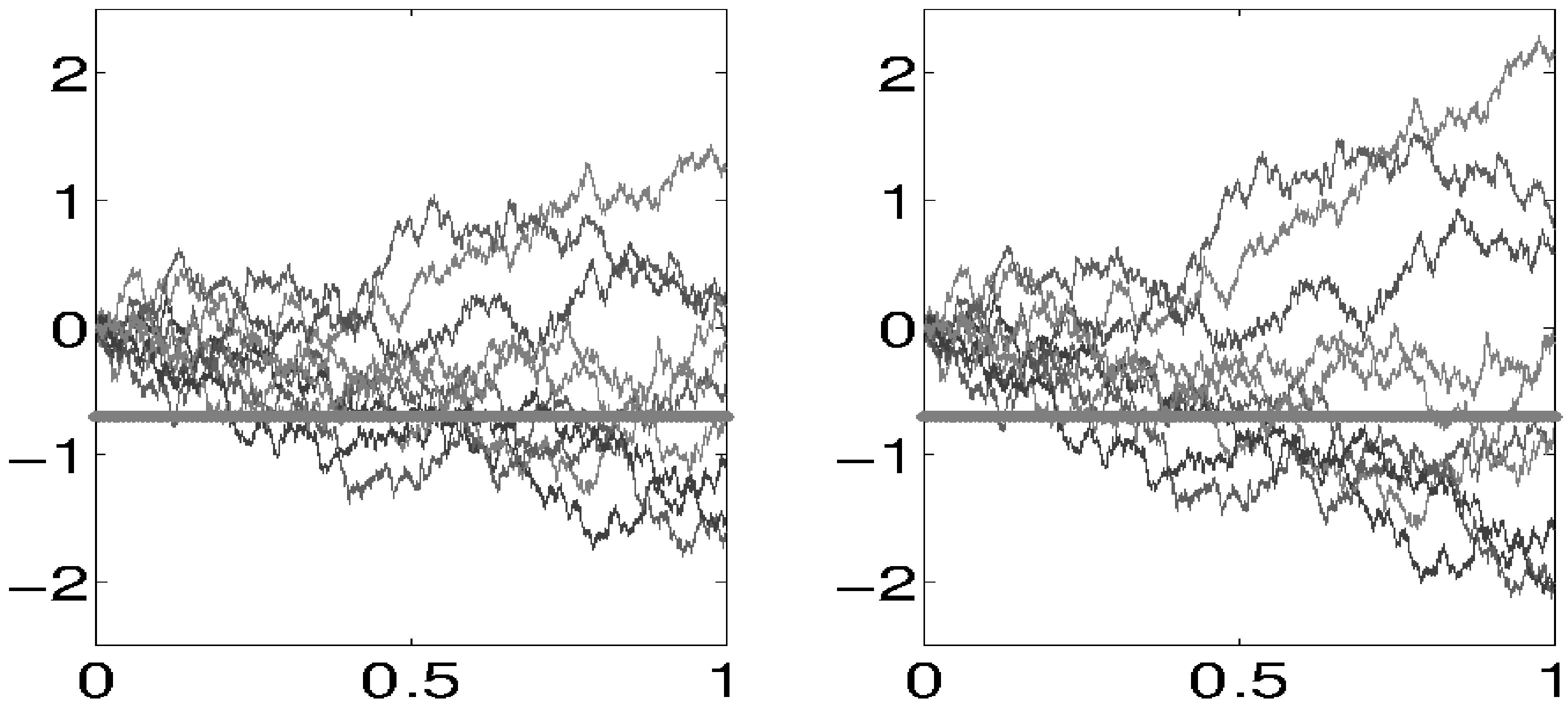
The overall **rate of borrowing and lending** a/N has been normalized by the number of banks.

Fully Connected Symmetric Network

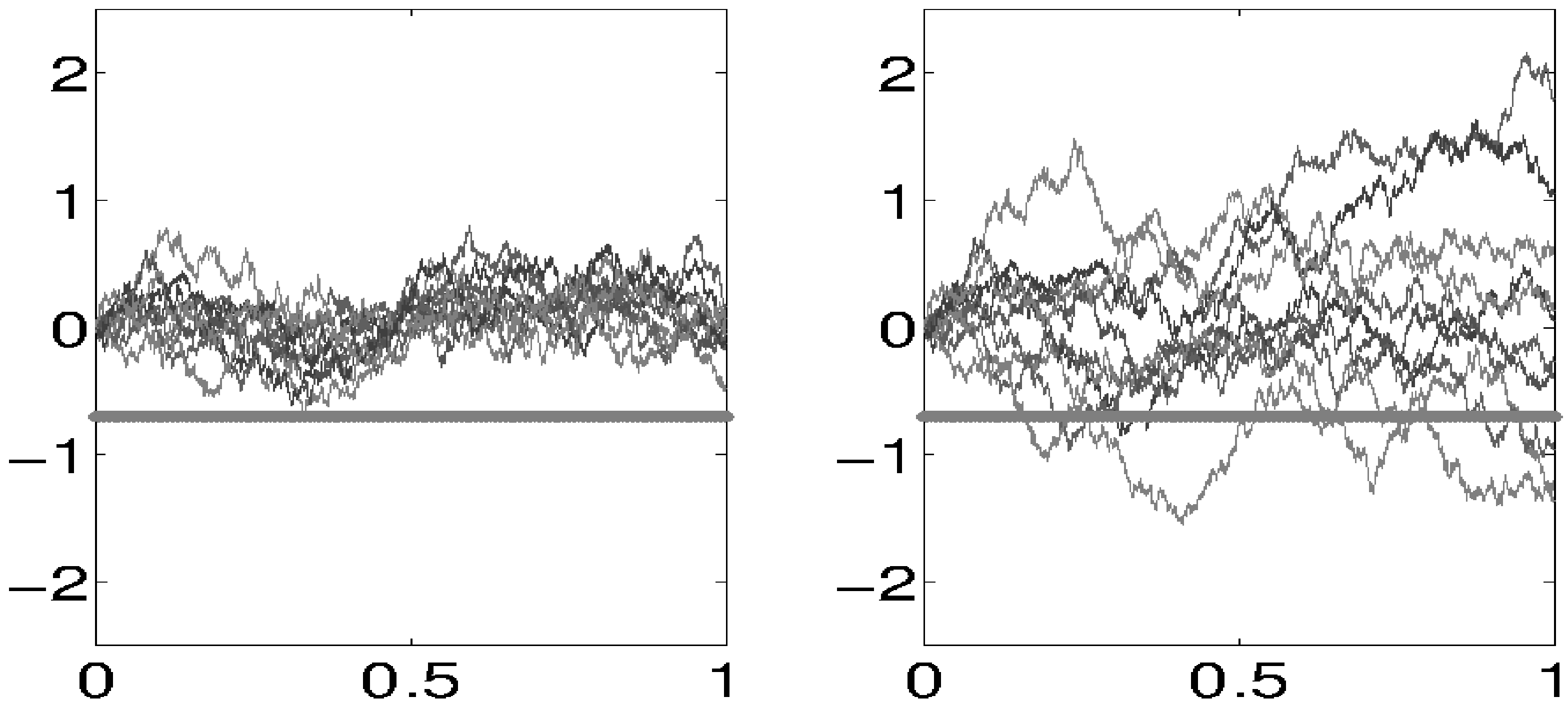
$$dX_t^{(i)} = \frac{a}{N}\sum_{j=1}^{N}(X_t^{(j)} - X_t^{(i)})\,dt + \sigma dW_t^{(i)}\,, \quad i = 1, \ldots, N.$$



