Convolution Methods for Exotic Options

Eric Reiner

UBS Warburg

Equities Quantitative Strategies

IPAM Financial Mathematics Workshop

11 January 2001

Outline

1. Standard Path-Dependent Options
2. Effects of Discrete Sampling
3. Discrete-Time Propagators and Factorization
4. Fast Fourier Transform Algorithm
5. Examples
    5.1. Knock-out (Barrier) Options
    5.2. Lookbacks
    5.3. Puttable Cliquets
6. Conclusions

Overview: The Standard Model

- Underlying driver:

  B.M. $z_t \sim N(0,t)$

- Asset price process:

  $$\frac{dS_t}{S_t} = (r_t - d_t)dt + \sigma_t dz_t$$

  with r, d, σ functions of t only

- "Black-Scholes+" framework

1. Standard Path-Dependent Options

- Vanilla Option Payoff (not path dependent): $C_T(S, K) = (S_T - K)^+$
- Path-Dependent Options: payoff depends on some function of the path taken by the asset from onset to expiry.
  - Extremum-Dependent Options: payoff depends on maximum or minimum price achieved on the path.
    - $M_{+\infty} \equiv \max\left(S(t), 0 \leq t \leq T\right)$
    - $M_{-\infty} \equiv \min\left(S(t), 0 \leq t \leq T\right)$
    - Barrier Options: vanilla option knocks in/out if barrier level is crossed during the option's life. (Merton, 1973; Rubinstein & Reiner, 1991)
      - Down-and-out call: $DOC_T(S, K; M_{-\infty}, B) = (S_T - K)^+ \, I_{M_{-\infty} - B}$
      - Up-and-out put: $UOP_T(S, K; M_{+\infty}, B) = (K - S_T)^+ \, I_{B - M_{+\infty}}$
      - Down-and-in call: $DIC_T(S, K; M_{-\infty}, B) = (S_T - K)^+ \, I_{B - M_{-\infty}}$
      - Down-and-out binary: $DOB_T(S; M_{-\infty}, B) = I_{M_{-\infty} - B}$

1. Standard Path-Dependent Options (continued)

- Double Barrier Options: knock in/out depends on both up <u>and</u> down barrier levels (and possibly order in which they're crossed). (Beaglehole, 1992; Kunitomo & Ikeda, 1992; Jamshidian, 1997)

  - Double barrier binary:

$$DBB_T(S; M_{-\infty}, B_-; M_\infty, B_+) = \left(I_{M_{-\infty}-B_-}\right)\left(I_{B_+-M_\infty}\right)$$

- Lookback Options: payoff involves extremum in place of strike or final spot. (Goldman, Sosin, & Gatto, 1979; Garman, 1989; Conze & Viswanathan, 1991)

  - Lookback call: $LC_T(S; M_{-\infty}) = S_T - M_{-\infty}$
  - Lookback put: $LP_T(S; M_{+\infty}) = M_{+\infty} - S_T$
  - Call on maximum: $C{+}_T(S, K; M_{+\infty}) = (M_{+\infty} - K)^+$

1. Standard Path-Dependent Options (continued)

- Averaging or Asian Options: payoff involves average in place of strike or final spot. (Boyle & Emanuel, 1985; Ingersoll, 1987; Ritchken, Sankarasubramaniam, & Vijh, 1989; Levy, 1990; Reiner, 1990; Geman & Yor, 1992)

  - $A \equiv M_1 \equiv \frac{1}{T}\int_0^T dt S(t)$
  - $G \equiv M_0 \equiv \exp\left(\frac{1}{T}\int_0^T dt \ln S(t)\right)$

  - Arithmetic average price call: $APC_T(S, K; A) = (A - K)^+$
  - Arithmetic average strike call: $ASC_T(S; A) = (S - A)^+$
  - Geometric average price call: $GPC_T(S, K; G) = (G - K)^+$

- Quasi-Path-Dependent Options: path dependence is endogenous to value (American Options).

2. Effects of Discrete Sampling

- Assumptions of continuous-sampling solutions are often unrealistic and/or impractical
  - Markets aren't always open
  - Intra-day prices may not be reliable or verifiable
  - Continuous monitoring and hedging are infeasible
  - Parameters may not be constant
- Most path-dependent option contracts are specified in terms of discrete, periodic samples
- The market has long recognized that discreteness has significant effects on value, particularly for extremum-dependent options
- There is a need to construct valuation tools that address this problem

2. Effects of Discrete Sampling (continued)

- Barrier Options
  - Monte Carlo and Lattice Calculations
    - Discrete sampling substantially affects values and deltas (Flesaker, 1992; Anderson & Brotherton-Ratcliffe, 1996; Cheuk & Vorst, 1996)
    - Convergence is slow and much tweaking needs to be done to get decent numbers in reasonable time (Broadie, Glasserman, & Kou, 1996)
  - Adjustment Techniques
    - Traders have long recognized that moving the barrier out by a factor $\exp(\beta\sigma\sqrt{dt})$ with $\beta \sim O(1)$ gives good agreement with simulation results
    - Broadie, Glasserman, & Kou (1996, 1997) show that $\beta = -\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$. This is quite accurate for frequent samples and barriers some distance from spot.
    - Term structures and fewer sample dates remain a problem.

