Mathematical Behavioural Finance

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Part 3:

Market Equilibrium and Asset Pricing

Section 1

An Arrow-Debreu Economy

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- Aggregate endowment is $(e_0, \tilde{e}_1) := \left(\sum_{i=1}^{I} e_{0i}, \sum_{i=1}^{I} \tilde{e}_{1i}\right)$

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- The preference of agent i over $(c_{0i}, \tilde{c}_{0i})$ is represented by

$$V_i(c_{0i}, \tilde{c}_{1i}) = u_{0i}(c_{0i}) + \beta_i \int u_{1i}(\tilde{c}_{1i}) d(w_i \circ \mathrm{P}),$$

where

  - u_{0i} is utility function for $t = 0$;
  - (u_{1i}, w_i) is the RDUT pair for $t = 1$;
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- The set of all feasible consumption plans is denoted by $\mathscr{C}$

Pricing Kernel

- The above economy is denoted by

$$\mathscr{E} := \left\{ (\Omega, \mathcal{F}, \mathrm{P}),\ (e_{0i}, \tilde{e}_{1i})_{i=1}^{I},\ \mathscr{C},\ \big(V_i(c_{0i}, \tilde{c}_{1i})\big)_{i=1}^{I} \right\}$$

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- A *pricing kernel* (or *state-price density, stochastic discount factor*) is an $\mathcal{F}$-measurable random variable $\tilde{\rho}$, with $\mathrm{P}(\tilde{\rho} > 0) = 1$, $\mathrm{E}[\tilde{\rho}] < \infty$ and $\mathrm{E}[\tilde{\rho}\tilde{e}_1] < \infty$, such that any claim $\tilde{x}$ tomorrow is priced at $\mathrm{E}[\tilde{\rho}\tilde{x}]$ today

Arrow-Debreu Equilibrium

An *Arrow–Debreu equilibrium* of $\mathscr{E}$ is a collection $\{\tilde{\rho}, (c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I\}$ consisting of a pricing kernel $\tilde{\rho}$ and a collection $(c^*_{0i}, \tilde{c}^*_{1i})_{i=1}^I$ of feasible consumption plans, that satisfies the following conditions:

Individual optimality : For every i, $(c^*_{0i}, \tilde{c}^*_{1i})$ maximises the preference of agent i subject to the budget constraint, that is,

$$V_i(c^*_{0i}, \tilde{c}^*_{1i}) = \max_{(c_{0i}, \tilde{c}_{1i}) \in \mathscr{C}} V_i(c_{0i}, \tilde{c}_{1i})$$

$$\text{subject to } c_{0i} + \mathrm{E}[\tilde{\rho}\tilde{c}_{1i}] \leq e_{0i} + \mathrm{E}[\tilde{\rho}\tilde{e}_{1i}]$$

Market clearing : $\sum_{i=1}^I c^*_{0i} = e_0$ and $\sum_{i=1}^I \tilde{c}^*_{1i} = \tilde{e}_1$

Literature

- Mainly on CPT economies, and on existence of equilibria
  - Qualitative structures of pricing kernel for both CPT and SP/A economies, assuming existence of equilibrium: Shefrin (2008)
  - Non-existence: De Giorgi, Hens and Riegers (2009), Azevedo and Gottlieb (2010)
  - Under specific asset return distribution: Barberis and Huang (2008)
  - One risky asset: He and Zhou (2011)
- RDUT economy with **convex** weighting function: Carlier and Dana (2008), Dana (2011) – existence

Standing Assumptions

- Agents have **homogeneous beliefs** P; $(\Omega, \mathcal{F}, \mathrm{P})$ admits no atom.
- For every i, $e_{0i} \geq 0$, $\mathrm{P}(\tilde{e}_{1i} \geq 0) = 1$, and $e_{0i} + \mathrm{P}(\tilde{e}_{1i} > 0) > 0$. Moreover, $\tilde{e}_1$ is **atomless**, $\mathrm{P}(\tilde{e}_1 > 0) = 1$, and $e_0 > 0$.
- For every i, $u_{0i}, u_{1i} : [0, \infty) \to \mathbb{R}$ are strictly increasing, strictly concave, continuously differentiable on $(0, \infty)$, and satisfy the **Inada** condition: $u'_{0i}(0+) = u'_{1i}(0+) = \infty$, $u'_{0i}(\infty) = u'_{1i}(\infty) = 0$. Moreover, $u_{1i}(0) = 0$.
- For every i, $w_i : [0, 1] \to [0, 1]$ is strictly increasing and continuously differentiable, and satisfies $w_i(0) = 0$, $w_i(1) = 1$.