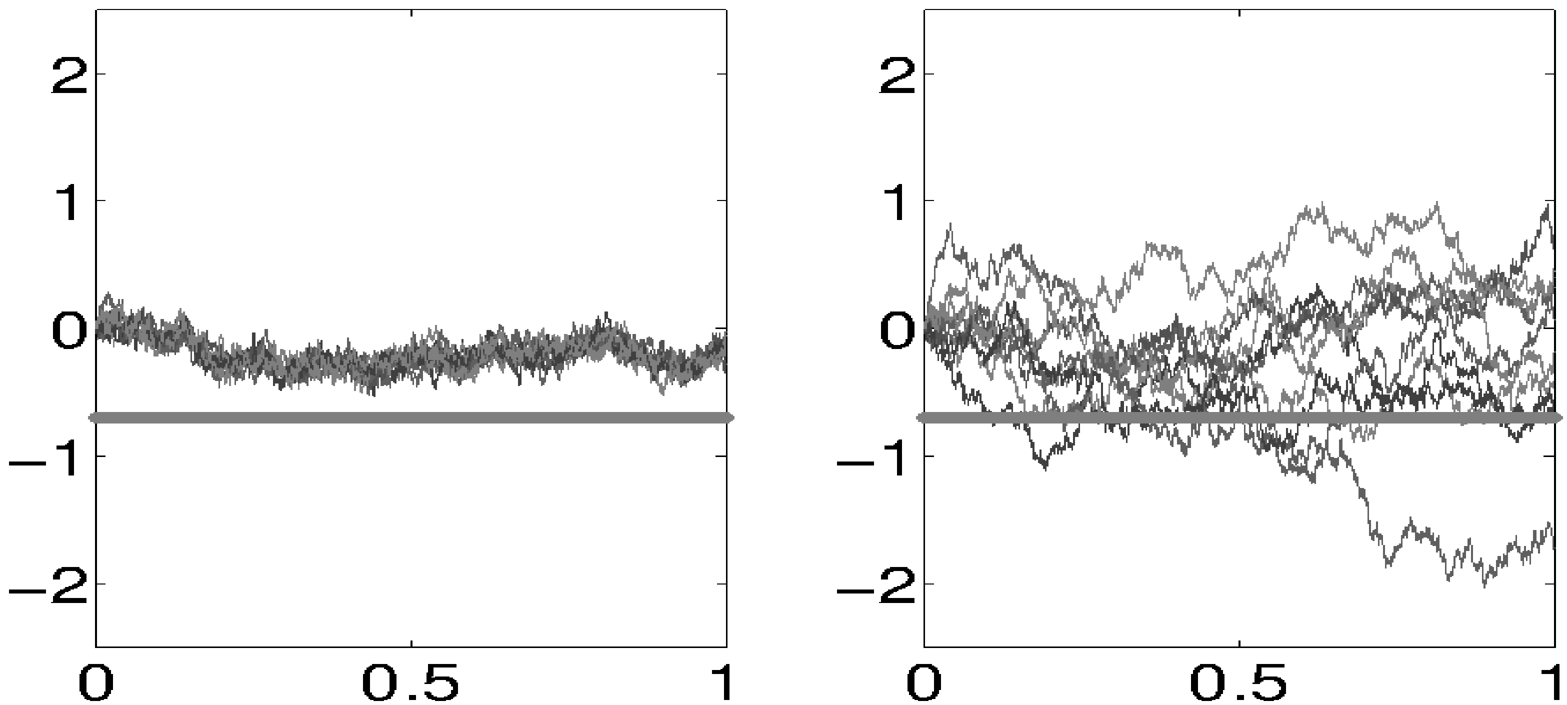
Denote the **default level** by $D < 0$ and simulate the system for various values of **a**: $\mathbf{0}, \mathbf{1}, \mathbf{10}, \mathbf{100}$ with fixed $\sigma = 1$



One realization of the trajectories of the coupled diffusions $X_t^{(i)}$ with **a = 1** (left plot) and trajectories of the independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $D = -0.7$



One realization of the trajectories of the coupled diffusions $X_t^{(i)}$ with **a = 10** (left plot) and trajectories of the independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $D = -0.7$



One realization of the trajectories of the coupled diffusions $X_t^{(i)}$ with **$a = 100$** (left plot) and trajectories of the independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. Solid horizontal line: default level $D = -0.7$

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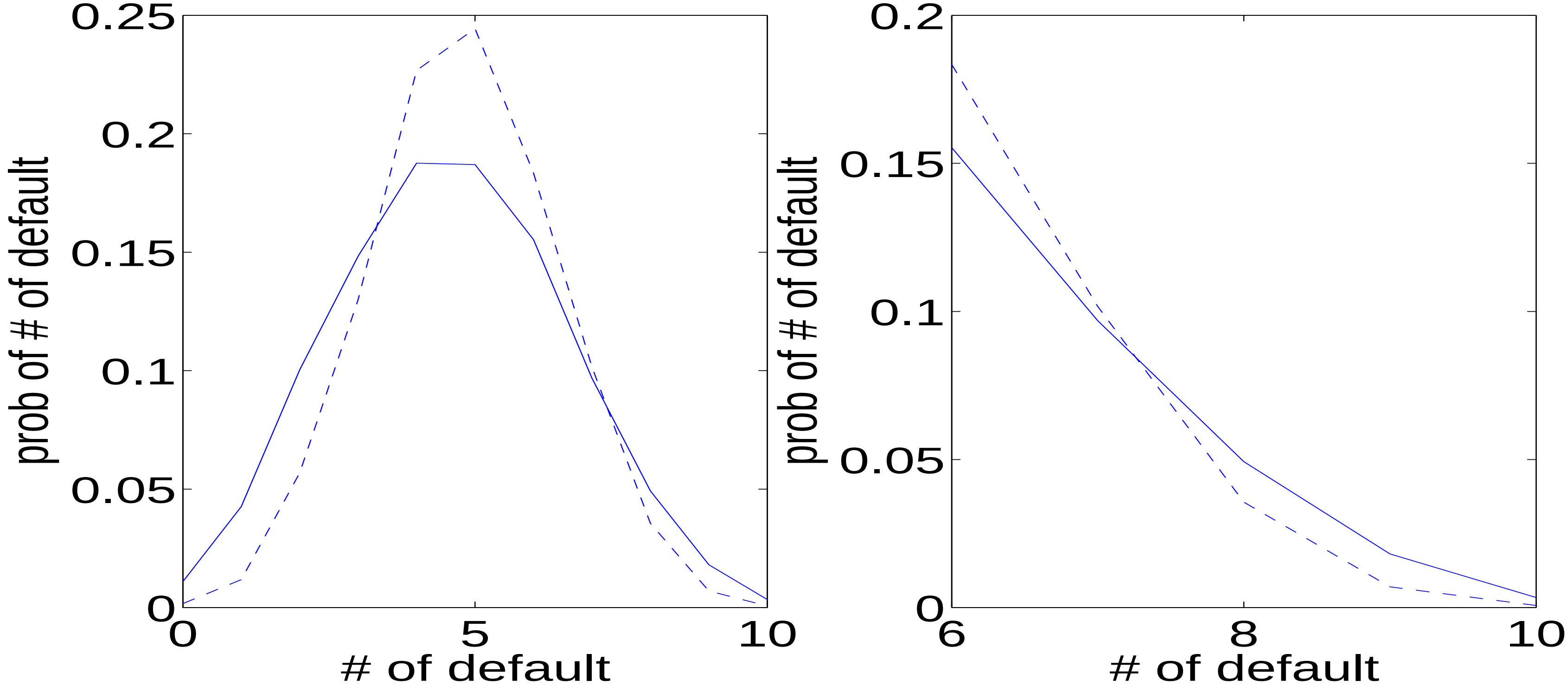
In the independent case, the loss distribution is Binomial(N, p) with parameter p given by