2. Effects of Discrete Sampling (continued)

- Exact Solutions

1. DOB Rebate at Maturity given sampling points $t_1, ..., t_n$

$$DOB_0(S_0) = e^{-r_n t_n} \int_B^{\infty} dS_1 \cdots \int_B^{\infty} dS_n \; P(S_1, ..., S_n)$$

$$\text{Let } x_i = \frac{\ln(S_i/S_0) - \tilde{\mu}_i t_i}{\sigma_i \sqrt{t_i}} \;, \;\; \tilde{\mu}_i = r_i - d_i - \sigma_i^2 \;, \;\; b_i = x_i|_{S_i = B}$$

$$\text{then } P(\bar{x}) = P(x_1, ..., x_n) = \exp(-\tfrac{1}{2}\bar{x} \cdot \bar{\bar{\rho}}^{-1} \cdot \bar{x})/(2\pi)^{n/2}$$

$$\text{with } \rho_{i,j} = \min\left(\frac{\sigma_i\sqrt{t_i}}{\sigma_j\sqrt{t_j}}, \frac{\sigma_j\sqrt{t_j}}{\sigma_i\sqrt{t_i}}\right)$$

$$\begin{aligned} DOB_0 &= e^{-r_n t_n} \int_{b_1}^{\infty} dx_1 \cdots \int_{b_n}^{\infty} dx_n \; P(\bar{x}) \\ &= e^{-r_n t_n} \int_{\infty}^{-b_1} dx_1 \cdots \int_{\infty}^{-b_n} dx_n \; P(\bar{x}) \\ &= e^{-r_n t_n} N_n(-b_1, ..., -b_n; \bar{\bar{\rho}}) \end{aligned}$$

2. Effects of Discrete Sampling (continued)

2. DIB Rebate at Maturity

$$DIB_0 = e^{-r_n t_n} \left[1 - N_n(-b_1, ..., -b_n; \bar{\bar{\rho}})\right]$$

2$'$. DIB Rebate at Maturity: alternative approach:

$$\begin{aligned} DIB_0 = e^{-r_n t_n} \Bigg[& \int_{-\infty}^{b_1} dx_1 \, P(x_1) + \int_{b_1}^{\infty} dx_1 \int_{-\infty}^{b_2} dx_2 \, P(x_1, x_2) + \\ & ... + \int_{b_1}^{\infty} dx_1 \cdots \int_{b_{n-1}}^{\infty} dx_{n-1} \int_{-\infty}^{b_n} dx_n \; P(x_1, ..., x_n) \Bigg] \\ = e^{-r_n t_n} \Big[& N_1(b_1) + N_2(-b_1, b_2; \bar{\bar{\rho}}^{(2*)}) + \\ & ... + N_n(-b_1, ..., -b_{n-1}, b_n; \bar{\bar{\rho}}^{(n*)}) \Big] \\ = e^{-r_n t_n} & \sum_{m=1}^{n} N_m(-b_1, ..., -b_{m-1}, b_m; \bar{\bar{\rho}}^{(m*)}) \end{aligned}$$

with $\rho_{i,j}^{(m*)} = (-1)_{\delta(i,m)} \, (-1)_{\delta(j,m)} \, \rho_{i,j}$

2. Effects of Discrete Sampling (continued)

3. DIB Rebate at Hit

$$DIB_0 = \sum_{m=1}^{n} e^{-r_m t_m} N_m(-b_1, ..., -b_{m-1}, b_m; \bar{\bar{\rho}}^{(m*)})$$

4. UOB Rebate at Maturity

$$UOB_0 = e^{-r_n t_n} N_n(b_1, ..., b_n; \bar{\bar{\rho}})$$

5. UIB Rebate at Maturity

$$\begin{aligned} UIB_0 &= e^{-r_n t_n} [1 - N_n[b_1, ..., b_n; \bar{\bar{\rho}})] \\ &= e^{-r_n t_n} \sum_{m=1}^{n} N_m(b_1, ..., b_{m-1}, -b_m; \bar{\bar{\rho}}^{(m*)}) \end{aligned}$$

6. UIB Rebate at Hit

$$UIB_0 = \sum_{m=1}^{n} e^{-r_m t_m} N_m(b_1, ..., b_{m-1}, -b_m; \bar{\bar{\rho}}^{(m*)})$$

7. Down-and-Out Call (assume $K \geq B$)

$$DOC_0 = e^{-r_n t_n} \int_B^\infty dS_1 \cdots \int_B^\infty dS_{n-1} \int_K^\infty dS_n \ P(\bar{S})\,(S_n - K)$$

$$= e^{-r_n t_n} \int_{b_1}^\infty dx_1 \cdots \int_{b_{n-1}}^\infty dx_{n-1} \int_{k_n}^\infty dx_n \ P(\bar{x}) \times$$

$$\left(S_0 e^{\tilde{\mu}_n t_n + \sigma_n \sqrt{t_n}\, x_n} - K\right) \quad \text{with } k_i = x_i|_{S_i = K}$$

- Second term evaluates immediately to:

  $-Ke^{-r_n t_n} N_n(-b_1, ..., -b_{n-1}, -k_n; \bar{\bar{\rho}})$

- After completing the square, first term becomes:

  $S_0 e^{-d_n t_n} N_n(\sigma_1\sqrt{t_1} - b_1, ..., \sigma_{n-1}\sqrt{t_{n-1}} - b_{n-1}, \sigma_n\sqrt{t_n} - k_n; \bar{\bar{\rho}})$

