Pricing and Hedging in Incomplete Markets

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forthcoming in JFE

IPAM Conference on the Mathematics of Finance

Friday, January 12, 2001 *UCLA*

The Issue

- For many reasons (eg. jumps, market closures, illiquidity) markets are fundamentally incomplete.
- This incompleteness invalidates the intuitively appealing notion of pricing contingent claims at their replication cost.
- Our concern is with recovering the spirit of this enormous development in asset pricing for the relevant context of incomplete markets.
- First, we survey the solutions offered so far.

Expected Utility Maximation (EUM)

- There is a large literature on pricing and hedging in incomplete markets by maximizing the investor's expected utility.

- This approach has a long history and a strong theoretical appeal grounded in economic theory. At the same time, it has had little acceptance in practice, its long history notwithstanding.

- Some question the embedded behavorial assumptions. We question the lack of a market perspective in this approach. Its primitives are internal to the investor, be they preferences, probabilities, or initial positions.

- Little recognition is given to the idea that prospective trades should be viewed favorably by market participants in general. Being totally individualistic in construction, EUM applies only to individuals who will never be subject to review by other market participants over the life of the trade.

- In this regard, we note that even Vice President Gore had to ultimately stop maximizing expected utility and concede that asking for recounts was no longer acceptable to the wider community of participants in the political market.

- We view EUM as potentially dangerous and ill advised.

- This is precisely why traders are reluctant to adopt such methodologies, as they are generally indefensible when confronted on review by other market participants.

Arbitrage Pricing Theory (APT)

- In contrast to EUM, APT is totally market driven.
- Assets in the market span are priced at their replication cost, no matter how disjointed this may appear from an EUM perspective.
- In APT, the EUM constructs (probabilities, preferences, and initial positions) never appear.
- Essentially, APT aggregates over all such constructs by requiring that trades be positively viewed by all such participants.
- APT relies on spanning and generally provides insufficiently tight bounds in the presence of incompleteness.

Selecting Pricing Kernels

- Minimizing distance from a prior subject to re-pricing liquid assets (Rubinstein (1994)),(Buchen and Kelley (1996), Stutzer(1996), Avellanada et. al (1997)).

- Parametric Calibration (Hull and White (1990), Heston (1993), Bates (1996), Madan, Carr, and Chang (1998)).

- Non-Parametric Calibration (Ait-Sahalia and Lo (1998), Dupire (1994)).

- The relevance of the selected measure may be called into question in these approaches. For example, when several methods are calibrated to vanilla option prices, they often predict widely differing exotic option prices.

Generalizing Arbitrage

- The main idea is that a trade is *acceptable* when all reasonable market participants view the benefits engendered by the gains as compensating for the costs imposed by the losses.

- One may regard these persons as potential counterparties willing to take the other side should it become necessary to unwind in the near future. In practice, these persons are operationalized as probability measures, which will be used to compute expected values in a single period context.

- Since an expected return of negative infinity is clearly not acceptable, acceptability requires that all expected returns must be bounded from below.

- For each probability measure, its associated lower bound (called a floor) must be nonpositive if all arbitrages are to be acceptable.

Acceptable Opportunity (AO)

- To judge potential investments in a portfolio context, we add in the costs and the cash flows of any related hedging activity.
- We also restrict attention to potential investments which are fully financed.
- A potential investment together with its financing and its (partial) hedge is termed an *opportunity*.
- An *acceptable opportunity* is then defined as an opportunity whose expected gains on a set of test measures weakly exceed their associated floors.

Coherent Risk Measures and AO's

- The fundamental notion of an acceptable opportunity is due to the path-breaking work of Artzner, Delbaen, Eber, and Heath on measuring risk.

- In their highly original paper, risk measures are termed *coherent* if they satisfy a riskfree condition, a monotonicity condition, and a diversification condition.

- In a significant advance, they characterize all coherent risk measures using a set of probability measures and associated constants which we call floors.

- A measure is coherent if it can be expressed as the maximum difference between the expected loss and a constant associated with each measure.

- They also define *acceptable positions* as those for which this difference is nonpositive, i.e. the minimum expected worth exceeds the floor for each measure.

- We consider the implications of their definition for pricing and hedging in incomplete markets.