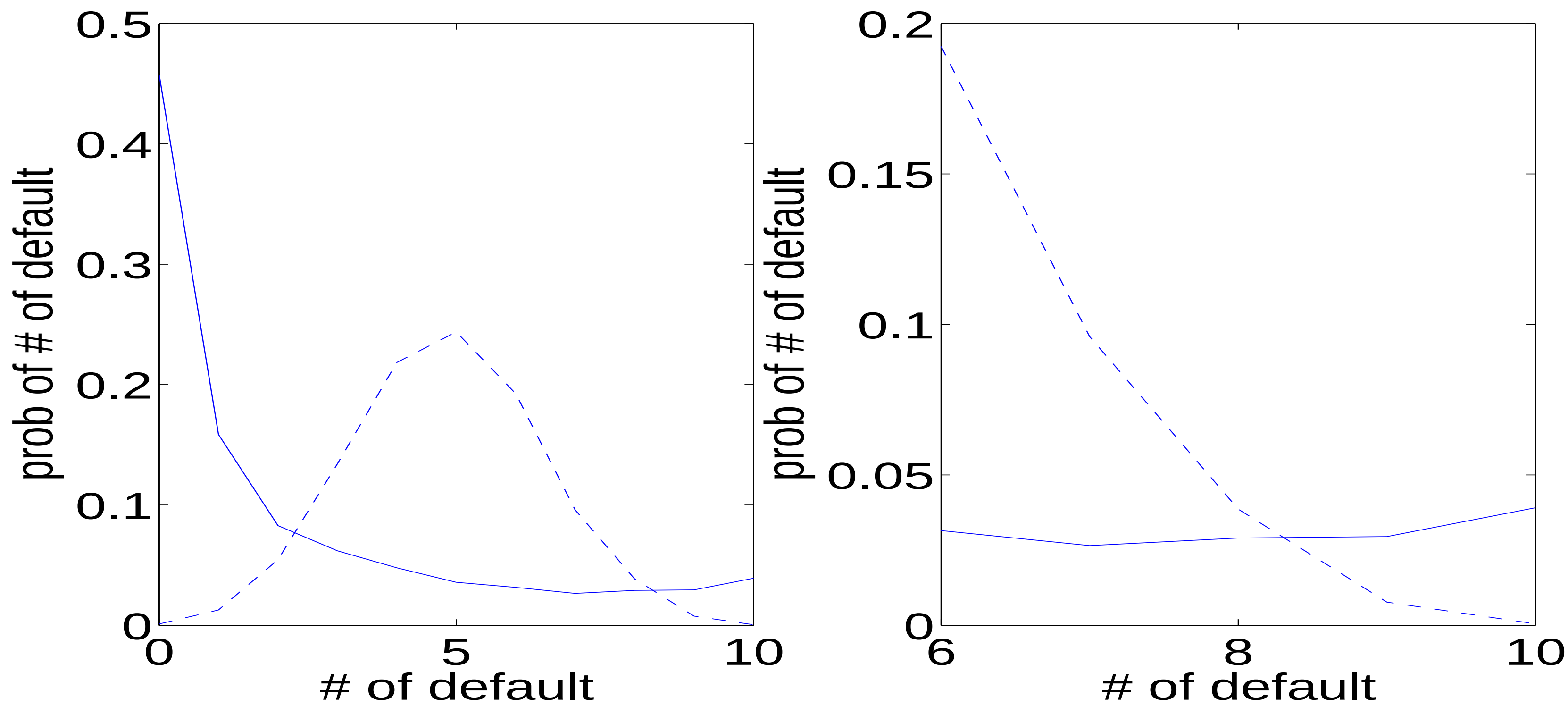
$$\begin{aligned} p &= \mathbb{P}\left(\min_{0 \le t \le T} (\sigma W_t) \le D\right) \\ &= 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\right), \end{aligned}$$

where Φ denotes the $\mathcal{N}(0,1)$ cdf.

With our choice of parameters, we have $p \approx 0.5$



On the left, we show plots of the loss distribution for the coupled diffusions with $\mathbf{a = 1}$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.



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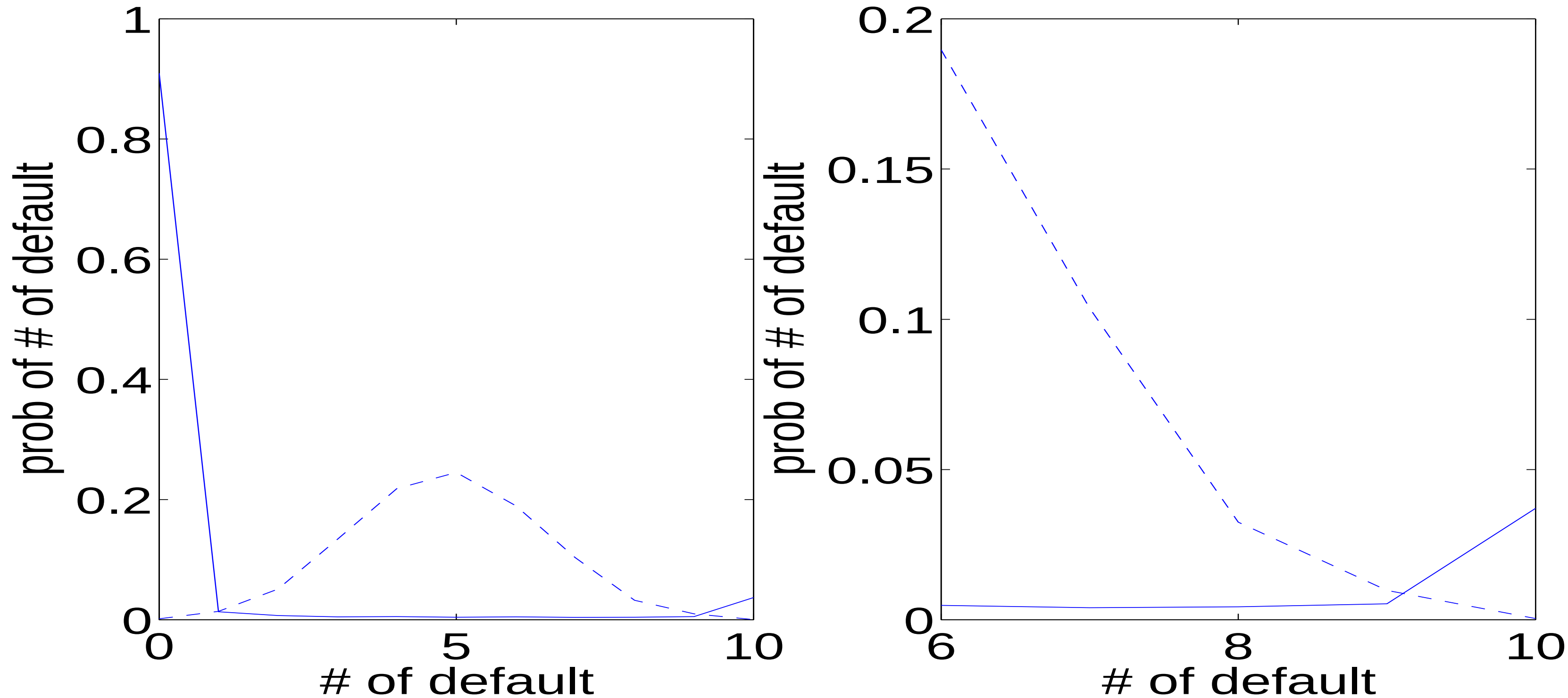