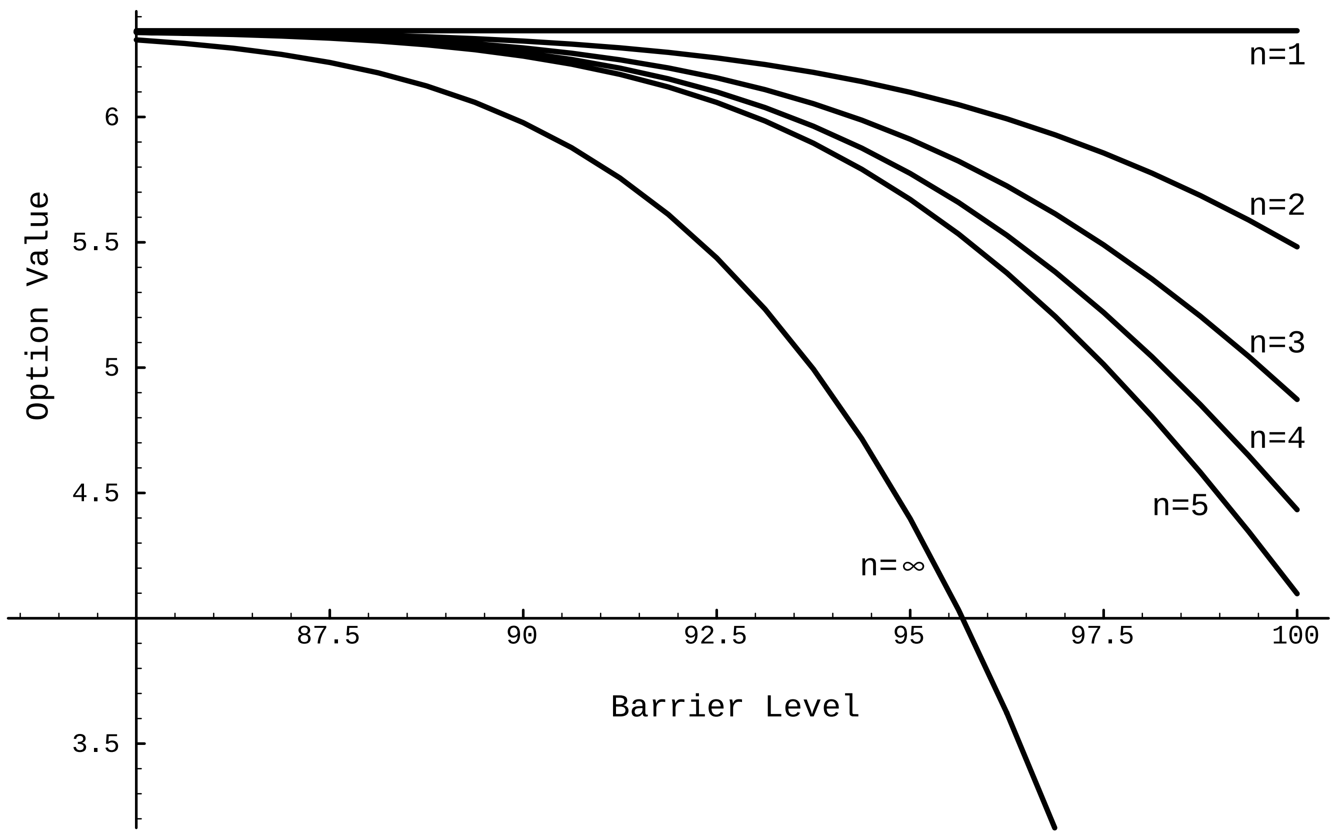
2. Effects of Discrete Sampling (continued)

- Comparison of down-and-out values ($S_0 = 100$, $K = 100$, $\sigma = 0.30$, $r = 0.10$, $d = 0.0$, $t_n = 0.2$):

Barrier	n = 1	n = 2	n = 3	n = 4	n = 5	n = ∞
85	6.344113	6.340525	6.339121	6.338034	6.336949	6.307575
86	6.344113	6.337903	6.335075	6.333051	6.331169	6.282648
87	6.344113	6.333696	6.328333	6.324650	6.321448	6.243846
88	6.344113	6.327154	6.317508	6.310990	6.305634	6.185283
89	6.344113	6.317284	6.300736	6.289562	6.280755	6.099473
90	6.344113	6.302820	6.275634	6.257115	6.242916	5.977242
91	6.344113	6.282212	6.239300	6.209664	6.187290	5.807772
92	6.344113	6.253629	6.188379	6.142611	6.108260	5.578772
93	6.344113	6.215000	6.119205	6.050986	5.999755	5.276814
94	6.344113	6.164077	6.028022	5.929839	5.855778	4.887793
95	6.344113	6.098541	5.911267	5.774722	5.671105	4.397503
96	6.344113	6.016117	5.765902	5.582235	5.442059	3.792265
97	6.344113	5.914725	5.589737	5.350551	5.167245	3.059563
98	6.344113	5.792627	5.381732	5.079834	4.848107	2.188607
99	6.344113	5.648567	5.142211	4.772477	4.489172	1.170793
100	6.344113	5.481901	4.872973	4.433121	4.097933	0.000000

2. Effects of Discrete Sampling (continued)

- Comparison of down-and-out values ($S_0 = 100$, $K = 100$, $\sigma = 0.30$, $r = 0.10$, $d = 0.0$, $t_n = 0.2$):



2. Effects of Discrete Sampling (continued)

- Lookback Options
  - Monte Carlo and Lattice Calculations
    - Discrete sampling substantially affects values and deltas (Reiner, 1991; Anderson & Brotherton-Ratcliffe, 1996; Levy & Mantion, 1997)
    - Convergence is slow; substantial tweaking and extrapolation techniques are necessary (Babbs, 1992; Broadie, Glasserman, & Kou, 1996)
  - Adjustment Techniques
    - Since lookbacks can be expressed as integrals of binary barrier options, Broadie, Glasserman, & Kou (1996) show that the barrier adjustment method may be applied. Again, this is quite accurate for frequent samples and strikes some distance from spot.
    - Term structures and fewer sample dates remain a problem.

2. Effects of Discrete Sampling (continued)

- Exact Solutions

8. Call on Maximum as a special case of the "Best of N Call"

$$C{+}_0(S_0) = e^{-r_n t_n}\left[\int_K^{\infty} dS_1\, S_1 \int_{-\infty}^{S_1} dS_2 \cdots \int_{-\infty}^{S_1} dS_n\ P(S_1, ..., S_n) + \right.$$

$$\left. ... - K\left(1 - \int_{-\infty}^{K} dS_1 \cdots \int_{-\infty}^{K} dS_n\ P(S_1, ..., S_n)\right)\right]$$

Let $k'_{i,j} = \dfrac{(r_i - d_i)t_i - (r_j - d_j)t_j}{\Sigma_{i,j}}$, $\Sigma_{i,j} = \sqrt{|\sigma_i^2 t_i - \sigma_j^2 t_j|}$

$$C{+}_0 = S_0 e^{-r_n t_n} \sum_{m=1}^{n} e^{(r_m - d_m)t_m} \times$$

$$N_n\left(\frac{\Sigma_{1,m}}{2} - k'_{1,m}, ..., \sigma_m\sqrt{t_m} - k_m, ..., \frac{\Sigma_{n,m}}{2} - k'_{n,m}; \bar{\bar{\rho}}^{(m\dagger)}\right)$$

$$-Ke^{-r_n t_n}[1 - N_n(k_1, ..., k_{n-1}, k_n; \bar{\bar{\rho}})]$$

with $\rho_{i,j}^{(m\dagger)}$ given by processes relative to m

9. Lookback Put in terms of Call on Maximum

Given an existing maximum $M_{+\infty,0}$

$$\max(S_1, ..., S_n, M_{+\infty,0}) - S_n = \left[\max(\bar{S}) - M_{+\infty,0}\right]^+ + M_{+\infty,0} - S_n$$

$$LP_0(S_0, M_{+\infty,0}) = C{+}_0(S_0, M_{+\infty,0}) + M_{+\infty,0}e^{-r_n t_n} - S_0 e^{-d_n t_n}$$