Section 2

Individual Optimality

Individual Consumptions

Consider

$$\begin{aligned} \underset{(c_0,\tilde{c}_1)\in\mathscr{C}}{\text{Max}} \quad & V(c_0,\tilde{c}_1) := u_0(c_0) + \beta\int_0^\infty w\left(\mathrm{P}\left(u_1(\tilde{c}_1) > x\right)\right)dx \\ \text{subject to} \quad & c_0 + \mathrm{E}[\tilde{\rho}\tilde{c}_1] \le \varepsilon_0 + \mathrm{E}[\tilde{\rho}\tilde{\varepsilon}_1] \end{aligned} \tag{1}$$

where $\tilde{\rho}$ is **exogenously** given, atomless, and ε_0 and $\tilde{\varepsilon}_1$ are endowments at $t=0$ and $t=1$ respectively

Quantile Formulation

- Recall the set of quantile functions of nonnegative random variables

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- Problem (1) can be reformulated as

$$\begin{aligned} \underset{c_0 \geq 0,\, G \in \mathbb{G}}{\text{Max}} \quad & U(c_0, G) := u_0(c_0) + \beta \int_0^1 u_1(G(p))\, d\bar{w}(p) \\ \text{subject to} \quad & c_0 + \int_0^1 F_{\tilde{\rho}}^{-1}(1-p) G(p)\, dp \leq \varepsilon_0 + \mathrm{E}[\tilde{\rho}\tilde{\varepsilon}_1], \end{aligned} \tag{2}$$

where $\bar{w}(p) = 1 - w(1-p)$

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where $\bar{w}(p) = 1 - w(1-p)$

- If $(c_0^*, G^*) \in [0,\infty) \times \mathbb{G}$ solves (2), then $(c_0^*, \tilde{c}_1^*)$, where $\tilde{c}_1^* = G^*(1 - F_{\tilde{\rho}}(\tilde{\rho}))$, solves (1)

Lagrange

Step 1. For a fixed Lagrange multiplier $\lambda > 0$, solve

$$\begin{aligned} \operatorname*{Max}_{c_0 \geq 0,\, G \in \mathbb{G}} \quad & u_0(c_0) + \beta \int_0^1 u_1(G(p))d\bar{w}(p) \\ & - \lambda \left(c_0 + \int_0^1 F_{\tilde{\rho}}^{-1}(1-p)G(p)dp - \varepsilon_0 - \mathrm{E}[\tilde{\rho}\tilde{\varepsilon}_1] \right). \end{aligned}$$

The solution (c_0^*, G^*) implicitly depends on λ

Step 2. Determine λ by

$$c_0^* + \int_0^{1-} F_{\tilde{\rho}}^{-1}(1-p)G^*(p)dp = \varepsilon_0 + \mathrm{E}[\tilde{\rho}\tilde{\varepsilon}_1]$$

Step 3. $\tilde{c}_1^* := G^*(1 - F_{\tilde{\rho}}(\tilde{\rho}))$

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&= \int_0^1 \left[u_1(G(p))w'(1-p) - \frac{\lambda}{\beta} F_{\tilde{\rho}}^{-1}(1-p)G(p) \right] dp
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- We have solved this problem ... provided that $M(z) = \frac{w'(1-z)}{F_{\tilde{\rho}}^{-1}(1-z)}$ satisfies some monotone condition!
- **Difficulty:** Such a condition (or literally any condition) is **not** permitted in our equilibrium problem!