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Mean-field Limit

Rewrite the dynamics as:

$$
\begin{aligned}
dX_t^{(i)} &= \frac{a}{N}\sum_{j=1}^{N}(X_t^{(j)} - X_t^{(i)})\,dt + \sigma dW_t^{(i)} \\
&= a\left[\left(\frac{1}{N}\sum_{j=1}^{N} X_t^{(j)}\right) - X_t^{(i)}\right] dt + \sigma dW_t^{(i)}.
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$$

The processes $X^{(i)}$'s are "OUs" **mean-reverting** to the **ensemble average** which satisfies

$$
d\left(\frac{1}{N}\sum_{i=1}^{N} X_t^{(i)}\right) = d\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right).
$$

Assuming for instance that $x_0^{(i)} = 0$, $i = 1, \ldots, N$, we obtain

$$\frac{1}{N}\sum_{i=1}^{N} X_t^{(i)} = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}, \quad \text{and consequently}$$

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In the limit $N \to \infty$, the strong law of large numbers gives

$$\frac{1}{N}\sum_{j=1}^{N} W_t^{(j)} \to 0 \quad a.s.\,,$$

and therefore, the processes $X^{(i)}$'s converge to independent OU processes with long-run mean zero.

In fact, $X_t^{(i)}$ is given explicitly by

$$X_t^{(i)} = \frac{\sigma}{N}\sum_{j=1}^{N} W_t^{(j)} + \sigma e^{-at}\int_0^t e^{as} dW_s^{(i)} - \frac{\sigma}{N}\sum_{j=1}^{N}\left(e^{-at}\int_0^t e^{as} dW_s^{(j)}\right),$$

and therefore, $X_t^{(i)}$ converges to $\sigma e^{-at}\int_0^t e^{as} dW_s^{(i)}$ which are independent OU processes.

This is a simple example of a **mean-field limit** and propagation of chaos studied in general by Sznitman (1991).

Systemic Risk

Using classical equivalent for the Gaussian cumulative distribution function, we obtain the *large deviation estimate*

$$\lim_{N\to\infty} -\frac{1}{N}\log I\!\!P\left(\min_{0\le t\le T}\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}\right)\le D\right) = \frac{D^2}{2\sigma^2 T}\,.$$

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$$\text{Since} \quad \frac{1}{N}\sum_{i=1}^{N} X_t^{(i)} = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^{(i)}, \quad \text{we identify}$$

$$\left\{\min_{\mathbf{0\le t\le T}}\left(\frac{\mathbf{1}}{\mathbf{N}}\sum_{\mathbf{i=1}}^{\mathbf{N}} \mathbf{X}_{\mathbf{t}}^{(\mathbf{i})}\right)\le \mathbf{D}\right\} \quad \text{as a } \textbf{systemic event}$$

Observe that this event does not depend on $a > 0$

The probability

$$\exp\left(-\frac{D^2 N}{2\sigma^2 T}\right)$$

of a systemic event does not depend on $a > 0$, in other words:

" Increasing stability by increasing the rate of borrowing and lending does not prevent a systemic event where a large number of banks default"

In fact, once in this event, increasing a creates even more defaults by **"flocking to default"**. This is illustrated in the simulation with $a = 100$ where the probability of systemic risk is roughly 3%.

Systemic Risk and Common Noise

$$dX_t^i = a\left(\frac{1}{N}\sum_{j=1}^{N} X_t^j - X_t^i\right)dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}\,dW_t^i\right), \quad i = 1, \cdots, N,$$

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The ensemble average:

$$\frac{1}{N}\sum_{i=1}^{N} X_t^i = \frac{\sigma}{N}\sum_{i=1}^{N} \widetilde{W}_t^i \quad = \quad \sigma\left(\rho W_t^0 + \frac{\sqrt{1-\rho^2}}{N}\sum_{i=1}^{N} W_t^i\right)$$

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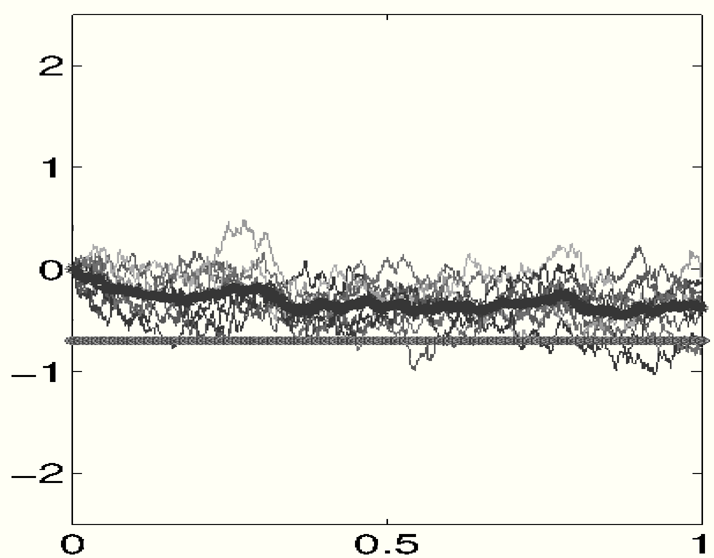
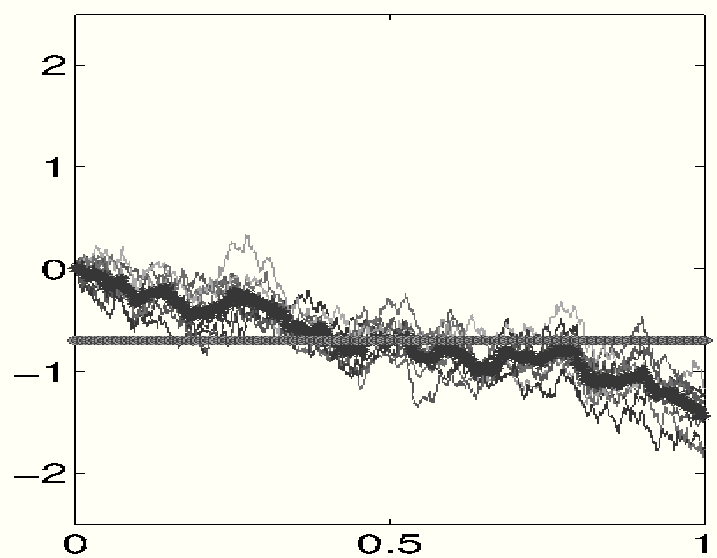
The **probability of the systemic** event becomes

$$\mathbb{P}\left(\min_{0\le s\le T}\frac{1}{N}\sum_{i=1}^N X_s^i < D\right) = 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\sqrt{\frac{N}{N\rho^2 + (1-\rho^2)}}\right) \to 2\Phi\left(\frac{D}{\sigma|\rho|\sqrt{T}}\right)$$

$\rho = 0$



$\rho = 0.5$



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What follows is from

Mean Field Games and Systemic Risk

by R. Carmona, J.-P. Fouque and L.-H. Sun (2013)

Mean Field Game

Banks are borrowing from and lending to a central bank:

$$dX_t^i \quad = \quad \alpha_t^i dt + \sigma dW_t^i, \quad i = 1, \cdots, N$$

where α^i is the control of bank i which wants to **minimize**

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left\{ \int_0^T f_i(X_t, \alpha_t^i)dt + g_i(X_T) \right\},$$

with **running cost**

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2 \right], \quad q^2 \leq \epsilon,$$

and **terminal cost** $\qquad g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2.$

This is an example of **Mean Field Game (MFG)** studied extensively by P.L. Lions and collaborators (see also the recent work of R. Carmona and F. Delarue).