10. Put on Minimum by symmetry

$$C{-}_0 = Ke^{-r_n t_n}[1 - N_n(-k_1, ..., -k_{n-1}, -k_n; \bar{\bar{\rho}})] -$$

$$S_0 e^{-r_n t_n} \sum_{m=1}^{n} e^{(r_m - d_m)t_m} \times$$

$$N_n(k'_{1,m} - \frac{\Sigma_{1,m}}{2}, ..., k_m - \sigma_m\sqrt{t_m}, ..., k'_{n,m} - \frac{\Sigma_{n,m}}{2}; \bar{\bar{\rho}}^{(m\dagger)})$$

11. Lookback Call in terms of Put on Minimum

By symmetry

$$LC_0(S_0, M_{-\infty,0}) = C{-}_0(S_0, M_{-\infty,0}) - M_{-\infty,0}e^{-r_n t_n} + S_0 e^{-d_n t_n}$$

2. Effects of Discrete Sampling (continued)

- Asian Options
  - Discrete sampling effects recognized from early studies (Ritchken, Sankarasubramaniam, & Vijh, 1989; Turnbull & Wakeman, 1990; Rubinstein, 1991; Curran, 1992) and are readily incorporated into simulations and approximation methods.
  - While substantial, effects of discreteness are not so pronounced as for extremum-dependent options.
  - For geometric averages, all results are closed-form univariate normals. For arithmetic averages, "exact" solutions are generally not possible, though see next point.
  - Term structures and fewer sample dates are naturally treated in all available approaches, but with one exception (Carverhill & Clewlow, 1990) a clean, robust, convergent (exact?) numerical method remains elusive.
- American (Bermuda) Options
  - Discrete sampling effects also long-known (Geske, 1979). "Exact" solutions available, but require self-consistent solution of stopping boundary (Sheikh, 1992) as for compound options.
  - Adjustment techniques (Ait-Sahlia, 1995) inspired recent barrier analysis.

2. Effects of Discrete Sampling (continued)

- Options with Intrinsically Discrete Path Dependence
  - Compound and Chooser Options
    - Call on Call: $COC_t(S_t) = (C_t(S_t, K_T) - K_t)^+$
    - Call on Put: $COP_t(S_t) = (P_t(S_t, K_T) - K_t)^+$
    - Put on Call: $POC_t(S_t) = (K_t - C_t(S_t, K_T))^+$
    - Put on Put: $POP_t(S_t) = (K_t - P_t(S_t, K_T))^+$
    - Chooser: $CORP_t(S_t) = \max(C_t(S_t, KC_T) - KC_t, P_t(S_t, KP_T) - KP_t, 0)$
    - All can be valued exactly using $N_2(\bullet,\bullet;\bullet)$ and self-consistent solution for exercise points.
  - Installment Options
    - Installment Call: $INC_i(S_{t_i}; t_i) = (INC_{i+1}(S_{t_i}; t_i) - K_i)^+$
    - Installment Put: $INP_i(S_{t_i}; t_i) = (INP_{i+1}(S_{t_i}; t_i) - K_i)^+$
    - Natural extensions of compound calls to n mandatory call dates.
    - All can be valued exactly using $N_n(\bar{\bullet}; \bar{\bar{\bullet}})$ and self-consistent solution for exercise points.

2. Effects of Discrete Sampling (continued)

- Puttable Cliquets
  - Consider the payoff of a series of forward starting options:

$$CLCFU_T = \sum_{m=1}^{n} (S_m - kS_{m-1})^+$$

$$CLCFN_T = \sum_{m=1}^{n} (S_m/S_{m-1} - k)^+$$

  - Final puttability:

$$PCLCFU_T = (CLCFU_T - K)^+$$

$$PCLCFN_T = (CLCFN_T - K)^+$$

  - Although optionlet returns are independent in Black-Scholes framework, valuation is analogous to arithmetic average option problem.
  - Puttability can be made “Bermudian,” i.e., K_m can be introduced for each reset date.

3. Discrete-Time Propagators and Factorization

- All of these valuation problems share a couple of interesting properties.
  - Sampling, reset, or exercise are limited to a discrete set of dates
  - Between these dates, return densities (and valuation densities) are independent of absolute levels and/or paths, but depend only on incremental returns themselves

- Whether we choose to work forward in time, propagating probability or backward in time, propagating values, these facts have significant implications.
  - Barrier Option
    - Begin at t_i with $P(S_i; t_i)$ (this has already been truncated if barrier crossed at t_i)
    - Convolve $P(S_i; t_i)$ with $\Pi(S_{i+1}, S_i; t_i, t_{i+1})$:

$$P(S_{i+1}; t_{i+1}) = \int_0^{+\infty} dS_i \; P(S_i; t_i) \; \Pi(S_{i+1}, S_i; t_i, t_{i+1})$$

    - Now, set to zero any probability outside of the barrier

3. Discrete-Time Propagators and Factorization (continued)

- Bermuda Option
  - Begin at t_i with $V(S_{i+1}; t_{i+1})$ (early exercise opportunities have already been evaluated)
  - Convolve $V(S_{i+1}; t_{i+1})$ with $e^{r_i t_i - r_{i+1} t_{i+1}} \Pi(S_{i+1}, S_i; t_i, t_{i+1})$:

$$V(S_i; t_i) = e^{r_i t_i - r_{i+1} t_{i+1}} \int_0^{+\infty} dS_{i+1}\; V(S_{i+1}; t_{i+1})\; \Pi(S_{i+1}, S_i; t_i, t_{i+1})$$