Contrasting AO, EUM, and APT

- AO lies between EUM and APT in terms of its input requirements:
  - APT requires only a specification of assets and state spaces; AO needs the test measures and floors as well.
  - EUM requires a full specification of statistical probability, preferences, and endowments; AO does not require these separately.
- AO also lies between EUM and APT in terms of its implications:
  - Going beyond APT, AO does select some risky investments as worth pursuing.
  - Falling short of EUM, AO does not determine optimal investments.
- So when it comes to pricing *theories*, there is no free lunch.

State Space Geometry & Acceptability

- Recall that each test measure is associated with a nonnegative floor. The test measures associated with zero floors are termed valuation measures, while the measures associated with negative floors are termed stress test measures.

- In the payoff space, the set of payoffs with zero expected value under a given measure is a hyperplane containing the origin. Each hyperplane splits the payoff space into a half space containing the positive orthant and a half space containing the negative othant.

- The intersection of all half spaces containing the positive orthant is a cone containing the positive orthant. This cone is the set of payoffs which meet the restrictions imposed by the valuation measures.

- The set of payoffs whose expected value equals a negative floor associated with a given measure is a hyperplane passing below the origin, i.e. entering into the negative orthant. Each such hyperplane splits the payoff space into a half space containing the positive orthant and its complement.

- The intersection of all the half spaces containing the positive orthant is a convex set. An opportunity is acceptable if its payoffs are a point lying in this convex set.

Market Efficiency Refined

- An opportunity is *strictly acceptable* if its expected payoff is positive under at least one valuation measure and nonnegative under all others.

- Our concept of market efficiency excludes not only arbitrages, but also all strictly acceptable opportunities.

- In attaining acceptability, stress test measures may be avoided by scaling down the position. Hence, market efficiency is concerned only with valuation measures.

- We show that the absence of strictly acceptable opportunities among the liquid assets is equivalent to the existence of a *representative state price density*, which is a strict convex combination of the valuation measures.

Expected Payoff Geometry & Acceptability

- Consider a space whose axes are the expected value under each valuation measure.
- Recall that a strictly acceptable opportunity (SAO) is an opportunity whose expected payoff is positive under at least one valuation measure and non-negative under all others.
- Thus, in this space, the set of expected values (EV's) from an SAO is a point lying in the positive orthant which is not at the origin.
- If the liquid assets do not generate any SAO's (our efficient market hypothesis or EMH), then the set of EV's generated by each liquid asset can be represented by a vector lying outside the positive orthant.
- Thus under our EMH, there exists vector(s) of positive weights whose inner product with each such EV vector vanishes.
- The more liquid assets there are relative to a fixed number of valuation measures, the more constraints there are on the vectors of positive weights.
- If the number of (linearly independent) liquid assets equals the number of valuation measures, then the vector of positive weights is uniquely determined.

Hedging Refined

- We refine the concept of a hedge by introducing the idea of acceptable completeness.
- The usual notion of completeness requires that the hedge residual be zero. As a result, the expected value of the residual vanishes for all probability measures.
- Acceptable completeness only requires zero expected value for the selected valuation measures, as opposed to all probability measures.
- We show that markets are acceptably complete if and only if the representative state price density is unique.
- In this case, we obtain unique prices and hedges, even if markets are classically incomplete.

Pricing Claims Outside the Acceptable Span

- When marginal trades are scaled up, the stress test measures (measures with negative floors) become relevant.
- We develop a theory for quoting bid ask spreads for non-marginal positions in claims whose payoffs lie outside the acceptable span.
- We do this by determining the costs of constructing hedges that make the hedged claims acceptable.
- The spreads increase with the scale of the claim to be hedged.

First Motivating Example

- Three states, two assets. This market is incomplete.
- The assets are a bond and a stock with time 1 payoffs:

	States		
Assets	ω_1	ω_2	ω_3
Bond	1	1	1
Stock	3	1	0

- Each asset is priced at unity and is financed by borrowing. After financing costs, the net payoffs from each liquid opportunity is:

	States		
Assets	ω_1	ω_2	ω_3
Bond	0	0	0
Stock	2	0	-1

- This market is arbitrage-free.
- Consider the following 2 valuation measures:

	States		
Measures	ω_1	ω_2	ω_3
1	1/3	1/3	1/3
2	0	0	1

- Under these measures, the bond is not *strictly* acceptable and neither is the stock, as it has a negative expected payoff under measure 2.