Nash Equilibria (FBSDE Approach)

The Hamiltonian:

$$
\begin{aligned}
&H^i(x, y^{i,1}, \cdots, y^{i,N}, \alpha^1(t,x), \cdots, \alpha^i_t, \cdots, \alpha^N(t,x)) \\
&= \sum_{k \neq i} \alpha^k(t,x) y^{i,k} + \alpha^i y^{i,i} \\
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Ansatz:

$$
Y^{i,j}_t = \eta_t \left(\frac{1}{N} - \delta_{i,j} \right) (\overline{X}_t - X^i_t),
$$

where η_t is a deterministic function satisfying the terminal condition $\eta_T = c$.

Forward-Backward Equations

Forward Equation:

$$\begin{aligned} dX_t^i &= \partial_{y^{i,i}} H^i dt + \sigma dW_t^i \\ &= \left[q + (1 - \frac{1}{N})\eta_t \right] (\overline{X}_t - X_t^i) dt + \sigma dW_t^i, \end{aligned}$$

with initial conditions $X_0^i = x_0^i$.

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Backward Equation:

$$\begin{aligned} dY_t^{i,j} &= -\partial_{x^j} H^i dt + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k \\ &= (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i) \left[q\eta_t - \frac{1}{N}(\frac{1}{N} - 1)\eta_t^2 + q^2 - \epsilon \right] dt \\ &\quad + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k, \qquad Y_T^{i,j} = c(\frac{1}{N} - \delta_{i,j})(\overline{X}_T - X_T^i). \end{aligned}$$

Solution to the Forward-Backward Equations

By summation of the forward equations: $d\overline{X}_t = \frac{\sigma}{N}\sum_{k=1}^{N} dW_t^k$.

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$$Z_t^{i,j,k} = \eta_t\sigma(\frac{1}{N} - \delta_{i,j})(\frac{1}{N} - \delta_{i,k}) \text{ for } \; k = 1, \cdots, N,$$

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and η_t **must satisfy the Riccati equation**

$$\dot{\eta}_t = 2q\eta_t + (1 - \frac{1}{N^2})\eta_t^2 - (\epsilon - q^2),$$

with the terminal condition $\eta_T = c$.

Solution to the Riccati Equation

$$\eta_t = \frac{-(\epsilon - q^2)\left(e^{(\delta^+ - \delta^-)(T-t)} - 1\right) - c\left(\delta^+ e^{(\delta^+ - \delta^-)(T-t)} - \delta^-\right)}{\left(\delta^- e^{(\delta^+ - \delta^-)(T-t)} - \delta^+\right) - c(1 - \frac{1}{N^2})\left(e^{(\delta^+ - \delta^-)(T-t)} - 1\right)},$$

with the notations

$$\begin{aligned}
\delta^{\pm} &= -q \pm \sqrt{R}, \\
R &= q^2 + \left(1 - \frac{1}{N^2}\right)(\epsilon - q^2) > 0.
\end{aligned}$$

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Observe that η_t is well defined for any $t \leq T$ since the denominator can be written as

$$-\left(e^{(\delta^+-\delta^-)(T-t)} + 1\right)\sqrt{R} - \left(q + c\left(1 - \frac{1}{N^2}\right)\right)\left(e^{(\delta^+-\delta^-)(T-t)} - 1\right),$$

which stays negative because $\delta^+ - \delta^- = 2\sqrt{R} > 0$, and in fact, using $q^2 \leq \epsilon$, we see that η_t is positive with $\eta_T = c$.

Financial Implications

1. Once the function η_t has been obtained, bank i implements its strategy by using its control

$$\hat{\alpha}_t^i = -Y_t^{i,i} + q(\overline{X}_t - X_t^i) = \left[q + (1 - \frac{1}{N})\eta_t\right](\overline{X}_t - X_t^i),$$

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Observe that the average $\overline{X}_t$ is given by $d\overline{X}_t = \frac{\sigma}{N}\sum_{k=1}^N dW_t^k$, and is identical to the average found in the uncontrolled case.

Therefore, systemic risk occurs in the same manner as in the case of uncontrolled dynamics.

Financial Implications

2. In fact, the controlled dynamics can be rewritten

$$dX_t^i = \left(q + (1 - \frac{1}{N})\eta_t\right) \frac{1}{N} \sum_{j=1}^{N} (X_t^j - X_t^i) dt + \sigma dW_t^i.$$

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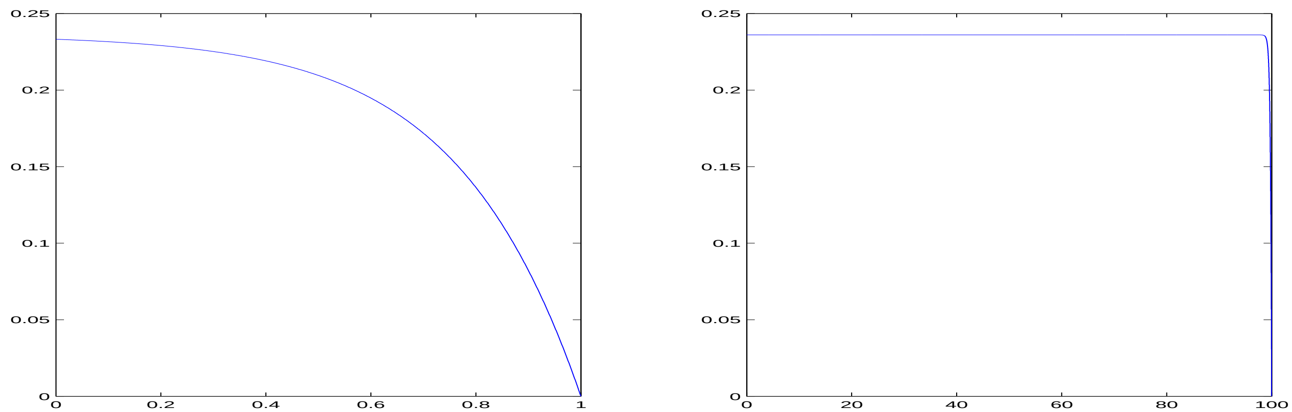
Under this equilibrium, the central bank is simply a **clearing house**, and the system is operating as if banks were borrowing from and lending to each other at the rate A_t, and the net effect is **creating liquidity** quantified by the rate of lending/borrowing.

Financial Implications

3. For T large (most of the time $T-t$ large), η_t is mainly constant. For instance, with $c=0$, $\lim_{T\to\infty}\eta_t = \frac{\epsilon-q^2}{-\delta^-} := \overline{\eta}$.



Plots of η_t with $c=0$, $q=1$, $\epsilon=2$ and $T=1$ on the left, $T=100$ on the right with $\overline{\eta}\sim 0.24$ (here we used $1/N\equiv 0$).