  - Now, apply early exercise condition

- In both cases, it is more convenient to work with log coordinates:

  Let $y_i = \ln(S_i/S_0)$

$$P(y_{i+1}; t_{i+1}) = \int_{-\infty}^{+\infty} dy_i\; P(y_i; t_i)\; \Pi(y_{i+1} - y_i; t_i, t_{i+1})$$

$$V(y_i; t_i) = e^{r_i t_i - r_{i+1} t_{i+1}} \int_{-\infty}^{+\infty} dy_{i+1}\; V(y_{i+1}; t_{i+1})\; \Pi(y_{i+1} - y_i; t_i, t_{i+1})$$

- Lookback Options
  - As is, neither approach appears applicable since $P(M_{+\infty,i+1})$ depends on both $M_{+\infty,i}$ and S_i
  - However, consider factorization:

$$\max(y_{n-1}, y_n) = y_{n-1} + \max(0, y_n - y_{n-1})$$

$$\begin{aligned}\max(y_{n-2}, y_{n-1}, y_n) &= \max(y_{n-2}, \max(y_{n-1}, y_n)) \\ &= \max(y_{n-2}, y_{n-1} + \max(0, y_n - y_{n-1})) \\ &= y_{n-2} + \max(0, y_{n-1} - y_{n-2} + \max(0, y_n - y_{n-1}))\end{aligned}$$

$$\begin{aligned}\max(y_1, ..., y_{n-1}, y_n) = y_1 + \max(0, y_2 - y_1 + \max(0, y_3 - y_2 + ... \\ + \max(0, y_{n-1} - y_{n-2} + \max(0, y_n - y_{n-1}))...))\end{aligned}$$

  - We can apply a sort of "backward" algorithm here, successively truncating the distribution at zero, then convolving with the distribution for the previous timestep.

$$P(m_{+\infty,i} - y_i) \sim \int_{-\infty}^{+\infty} d(y_{i+1} - y_i)\ P(m_{+\infty,i+1} - y_{i+1})\ \Pi(y_{i+1} - y_i)$$

3. Discrete-Time Propagators and Factorization (continued)

- Implementation Issues

  - Consider n timesteps and a discrete approximation to P and Π on a grid of M points

  - Generally, the integration of P and Π is an $O(M^2)$ process (analogous to matrix $\cdot$ vector)

  - The total operation count appears to be $O(nM^2)$. What accuracy can we get with some reasonable integration rule (trapezoidal? higher order?)?

  - Why are we doing nothing to take into account the independent convolution property of our integrands?

4. Fast Fourier Transform Algorithm

- Convolution integrals have one particularly useful property:
  - Define the characteristic function or Fourier transform:

    $$\hat{P}(k) = E[e^{\iota k y}] = \int_{-\infty}^{+\infty} dy\; e^{\iota k y}\; P(y)$$

  - Then if $P(y) = \displaystyle\int_{-\infty}^{+\infty} dz\; P'(y-z)\;\Pi(z)$

    $$\hat{P}(k) = \hat{P}'(k)\hat{\Pi}(k)$$

  - This is just the product rule for the characteristic function of the sum of two independent random variables.
- Although this tells us that Fourier space is a very convenient place to evaluate convolution integrals, it doesn't tell us anything about how to get there.
- Once we've discretized, it appears that we've just replaced one quadratic-time algorithm by another and added complex arithmetic to the soup.
- Let's pursue this anyway...

4. Fast Fourier Transform Algorithm (continued)

- Approximate distribution by a set of M equally spaced points separated by Δy.

- Choose Δy large enough that truncation of tails is arbitrarily small ($M/2\Delta y$ = 8 s.d. or so). We will want to do this so that our points are indexed $-M/2, -(M/2-1), ..., -1, 0, 1, ..., M/2-1$, but for calculational reasons it's better to pretend that the negative indices $-i$ into $M-i$ so that our actual indices are $0, 1, ..., M/2-1, M/2, ...M$. As we'll see later, that's ok.

- Now consider the characteristic function of a discrete distribution (and how it approximates that of a continuous distribution):

$$\hat{P}(k) = \int_{-\infty}^{+\infty} dy \; e^{\iota ky} \; P(y) \sim \sum_{i=0}^{M} [P(i\Delta y)\Delta y] \, e^{\iota ki\Delta y} = \Delta y \sum_{i=0}^{M} [P_i] \, e^{\iota ki\Delta y}$$

- The inversion relationship to recover $P(y)$ is:

$$P(y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \; e^{-\iota ky} \; \hat{P}(k)$$

- How should we choose (discretely) to represent the characteristic function? What's our best approximation?

4. Fast Fourier Transform Algorithm (continued)

- Key observation is that:

$$\hat{P}\left(k+\frac{2\pi}{\Delta y}\right)=\sum_{i=0}^{M} P_i \exp\left[\iota\left(k+\frac{2\pi}{\Delta y}\right) i\Delta y\right]=\sum_{i=0}^{M} P_i \exp\left[\iota k i\Delta y\right]=\hat{P}\left(k\right)$$

  - So, we can restrict our representation to the region $k \in [0, 2\pi/\Delta y)$.
  - We know that in some sense there are only M pieces of information, so this suggests we choose a discrete representation:

$$k_j=\frac{2\pi\iota j}{M\Delta y}; j=0,...,M-1$$

  - Then we approximate $P_i=\Delta y P(y_i)$ by the trapezoidal rule:

$$\begin{aligned}
P_i &= \frac{1}{2\pi}\int_0^{\frac{2\pi}{\Delta y}} dk\, e^{-\iota k i\Delta y}\tilde{P}(k) \\
&\sim \tfrac{1}{2\pi}\Big(\tfrac{2\pi}{M\Delta y}\Big)\Big[\tfrac{1}{2}\tilde{P}(0)+\exp\Big(\tfrac{-2\pi\iota i}{M}\Big)\tilde{P}\Big(\tfrac{2\pi}{M\Delta y}\Big)+\exp\Big(\tfrac{-4\pi\iota i}{M}\Big)\tilde{P}\Big(\tfrac{4\pi}{M\Delta y}\Big)+... \\
&\qquad +\exp\Big(\tfrac{-2(M-1)\pi\iota i}{M}\Big)\tilde{P}\Big(\tfrac{2(M-1)\pi}{M\Delta y}\Big)+\tfrac{1}{2}\tilde{P}\left(\tfrac{2\pi}{\Delta y}\right)\Big] \\
&= \tfrac{1}{M\Delta y}\Big[\tilde{P}(0)+...+\exp\Big(\tfrac{-2(M-1)\pi\iota i}{M}\Big)\tilde{P}\Big(\tfrac{2(M-1)\pi}{M\Delta y}\Big)\Big]
\end{aligned}$$