Calculus of Variation

- Set

  $$\mathbb{G}_0 = \{G : [0,1) \to [0,\infty] \,|\, G \in \mathbb{G} \text{ and } G(p) > 0 \text{ for all } p \in (0,1)\}$$

- Calculus of variation shows that solving (3) is equivalent to finding $G \in \mathbb{G}_0$ satisfying

$$\begin{cases} \int_q^1 u_1'(G(p))d\bar{w}(p) - \frac{\lambda}{\beta}\int_q^1 F_{\tilde{\rho}}^{-1}(1-p)dp \le 0 \quad \forall q \in [0,1), \\ \int_0^1 \left(\int_q^{1-} u_1'(G(p))d\bar{w}(p) - \frac{\lambda}{\beta}\int_q^1 F_{\tilde{\rho}}^{-1}(1-p)dp\right) dG(q) = 0 \end{cases} \tag{4}$$

Equivalent Condition

Previous condition is equivalent to

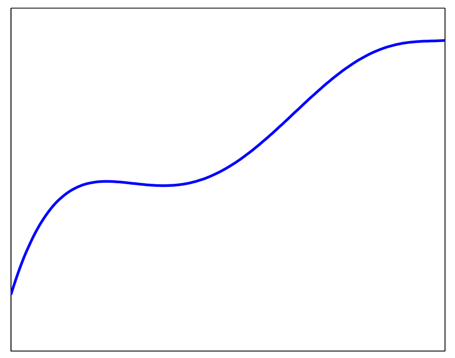
$$\begin{cases} K(q) \geq \dfrac{\lambda}{\beta} N(q) \quad \text{for all } q \in (0,1), \\ K \text{ is affine on } \left\{ q \in (0,1) : K(q) > \dfrac{\lambda}{\beta} N(q) \right\}, \\ K(0) = \dfrac{\lambda}{\beta} N(0),\ K(1-) = N(1-) \end{cases} \tag{5}$$

where

$$\begin{cases} K(q) = -\displaystyle\int_q^1 u_1'(G(\bar{w}^{-1}(p)))dp \\ N(q) = -\displaystyle\int_q^1 F_{\tilde{\rho}}^{-1}(1-\bar{w}^{-1}(p))d\bar{w}^{-1}(p) \end{cases} \tag{6}$$

for all $q \in [0,1)$

How Concave Envelope is Formed



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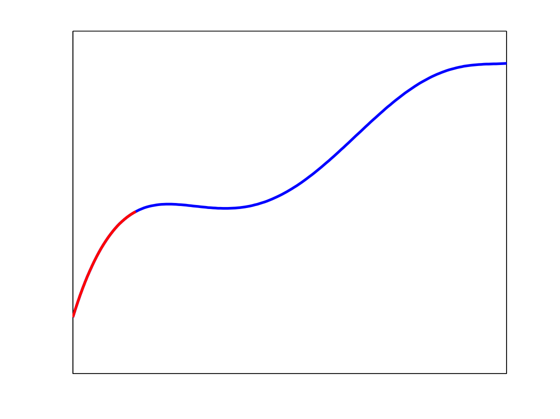

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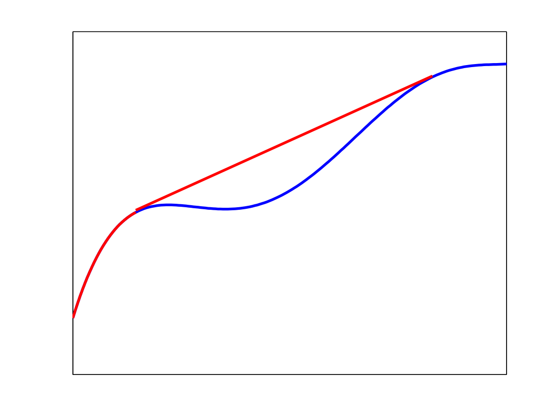

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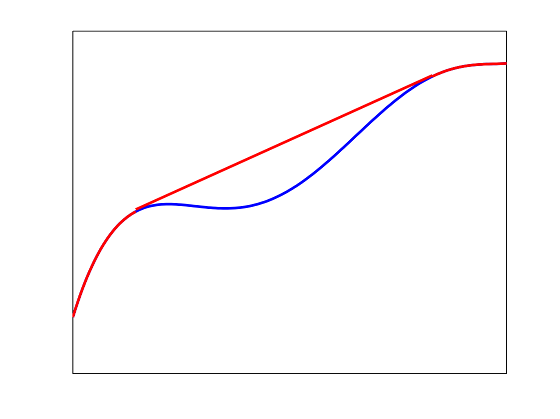