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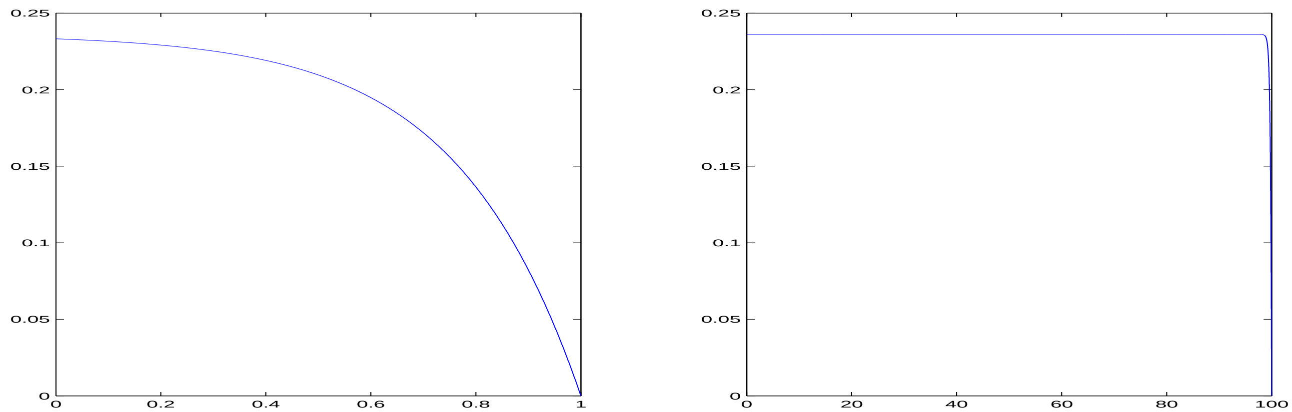

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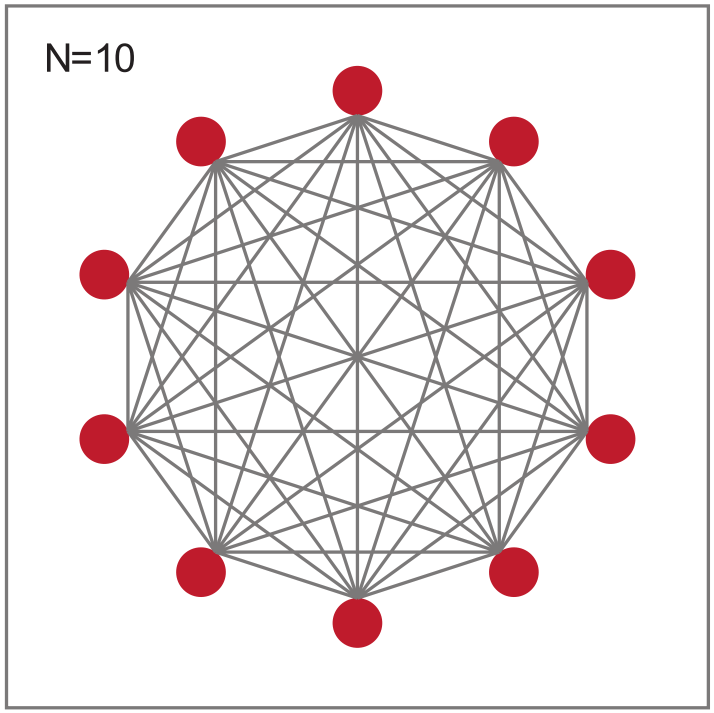
Therefore, in this infinite-horizon equilibrium, banks are borrowing and lending to each other at the constant rate

$$A := q + (1-\frac{1}{N})\overline{\eta} = q + \overline{\eta} \quad \text{in the } \textbf{Mean Field Limit}.$$

Equilibrium: Fully Connected Symmetric Network

$$dX_t^{(i)} = \frac{A}{N}\sum_{j=1}^{N}(X_t^{(j)} - X_t^{(i)})\,dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}\,dW_t^{(i)}\right), \quad i = 1, \ldots, N.$$



But, how to generate a centrally connected network?

Game with a Central Bank

Banks ($i = 2, \cdots, N$) are borrowing from and lending to a central bank ($i = 1$):

$$\begin{aligned} dX_t^i &= \alpha_t^i dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}\, dW_t^i\right), \quad i = 2, \cdots, N \\ dX_t^1 &= -\left(\sum_{i=2}^{N} \alpha_t^i\right) dt + \sigma_1 \left(\rho_1 dW_t^0 + \sqrt{1-\rho_1^2}\, dW_t^1\right) \end{aligned}$$

where α^i is the control of bank i which wants to **minimize**

$$J^i(\alpha^2, \cdots, \alpha^N) = \mathbb{E}\left\{\int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T)\right\},$$

with **running cost**

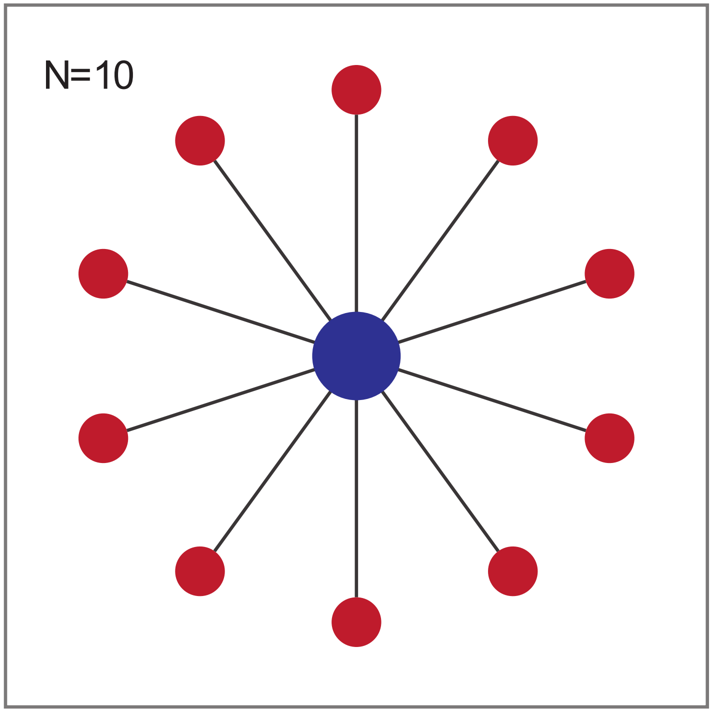
$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\mathbf{x^1} - x^i) + \frac{\epsilon}{2}(\mathbf{x^1} - x^i)^2\right], \quad q^2 \leq \epsilon,$$

and **terminal cost** $\qquad g_i(x) = \frac{c}{2}\left(\mathbf{x^1} - x^i\right)^2.$

Equilibrium (skipping details of the derivation)

$$dX_t^i = \mathbf{A}(\mathbf{X^1} - \mathbf{X_t^i})dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}\, dW_t^i\right), \quad i = 2, \cdots, N$$

$$dX_t^1 = -\mathbf{A}\sum_{\mathbf{i=2}}^{\mathbf{N}}(\mathbf{X_t^i} - \mathbf{X_t^1})dt + \sigma_1\left(\rho_1 dW_t^0 + \sqrt{1-\rho_1^2}\, dW_t^1\right)$$



Comparison using Systemic Risk Measures

Joint work in progress with F. Biagini, M. Frittelli and T. Meyer-Brandis

Let $X = (X_1, \cdots, X_N)$ be a risky position.