4. Fast Fourier Transform Algorithm (continued)

- Hence:

$$P_i = \frac{1}{M}\sum_{j=0}^{M-1} \exp\left(\frac{-2\pi\iota ji}{M}\right)\left[\frac{1}{\Delta y}\tilde{P}\left(\frac{2\pi j}{M}\right)\right] = \frac{1}{M}\sum_{j=0}^{M-1} \exp\left(\frac{-2\pi\iota ji}{M}\right)\left[\tilde{P}_j\right]$$

  - But this is an exact inversion relationship!

- All this corresponds to is a mapping from y_i to k_j with basis:

$$k_j = \exp\left(\frac{2\pi\iota jy}{M\Delta y}\right); j = 0, ..., M-1$$

  - Orthogonal elements
  - Natural embedding on a circle
  - Invertible transformation with $y_j = \frac{1}{M}\exp\left(\frac{-2\pi\iota jy}{M\Delta y}\right)$
  - Consistent with trapezoidal rule, or for smooth integrands higher order integration rules
  - Construction of discrete approximation to characteristic function still appears to be a matrix multiplication problem ... except if we choose M carefully!

4. Fast Fourier Transform Algorithm (continued)

- If M has a factor of 2, then we can write (Numerical Recipes):

$$
\begin{aligned}
\hat{P}_k &= \sum_{j=0}^{M-1} \exp\left(\frac{2\pi\iota kj}{M}\right) P_j \\
&= \sum_{j=0}^{M/2-1} \exp\left(\frac{2\pi\iota k(2j)}{M}\right) P_{2j} + \sum_{j=0}^{M/2-1} \exp\left(\frac{2\pi\iota k(2j+1)}{M}\right) P_{2j+1} \\
&= \sum_{j=0}^{M/2-1} \exp\left(\frac{2\pi\iota kj}{M/2}\right) P_{2j} + \exp\left(\frac{2\pi\iota k}{M}\right) \sum_{j=0}^{M/2-1} \exp\left(\frac{2\pi\iota kj}{M/2}\right) P_{2j+1}
\end{aligned}
$$

- Evidently, we can replace a transform of $O(M)$ by two transforms of $O(M/2)$
- If M is a power of 2, we can do this $\log_2(M)$ times, leaving a single complex multiplication
- Total operation count for the transform is $O(M\log_2(M))$, not $O(M^2)$!!!

4. Fast Fourier Transform Algorithm (continued)

- Resulting pricing algorithm is $O(n\, M \log_2(M)))$

- Asymptotically for large M, accuracy is limited only by truncation steps and evaluation of non-smooth payoffs. With some attention to integration rules at discontinuities, convergence can be accelerated.

- These algorithms are natural candidates for Richardson extrapolation.

- In addition to option values, we have access to probability densities for discrete knockout, extremum, etc. processes!

5. Examples

- Knock-out (Barrier) Options
- Lookbacks
- Puttable Cliquets

5.1. Knock-out Options

- Convergence studies
- Choose parameters from Broadie, Glasserman, and Kou (1996)

  $DOC(S = 100, K = 100, H = 95, t_n = 0.2, \sigma = 0.6, r = 0.1, d = 0.0, n = 4)$