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- $G^*(q) = (u_1')^{-1}\left(\frac{\lambda}{\beta}\hat{N}'(1 - w(1-q))\right)$
- $\tilde{c}_1^* = G^*(1 - F_{\tilde{\rho}}(\tilde{\rho})) = (u_1')^{-1}\left(\frac{\lambda}{\beta}\hat{N}'\left(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\right)\right)$

Complete/Explicit Solution to Individual Consumption

Theorem

(Xia and Zhou 2012) *Assume that $\tilde{\rho} > 0$ a.s., atomless, with $E[\tilde{\rho}] < +\infty$. Then the optimal consumption plan is given by*

$$\begin{cases} c_0^* = (u_0')^{-1}(\lambda) \\ \tilde{c}_1^* = (u_1')^{-1}\left(\dfrac{\lambda}{\beta}\hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big)\right), \end{cases}$$

where λ is determined by

$$(u_0')^{-1}(\lambda) + \mathbb{E}\left[\tilde{\rho}(u_1')^{-1}\left(\frac{\lambda}{\beta}\hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big)\right)\right] = \varepsilon_0 + \mathbb{E}[\tilde{\rho}\tilde{\varepsilon}].$$

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- It recovers one of the results in Part 2!

Section 3

Representative RDUT Agent

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where λ_i^* satisfies

$$(u_{0i}')^{-1}(\lambda_i^*) + \mathrm{E}\left[\tilde{\rho}(u_{1i}')^{-1}\left(\frac{\lambda_i^*}{\beta_i}\hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big)\right)\right] = e_{0i} + \mathrm{E}[\tilde{\rho}\tilde{e}_{1i}]$$

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- Aggregate consumption is

$$c_0^* = \sum_{i=1}^{I}(u_{0i}')^{-1}(\lambda_i^*),\ \tilde{c}_1^* = \sum_{i=1}^{I}(u_{1i}')^{-1}\left(\frac{\lambda_i^*}{\beta_i}\hat{N}'\Big(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\Big)\right)$$

A Representative Agent

- For $\lambda_1 > 0, \ldots, \lambda_I > 0$, set $\lambda = (\lambda_1, \ldots, \lambda_I)$ and

$$h_{0\lambda}(y) := \sum_{i=1}^{I} (u'_{0i})^{-1}(\lambda_i y), \; h_{1\lambda}(y) := \sum_{i=1}^{I} (u'_{1i})^{-1}\left(\frac{\lambda_i y}{\beta_i}\right)$$

- Define $u_{t\lambda}(x) = \int_0^x h_{t\lambda}^{-1}(z)dz$, $t = 0, 1$
- Then

$$c_0^* = (u'_{0\lambda^*})^{-1}(1), \; \tilde{c}_1^* = (u'_{1\lambda^*})^{-1}\left(\hat{N}'\left(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\right)\right)$$

- Consider an **RDUT** agent, indexed by λ^*, whose preference is

$$V_{\lambda^*}(c_0, \tilde{c}_1) := u_{0\lambda^*}(c_0) + \int u_{1\lambda^*}(\tilde{c}_1) d(w \circ \mathrm{P}) \qquad (7)$$

  and whose endowment is the aggregate endowment $(e_0, \tilde{e}_1)$
- This representative agent's optimal consumption plan is the aggregate consumption plan

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- Use existing results for EUT economy

Section 4

Asset Pricing

Explicit Expression of Pricing Kernel

Theorem

(Xia and Zhou 2012) *If there exists an equilibrium of economy* $\mathscr{E}$ *where the pricing kernel* $\tilde{\rho}$ *is atomless and* λ^* *is the corresponding Lagrange vector, then*

$$\boxed{\tilde{\rho} = w'(1 - F_{\tilde{e}_1}(\tilde{e}_1)) \frac{u'_{1\lambda^*}(\tilde{e}_1)}{u'_{0\lambda^*}(e_0)} \quad \text{a.s.}.} \tag{8}$$