"Aggregate first" versus *"Allocate first"*

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- **Aggregate first:**

$$\rho(X) = \inf \left\{ m \in \mathbb{R} \mid \Lambda_1 (\sum_{i=1}^{N} X_i + m) \in \mathcal{A}^\gamma \right\}.$$

For instance: $\Lambda_1(Y) = (Y - d)^-$ and $\mathcal{A}^\gamma = \{Z \in L^0 \mid \mathbb{E}(Z) \leq \gamma\}$.

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- **Allocate first:**

$$\rho(X) = \inf \left\{ \sum_{i=1}^{N} m_i \mid \Lambda(X + m) \in \mathcal{A}^{\gamma}, m = (m_1, \cdots, m_N) \in I\!R^N \right\}.$$

For instance: $\Lambda(Y) = \sum_{i=1}^{N} (Y_i - d_i)^-$ and $\mathcal{A}^{\gamma} = \{Z \in L^0 \mid I\!E(Z) \leq \gamma\}$.

The Gaussian Case

Let $X = (X_1, \cdots, X_N)$ be a Gaussian vector with marginal distributions $\mathcal{N}(\mu_i, s_i^2)$, then by **allocating first** we get:

$$\rho(X) = \inf \left\{ \sum_{i=1}^{N} m_i \mid \sum_{i=1}^{N} \mathbb{E}[(X_i + m_i - d_i)^-] \leq \gamma \right\}.$$

$$\Psi(m) := \sum_{i=1}^{N} \mathbb{E}[(X_i + m_i - d_i)^-]$$

$$= \sum_{i=1}^{N} \left[-\frac{s_i}{\sqrt{2\pi}} \exp\left(-\frac{(d_i - \mu_i - m_i)^2}{2s_i^2} \right) + (d_i - \mu_i - m_i)\Phi\left(\frac{d_i - \mu_i - m_i}{s_i} \right) \right].$$

Optimal Allocation

Noting that

$$\frac{\partial \Psi(m)}{\partial m_i} = -\Phi\left(\frac{d_i - \mu_i - m_i}{s_i}\right),$$

a direct computation via Lagrange multiplier gives:

$$m_i^* = d_i - \mu_i - s_i R$$

where R solves

$$P(R) := \frac{1}{\sqrt{2\pi}} \exp(-R^2/2) - R\Phi(R) = -\frac{\gamma}{\sum_{i=1}^N s_i}$$

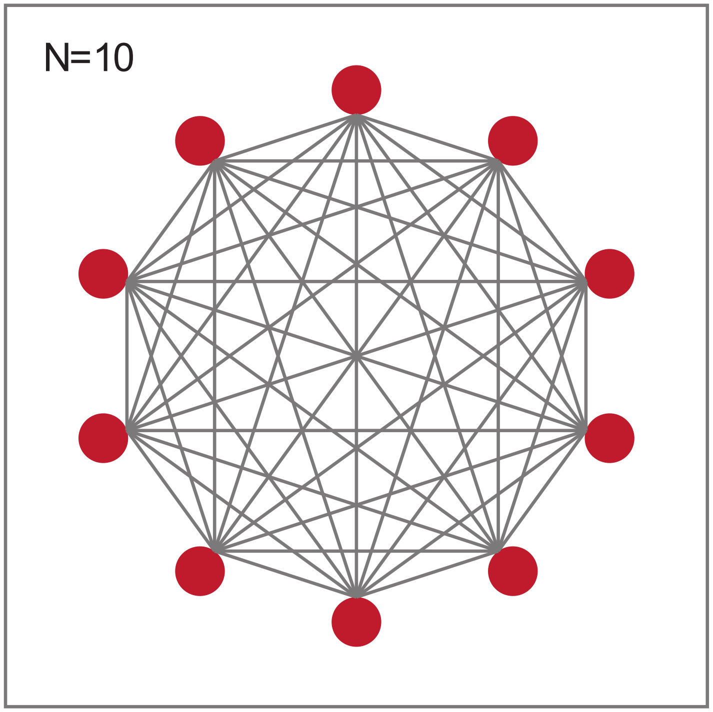
Note that this **"allocate first"** approach gives a way to rank banks in terms of their contributions to systemic risk.

A direct analysis of $\frac{\partial m_i}{\partial \mu_i}$ and $\frac{\partial m_i}{\partial \sigma_i}$ shows that m_i^* decreases with μ_i and increases with s_i.

Fully Connected Symmetric Network

Here we consider the fully homogeneous case where $x_0^i = x_0$, $p_{i,j} = a/N$, $\sigma_i = \sigma$, $\rho_i = \rho$, $d_i = d$, so that the model becomes

$$dX_t^i = \left[\frac{a}{N} \sum_{j=1}^{N} (X_t^j - X_t^i) \right] dt + \sigma \left(\rho dW_t^0 + \sqrt{1-\rho^2} dW_t^i \right), \quad i = 1, \cdots, N,$$



Fully Connected Symmetric Network (continued)

$X_t = (X_t^1, \cdots, X_t^N)$ is Gaussian with marginal distributions characterized by the means

$$\mu_i = \mathbb{E}(X_t^i) = x_0,$$

and the variances

$$s_i^2 = \sigma^2(1-\rho^2)(1-\frac{1}{N})\left(\frac{1-e^{-2at}}{2a}\right) + \sigma^2\left(\rho^2 + \frac{1-\rho^2}{N}\right)t.$$

Central Clearing Symmetric Network

$$dX_t^i = \mathbf{a}(\mathbf{X^1} - \mathbf{X_t^i})dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}\, dW_t^i\right), \quad i = 2, \cdots, N$$

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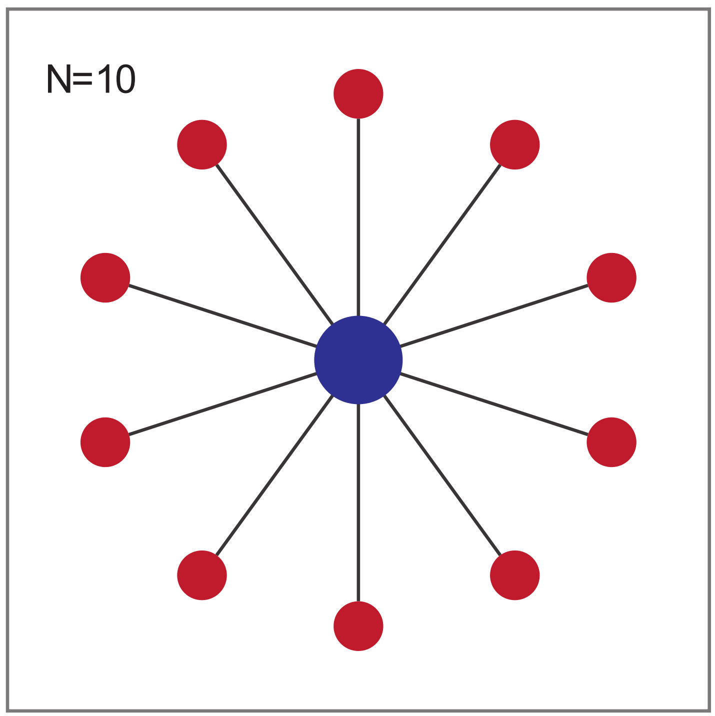

Central Clearing Symmetric Network (continued)

$X_t = (X_t^1, \cdots, X_t^N)$ is again Gaussian. Assuming $X_0^i = x_0, i = 1, \cdots, N$, a tedious computation gives $\mathbb{E}(X_t^i) = x_0$ and the marginal variances for $i \geq 2$:

$$
\begin{aligned}
s_i^2 &= \sigma^2(1-\rho^2)(1-\frac{1}{N})\left(\frac{1-e^{-2at}}{2a}\right) + \sigma^2\left(\rho^2 + \frac{1-\rho^2}{N}\right)t \\
&+ \frac{2\sigma\rho t}{N}\left[\sigma_1\rho_1 - \sigma\rho\right] + \mathcal{O}(\frac{1}{N^2}),
\end{aligned}
$$

which shows that the **sign of** $[\sigma_1\rho_1 - \sigma\rho]$ decides which network is more risky (at least at the order $1/N$).