  Exact Value: 9.49053470836

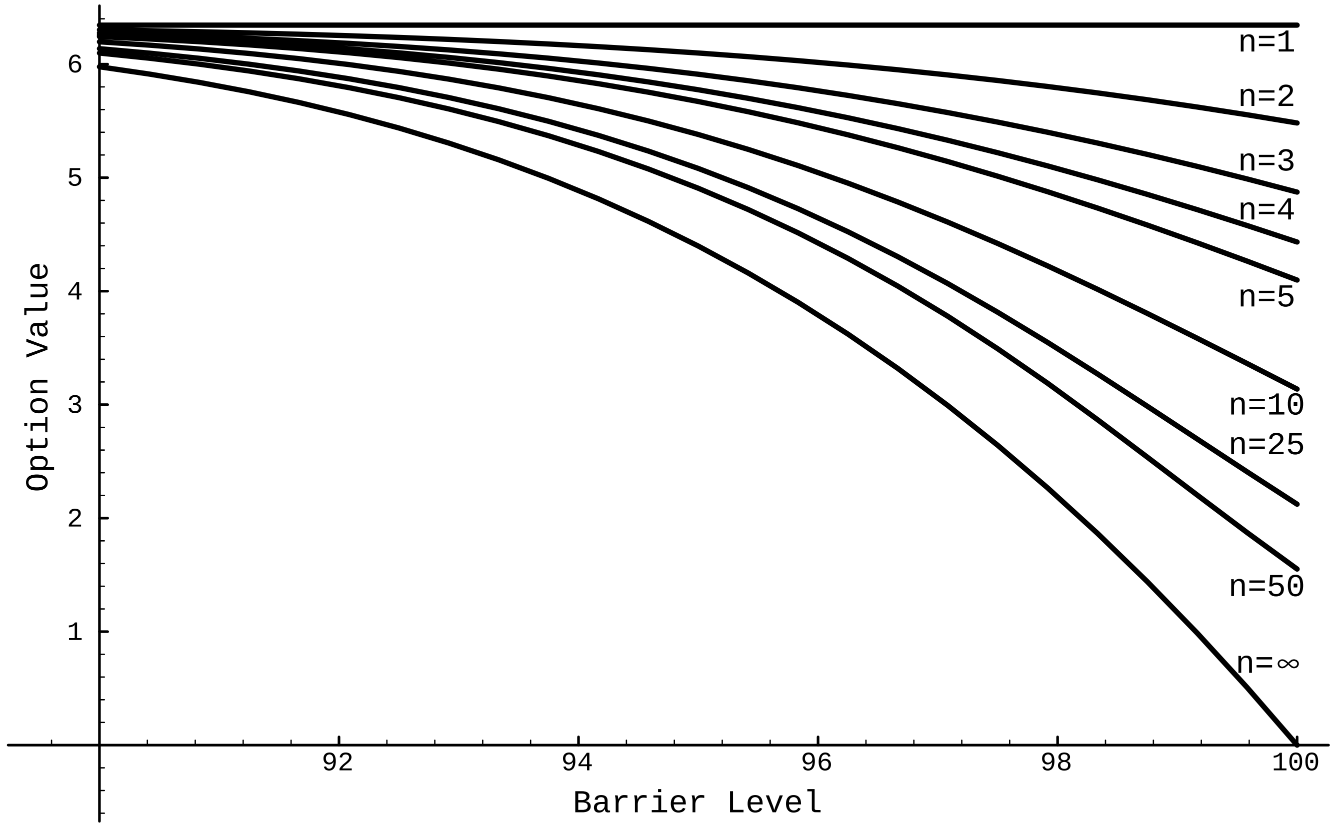
n	ET	Extrap	M	Trap Rule	2-pt Extrap	Cubic Fit	2-pt Extrap
			64	9.361364785		9.480920962	
			128	9.456771994	9.488574397	9.489909126	9.490508337
256	9.49690		256	9.481978524	9.490380701	9.490506658	9.490546494
504	9.49349	9.4899	512	9.488396701	9.490536093	9.490532960	9.490534713
1240	9.49189	9.4907	1024	9.490000875	9.490535599	9.490534600	9.490534709
2308	9.49124	9.4905	2048	9.490401492	9.490535031	9.490534702	9.490534708
4524	9.49090	9.4905	4096	9.490501418	9.490534727	9.490534708	9.490534708
8632	9.49072	9.4905	8192	9.490526385	9.490534708	9.490534708	9.490534708
			16384	9.490532627	9.490534708	9.490534708	9.490534708

5.1. Knock-out Options (continued)

- Effects of Number of Sampling Points
- Choose parameters from Broadie, Glasserman, and Kou (1997)
  $DOC(S = 100, K = 100, t_n = 0.2, \sigma = 0.3, r = 0.1, d = 0.0)$

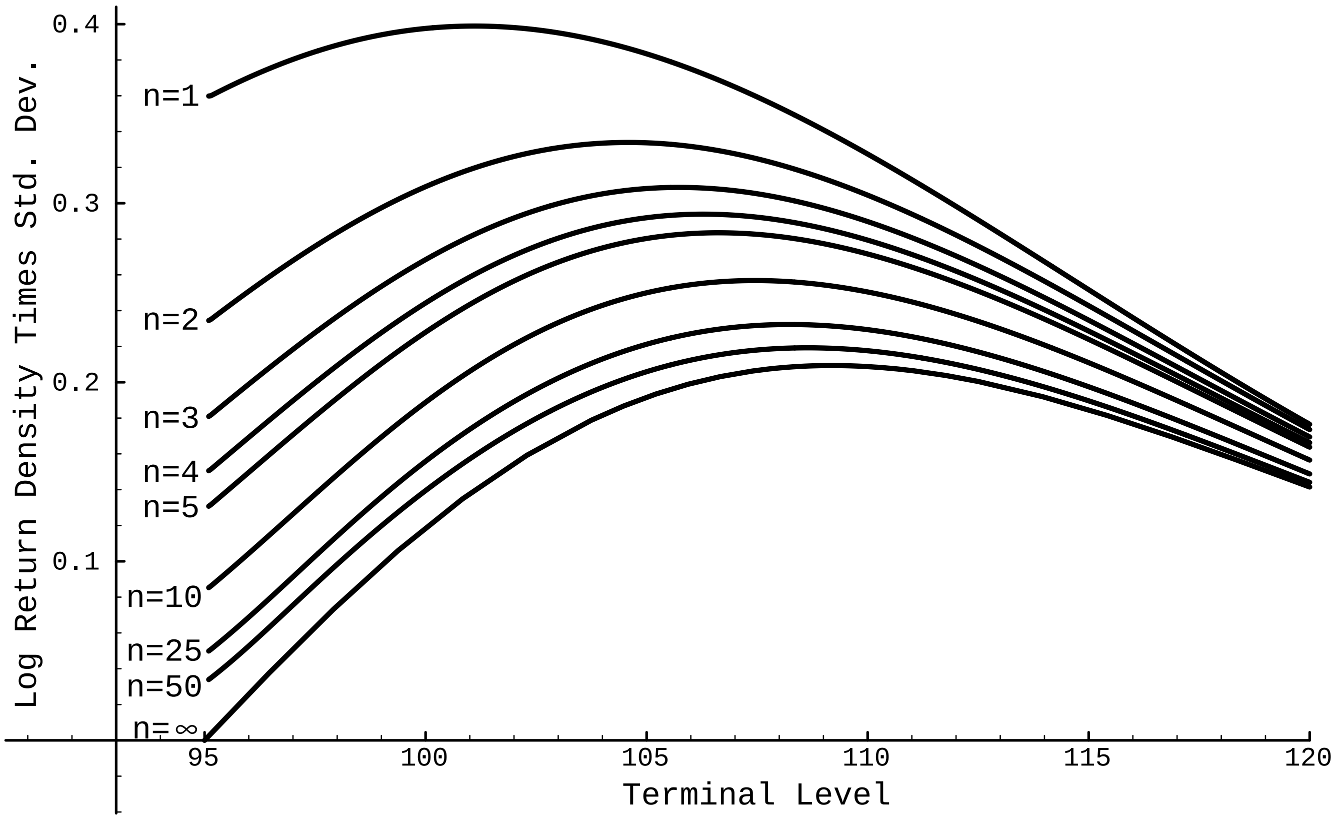
Barrier	n = 1	n = 2	n = 3	n = 4	n = 5	n = 10	n = 25	n = 50	n = ∞
90	6.3441	6.3028	6.2756	6.2571	6.2429	6.1971	6.1368	6.0982	5.9772
91	6.3441	6.2822	6.2393	6.2097	6.1873	6.1188	6.0320	5.9771	5.8078
92	6.3441	6.2536	6.1884	6.1426	6.1083	6.0076	5.8860	5.8097	5.5788
93	6.3441	6.2150	6.1192	6.0510	5.9998	5.8538	5.6875	5.5843	5.2768
94	6.3441	6.1641	6.0280	5.9298	5.8558	5.6477	5.4239	5.2879	4.8878
95	6.3441	6.0985	5.9113	5.7747	5.6711	5.3804	5.0814	4.9068	4.3975
96	6.3441	6.0161	5.7659	5.5822	5.4421	5.0463	4.6475	4.4266	3.7923
97	6.3441	5.9147	5.5897	5.3506	5.1672	4.6447	4.1158	3.8340	3.0596
98	6.3441	5.7926	5.3817	5.0798	4.8481	4.1822	3.4941	3.1263	2.1886
99	6.3441	5.6486	5.1422	4.7725	4.4892	3.6728	2.8124	2.3364	1.1708
100	6.3441	5.4819	4.8730	4.4331	4.0979	3.1371	2.1227	1.5513	0.0000