Idea of proof. Market clearing – $\tilde{e}_1 = \tilde{c}_1^* = (u'_{1\lambda^*})^{-1}\left(\hat{N}'\left(1 - w(F_{\tilde{\rho}}(\tilde{\rho}))\right)\right)$ – manipulate quantiles

Interpretations

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- Pricing kernel is a **weighted** marginal rate of substitution between initial and end-of-period consumption
- The weight is $w'(1 - F_{\tilde{e}_1}(\tilde{e}_1))$
- An inverse-S shaped weighting w leads to a **premium** when evaluating assets in both very high and very low future consumption states

Implied Utility Function

- Define u_w by

$$u'_w(x) = w'(1 - F_{\tilde{e}_1}(x))u'_{1\lambda^*}(x)$$

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- u_w: *implied utility function*

Implied Relative Risk Aversion

- Implied relative index of risk aversion

$$R^w(x) := -\frac{xu_w''(x)}{u_w'(x)} = -\frac{xu_{1\lambda^*}''(x)}{u_{1\lambda^*}'(x)} + \frac{xw''(1 - F_{\tilde{e}_1}(x))}{w'(1 - F_{\tilde{e}_1}(x))} f_{\tilde{e}_1}(x) \tag{9}$$

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- It represents **overall** degree of risk-aversion (or risk-loving) of RDUT agent, combining outcome utility and probability weighting

Existence of Equilibria

Theorem

(Xia and Zhou 2012) *If* $\Psi_\lambda(p) \equiv w'(p)\, u'_{1\lambda}\left(F_{\tilde{e}_1}^{-1}(1-p)\right)$ *is strictly increasing for any* λ*, and*

$$\begin{cases} \mathrm{E}[w'(1-F_{\tilde{e}_1}(\tilde{e}_1))u_{1i}(\tilde{e}_1)] < \infty \\ \mathrm{E}\left[w'(1-F_{\tilde{e}_1}(\tilde{e}_1))u'_{1i}\left(\dfrac{\tilde{e}_1}{I}\right)\right] < \infty \end{cases}$$

for all $i = 1, \dots, I$*, then there exists an Arrow-Debreu equilibrium of economy* $\mathscr{E}$ *where the pricing kernel is atomless. If in addition*

$$-\frac{cu''_{1i}(c)}{u'_{1i}(c)} \le 1 \text{ for all } i = 1, \dots, I \text{ and } c > 0,$$

then the equilibrium is unique.

Monotonicity of Ψ_λ

- It is defined through model primitives:
  $\Psi_\lambda(p) = w'(p)\, u'_{1\lambda}\left(F_{\tilde{e}_1}^{-1}(1-p)\right)$
- Monotonicity of Ψ_λ for any λ requires a **concave** implied utility function for any initial distribution of the wealth.
- Automatically satisfied when w is convex
- Possibly satisfied when w is concave or inverse-S shaped

Monotonicity of Ψ_λ: An Example

Example. Take $w(p) = p^{1-\alpha}$ where $\alpha \in (0,1)$, $u_{1\lambda}(c) = \frac{c^{1-\beta}}{1-\beta}$ where $\beta \in (0,1)$, and $\tilde{e}_1$ follows the Parato distribution

$$F_{\tilde{e}_1}(x) = \begin{cases} 1 - \left(\frac{x_m}{x}\right)^\gamma & x \geq x_m \\ 0 & x < x_m. \end{cases}$$

In this case

$$\Psi_\lambda(p) = w'(p) u'_{1\lambda}\left(F_{\tilde{e}_1}^{-1}(1-p)\right) = (1-\alpha) x_m^{-\beta} p^{\frac{\beta}{\gamma}-\alpha}.$$

This is a strictly increasing function if and only if $\alpha < \frac{\beta}{\gamma}$.