Mean Field Game ($N \to \infty$)

In most cases the finite player game will not be solved explicitly.
For instance, with a slight generalization such as:

$$dX_t^i \quad = \quad a(\overline{X}_t)\,\alpha_t^i dt + \sigma dW_t^i, \quad i = 1, \cdots, N$$

This is where MFG is needed.

Mean Field Game ($N \to \infty$) with Common Noise

1. Fix $(m_t)_{t \geq 0}$ (the limit of $\overline{X}_t$ as $N \to \infty$ which depends on W^0)

Mean Field Game ($N \to \infty$)

- Fix $(m_t)_{t\geq 0}$ (the limit of $\overline{X}_t$ as $N \to \infty$ which depends on W^0)
- Solve the control problem

$$\inf_{(\alpha_t)} \mathbb{E}\left\{\int_0^T \left[\frac{1}{2}(\alpha_t)^2 - q\alpha_t(m_t - X_t) + \frac{\epsilon}{2}(m_t - X_t)^2\right] dt + \frac{c}{2}(m_T - X_T)^2\right\}$$

subject to: $dX_t = \alpha_t dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}\, dW_t\right)$

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Hamiltonian:

$$H(t,x,y,\alpha) = \alpha y + \frac{1}{2}\alpha^2 - q\alpha(m_t - x) + \frac{\epsilon}{2}(m_t - x)^2$$

$$\frac{\partial H}{\partial \alpha} \longrightarrow \hat{\alpha} = q(m_t - x) - y$$

See also the recent paper by Carmona-Delarue-Lacker.

Adjoint Equations

$$
\begin{aligned}
dX_t &= [q(m_t - X_t) - Y_t]\,dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t\right), \qquad X_0 = \xi \\
dY_t &= -\frac{\partial H}{\partial x}dt + Z_t^0 dW_t^0 + Z_t dW_t\,, \qquad\qquad Y_T = c(X_T - m_T) \\
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Taking conditional expectation given $(W_s^0)_{s\le t}$ in the second equation and using $m_t = m_t^X$ for all $t \le T$ and consequently $m_T^Y = c(m_T^X - m_T) = 0$, gives:

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$$

Then, taking conditional expectations in the first equation gives:

$$
dm_t^X = -m_t^Y dt + \rho\sigma dW_t^0.
$$

Adjoint Equations (continued)

Ansatz: $Y_t = -\eta_t(m_t - X_t)$ with η_t deterministic.

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Differentiating this ansatz and using the forward equation leads to

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Plugging the ansatz in the backward equation gives

$$dY_t = \left[-q\eta_t + (\epsilon - q^2)\right](m_t - X_t)dt + Z_t^0 dW_t^0 + Z_t dW_t.$$

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From $m_t^Y = -\int_t^T e^{q(s-t)} Z_s^0 dW_s^0$, we obtain $m_t^Y = 0$.

Mean Field Game Solution

Equating the drifts in the two Itô decompositions, we get

$$\dot{\eta}_t = \eta_t^2 + 2q\eta_t - \left(\epsilon - q^2\right), \quad \eta_T = c,$$

which is the same Riccati equation as before but with "$N = \infty$".

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From $m_t^Y = 0$, we deduce that $m_t^X = \mathbb{E}(\xi) + \rho\sigma W_t^0$ which will enter in the optimal control $(q + \eta_t)(m_t^X - X_t)$.

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Once a solution to the MFG is found, on can use it to construct approximate Nash equilibria for the finitely many players games. Here, if one assumes that each player is given the information $\overline{X}_t$, and if player i uses the strategy $\alpha_t^i = (q + \eta_t)(\overline{X}_t - X_t^i)$, which is the limit as $N \to \infty$ of the strategy used in the finite players game, one sees how solving the limiting MFG problem can provide approximate Nash equilibria for which the financial implications are identical as the ones given for the exact Nash equilibria.

Mean Field Equilibrium

As $N \to \infty$, banks become independent,
$\hat{\alpha} = q(0 - x) - y = -(q + \eta)x$, and they follow the dynamics

$$dX_t = -(q + \eta_t)X_t dt + \sigma dW_t$$

But observe that **the large deviation probability of systemic risk has been lost.**

MFG with Common Noise / HJB Approach

For Markovian strategies $\alpha(t,x)$, the dynamics are given by

$$dX_t = \alpha(t,X_t)dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t\right).$$

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$$dX_t = \alpha(t, X_t)dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t\right).$$

Forward equation for the conditional density of X_t given W^0:

$$dp_t = \left\{-\partial_x\left[\alpha(t,x)p_t\right] + \frac{1}{2}\sigma^2(1-\rho^2)\partial_{xx}p_t\right\}dt - \rho\sigma(\partial_x p_t)dW_t^0$$

with $\alpha(t,x)$ given and $m_t = \int x p_t(x)dx$.

The conditional mean m_t of X_t given W^0 is Markovian with infinitesimal generator denoted by $\mathcal{L}^m dt + \rho\sigma(\partial_m)dW_t^0$.

Stochastic HJB Equation

The HJB equation for the value function $V(t, x, m)$ is

$$\begin{aligned} dV &+ \left[\frac{1}{2}\sigma^2(1-\rho^2)\partial_{xx}V + \mathcal{L}^m V + (\partial_{xm}V)\frac{d\langle m, X\rangle}{dt}\right] dt \\ &+ \inf_{\alpha}\left\{\alpha\partial_x V + \frac{\alpha^2}{2} - q\alpha(m-x) + \frac{\epsilon}{2}(m-x)^2\right\} dt \\ &+ \rho\sigma(\partial_m V)dW_t^0 + \rho\sigma(\partial_x V)dW_t^0 = 0. \end{aligned}$$

- Minimize in α to get $\hat{\alpha} = q(m-x) - \partial_x V$
- Make the ansatz $V(t, x, m) = \frac{\eta_t}{2}(m-x)^2 + \mu_t$
- Plug in the forward equation for p_t, multiply by x, and integrate with respect to x gives:

$$dm_t = \rho\sigma dW_t^0$$

Mean Field Game Solution

Therefore, conditionally in W^0, $\mathcal{L}^m V = 0$ and $d\langle m, X\rangle = 0$.

Then, verifying that the ansatz satisfies the HJB equation, by canceling terms in $(m-x)^2$ we obtain that η_t must satisfy the same Riccati equation as before (not affected by ρ):

$$\dot{\eta}_t = \eta_t^2 + 2q\eta_t - (\epsilon - q^2), \quad \eta_T = 0.$$

Canceling state-independent terms leads to $\dot{\mu}_t = -\frac{1}{2}\sigma^2(1-\rho^2)\eta_t$ and therefore

$$\mu_t = \frac{1}{2}\sigma^2(1-\rho^2)\int_t^T \eta_s ds.$$

THANKS FOR YOUR ATTENTION