The Fundamental Theorems Revised

- For any portfolio of κ bonds and λ stocks, the expected payoff under measure 1 is $\lambda/3$, and under measure 2 is $-\lambda$. Thus, there are no strictly acceptable opportunities.

- By our analog of the first fundamental theorem, a convex combination of these valuation measures reprices the assets. The first measure prices the financed stock at 1/3, while the second measure gives -1. Thus, the weight w on the expected payoff given by the first measure solves:

$$\begin{aligned} \frac{w}{3} - (1-w) &= 0 \\ w &= 3/4. \end{aligned}$$

- By our analog of the second fundamental theorem, the representative state price density $[\frac{3}{4}, \frac{1}{4}]$ is unique.

- Under the first measure, the expected payoff from a call struck at 2 is $\frac{1}{3}(3-2) + \frac{1}{3}0 + \frac{1}{3}0 = \frac{1}{3}$. Under the second measure, the call's expected payoff is 0. Thus, the unique value of the call is $\frac{3}{4}\frac{1}{3} + \frac{1}{4}0 = \frac{1}{4}$.

Second Motivating Example

- This example illustrates a method for constructing test measures in a manner consistent with EUM.

- It also is consistent with U-shaped measure changes, which been empirically documented in Carr, Geman, Madan and Yor (2000).

- Five States, Three Assets. The assets are a bond, stock and a straddle. The time 1 cash flows are given in the table below.

	States				
Assets	ω_1	ω_2	ω_3	ω_4	ω_5
Bond	1	1	1	1	1
Stock	80	90	100	110	120
Straddle	20	10	0	10	20

- Denote this 3×5 matrix of cash flows by A.

- The time one asset prices are

Asset	Price
Bond	.9091
Stock	88.1899
Straddle	12.3173

- Denote this vector by π.

- One can verify that there are no arbitrage opportunities.

The Test Measures

- Consider 3 valuation measures defined by uniform priors and uniform preferences (power utility with relative risk aversion 5) with the following positions given by the bond for the 1st measure, the stock for the 2nd measure, and a long-bond/short-stock position for the 3rd measure:

	Positions				
Individuals	ω_1	ω_2	ω_3	ω_4	ω_5
1	100	100	100	100	100
2	80	90	100	110	120
3	120	110	100	90	80

- The 3 unnormalized valuation measures are obtained by evaluating the marginal utility at these positions and multiplying by subjective probability:

States	Measures		
ω_1	1	3.0518	.4019
ω_2	1	1.6935	.6209
ω_3	1	1	1
ω_4	1	.6209	1.6935
ω_5	1	.4019	3.0518

- Denote this 5×3 matrix by B. Consider in addition, 2 stress test measures that require cash flows in states ω_1 and ω_5 to exceed -50. Acceptability requires:

$$\begin{aligned} x'B &\geq 0 \\ x'e_1 &\geq -50 \\ x'e_5 &\geq -50. \end{aligned}$$

Fundamental Theorems Revised

- If there are no strictly acceptable opportunities α, then when $\alpha'\pi = 0$, it is not the case that:

$$\begin{aligned} \alpha' AB &\geq 0 \\ \alpha' AB &\neq 0 \end{aligned}$$

- It follows from classical arguments that there exists $w > 0$ such that:

$$\pi = ABw.$$

- We may verify that:

$$w = [.0085, .085, .0427].$$

- Furthermore, the representative state price density (RSPD) is unique and is given by:

$$\begin{aligned} q &= Bw \\ &= [.2861, .1796, .1366, .1338, .1731]. \end{aligned}$$

- This RSPD is U-shaped.

- This shape results from the positive weight given to the short position in the representative state price function.

Third Motivating Example

- This example shows that we can price and hedge uniquely even with a continuum of states.
- Consider a single period economy with length T.
- One can initially trade in a stock priced at S_0 and a bond priced at e^{-rT}, where r is the continuously compounded interest rate.
- The terminal outcomes for the stock are the positive half line $(S, S \geq 0)$, while the bond has a payoff of one dollar.

The Test Measures

- We employ 2 valuation measures given by lognormal distributions for the stock with mean continuously compounded rates of $\mu_d < r < \mu_u$ and volatilities $\sigma_d < \sigma_u$.

- There are no stress test measures.