Section 5

CCAPM and Interest Rate

Consumption-Based CAPM

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- A rank-dependent consumption-based CAPM (CCAPM):

$$\boxed{\bar{r} - r_f \approx \left[\alpha + \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0\right] \mathbf{Cov}(\tilde{g}, \tilde{r})}$$

where $\alpha := -\frac{e_0 u''_{1\lambda*}(e_0)}{u'_{1\lambda*}(e_0)}$ and $f_{\tilde{e}_1}$ is density function of $\tilde{e}_1$

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- Classical EUT based CCAPM: $\bar{r} - r_f \approx \alpha \mathbf{Cov}(\tilde{g}, \tilde{r})$

Prices and Expected Consumption Growth

- Again $\bar{r} - r_f \approx \left[\alpha + \frac{w''(1-F_{\tilde{e}_1}(e_0))}{w'(1-F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0)e_0\right] \mathbf{Cov}(\tilde{g}, \tilde{r})$

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- Recall $1 - F_{\tilde{e}_1}(e_0) = P(\tilde{e}_1 > e_0)$
- The subjective expectation (or belief) on general consumption growth should be priced in for individual assets

Consumption-Based Real Interest

- A rank-dependent consumption-based real interest rate formula:

$$\boxed{1 + r_f \approx \frac{1}{\beta w'(1 - F_{\tilde{e}_1}(e_0))} \left[1 + \alpha \bar{g} + \frac{w''(1 - F_{\tilde{e}_1}(e_0))}{w'(1 - F_{\tilde{e}_1}(e_0))} f_{\tilde{e}_1}(e_0) e_0 \bar{g} \right]}$$

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- Classical EUT based real interest rate theory: $1 + r_f \approx \frac{1+\alpha\bar{g}}{\beta}$

Section 6

Equity Premium and Risk-Free Rate Puzzles

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- Equity premium puzzle (Mehra and Prescott 1985): observed equity premium is too high to be explainable by classical CCAPM
- Risk-free rate puzzle (Weil 1989): observed risk-free rate is too low to be explainable by classical CCAPM

Economic Data 1889–1978 (Mehra and Prescott 1985)

	Consumption growth		riskless return		equity premium		S&P 500 return	
Periods	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
1889–1978	**1.83**	**3.57**	**0.80**	**5.67**	**6.18**	**16.67**	**6.98**	**16.54**
1889–1898	2.30	4.90	5.80	3.23	1.78	11.57	7.58	10.02
1899–1908	2.55	5.31	2.62	2.59	5.08	16.86	7.71	17.21
1909–1918	0.44	3.07	-1.63	9.02	1.49	9.18	-0.14	12.81
1919–1928	3.00	3.97	4.30	6.61	14.64	15.94	18.94	16.18
1929–1938	-0.25	5.28	2.39	6.50	0.18	31.63	2.56	27.90
1939–1948	2.19	2.52	-5.82	4.05	8.89	14.23	3.07	14.67
1949–1958	1.48	1.00	-0.81	1.89	18.30	13.20	17.49	13.08
1959–1968	2.37	1.00	1.07	0.64	4.50	10.17	5.58	10.59
1969–1978	2.41	1.40	-0.72	2.06	0.75	11.64	0.03	13.11

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- A measure of 30 means indifference between a gamble equally likely to pay \$50,000 or \$100,000 and a certain payoff of \$51,209
- No human is *that* risk averse

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- Expected rate of return provided by our model is larger than that by EUT

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- The presence of a suitable probability weighting function will *simultaneously* increase equity premium and decrease risk-free rate under RDUT, diminishing the gap seen under EUT

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- In general, at times when most people believe that economy is in a downturn, expected rate of return provided by RDUT is smaller than that provided by EUT model
- Hence we should investigate asset pricing by differentiating periods of economic growth from those of economic depression

Section 7

Summary and Further Readings

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- Motivated to re-study RDUT portfolio choice problem **without** any monotonicity condition
- At equilibrium one cannot distinguish between RDUT and EUT economies; however, representative risk aversion level is (possibly substantially) altered

Summary

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- Probability weighting may offer a new way of thinking in explaining many economic phenomena

Essential Readings

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Section 8

Final Words

Two Revolutions in Finance

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  - It seeks to explore the consistency and predictability in human flaws so that such flaws can be avoided or even exploited for profit

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  - Robert Shiller (2006), "the two ... have always been interwind, and some of the most important applications of their insights will require the use of both approaches"

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- *Mathematical behavioural finance*: research is in its infancy, yet potential is unlimited – or so we believe