- The matrix of asset valuation measure outcomes, valuing both assets on both measures is the matrix:

$$C = \begin{bmatrix} e^{-rT} & e^{-rT} \\ S_0 e^{(\mu_u - r)T} & S_0 e^{(\mu_d - r)T} \end{bmatrix}$$

NSAO for the Lognormal Economy

- For any zero cost trading strategy, we must have:

$$\begin{aligned} \alpha_0 e^{-rT} + \alpha_1 S_0 &= 0 \\ \alpha_0 &= -\alpha_1 S_0 e^{rT}. \end{aligned}$$

- Hence, we have that:

$$a'C = \alpha_1 S_0 \left[e^{(\mu_u - r)T} - 1, e^{(\mu_d - r)T} - 1 \right].$$

- Since $\mu_d < r < \mu_u$, there are no strictly acceptable opportunities.

RSPF for Lognormal Economy

- By our version of the 1st fundamental theorem, there exists an RSPF:

$$q(S,T) = w_d \frac{1}{S} n\left(\frac{\ln(S/S_0) - (\mu_d - \sigma_d^2/2)T}{\sigma_d T^{1/2}}\right)$$
$$+ w_u \frac{1}{S} n\left(\frac{\ln(S/S_0) - (\mu_u - \sigma_u^2/2)T}{\sigma_u T^{1/2}}\right),$$

where $n(x)$ is the standard normal density.

- The number of assets equals the number of valuation measures, and so by our version of the 2nd fundamental theorem, the RSPF is unique, and is obtained on solving:

$$\begin{bmatrix} e^{-rT} \\ S_0 \end{bmatrix} = Cw,$$

with the solution $w_u = w$, $w_d = 1 - w$ and:

$$w = \frac{e^{rT} - e^{\mu_d T}}{e^{\mu_u T} - e^{\mu_d T}}.$$

- European call options are then uniquely priced using this RSPF by a price $C(K)$ for strike K given by:

$$\begin{aligned} C(K) &= wBS(\sigma_u) + (1-w)BS(\sigma_d) \\ &= wBS_u + (1-w)BS_d, \end{aligned}$$

where $BS(\sigma)$ is the Black Scholes formula.

The Lognormal Hedge Ratio

- For a hedge, we must have positions of α stocks and β bonds such that the residual is just acceptable. This requires that we solve:

$$[\beta, \alpha]\, C = \left[\; BS_u,\;\; BS_d \;\right].$$

- The solution is:

$$\begin{aligned} \alpha &= \frac{BS_u - BS_d}{S_0\left(e^{(\mu_u - r)T} - e^{(\mu_d - r)T}\right)} \\ \beta &= \frac{-e^{(\mu_d - r)T} BS_u + e^{(\mu_u - r)T} BS_d}{S_0\left(e^{(\mu_u - r)T} - e^{(\mu_d - r)T}\right)}. \end{aligned}$$

- We note that the hedge position in the stock is a proper delta type calculation, where the deltas or changes in prices are across measures, rather than across states.

Summary and Conclusions

- Most practical problems require the specification of finite state spaces on which one must define the class of test measures.

- For derivatives on a single underlying monitored at regularly sampled intervals, one must define measures on the joint density of the asset price path sampled at the monitoring intervals.

- We need to begin explorations of research design for the acceptability set in this context, and its implications for pricing, quoting, and hedging of simple products. The intent is to come within market quoted spreads on these items.

Generalizations to Continuous time Models

- Much of the theoretical understanding of dynamic trading strategies is done in the context of continuous time semimartingale models.

- We need to generalize the ideas of acceptability to this larger context, where the specification of the state space requires the specification of a reference measure P to begin with. We must ensure that this measure is broad enough to permit sufficient diversity in the test measures that are to absolutely continuous with respect to P.

- We note that in this regard geometric Brownian motion is an undesirable measure as the set of measures that are absolutely continuous with respect to it is extremely narrow, permitting no change in the volatility structure.

Pricing Exotics Acceptably

- It is being increasingly realized that the pricing of exotic derivatives is on weak foundations, as one may calibrate a variety of models to the vanilla options surface and get widely differing prices for even the simpler exotics.
- In such a situation, the proposed price has little standing.
- A possible avenue of resolution is to define the acceptable hedge and price or quote at the cost of attaining this cover.