- Effects of Number of Sampling Points



- Probability Density vs. Number of Sampling Points

$$DO(S = 100, H = 95, t_n = 0.2, \sigma = 0.3, r = 0.1, d = 0.0)$$



5.1. Knock-out Options (continued)

- Effects of Number of Sampling Points on Double Barrier Binary
- Adapt parameters from Broadie, Glasserman, and Kou (1997)

  $100\, DBB(S = 100,\ H_+ = 105,\ H_- = 95,\ t_n = 0.2,\ \sigma = 0.3,\ r = 0.1,\ d = 0.0)$

n	Value
1	28.395201390
2	15.168622926
3	9.758497477
4	6.952587190
5	5.287021405
10	2.191938686
25	0.697448889
50	0.317632619
100	0.161476402
∞	0.017493814

5.2. Lookback Options

- Convergence studies
- Choose parameters from Broadie, Glasserman, and Kou (1996)

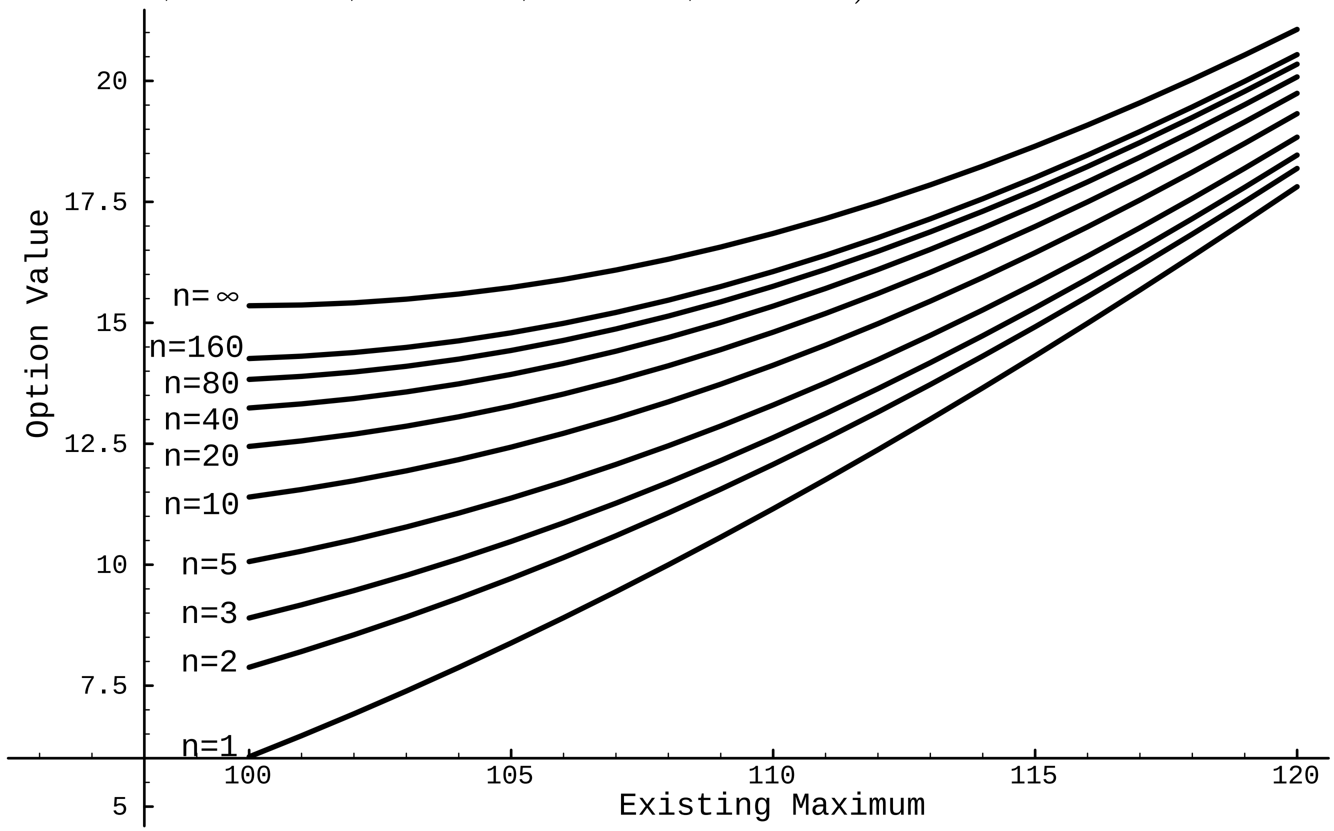
  $LP(S = 100, K = 100, t_n = 0.2, \sigma = 0.3, r = 0.1, d = 0.0, n = 4)$

  Exact Value: 6.5743660937

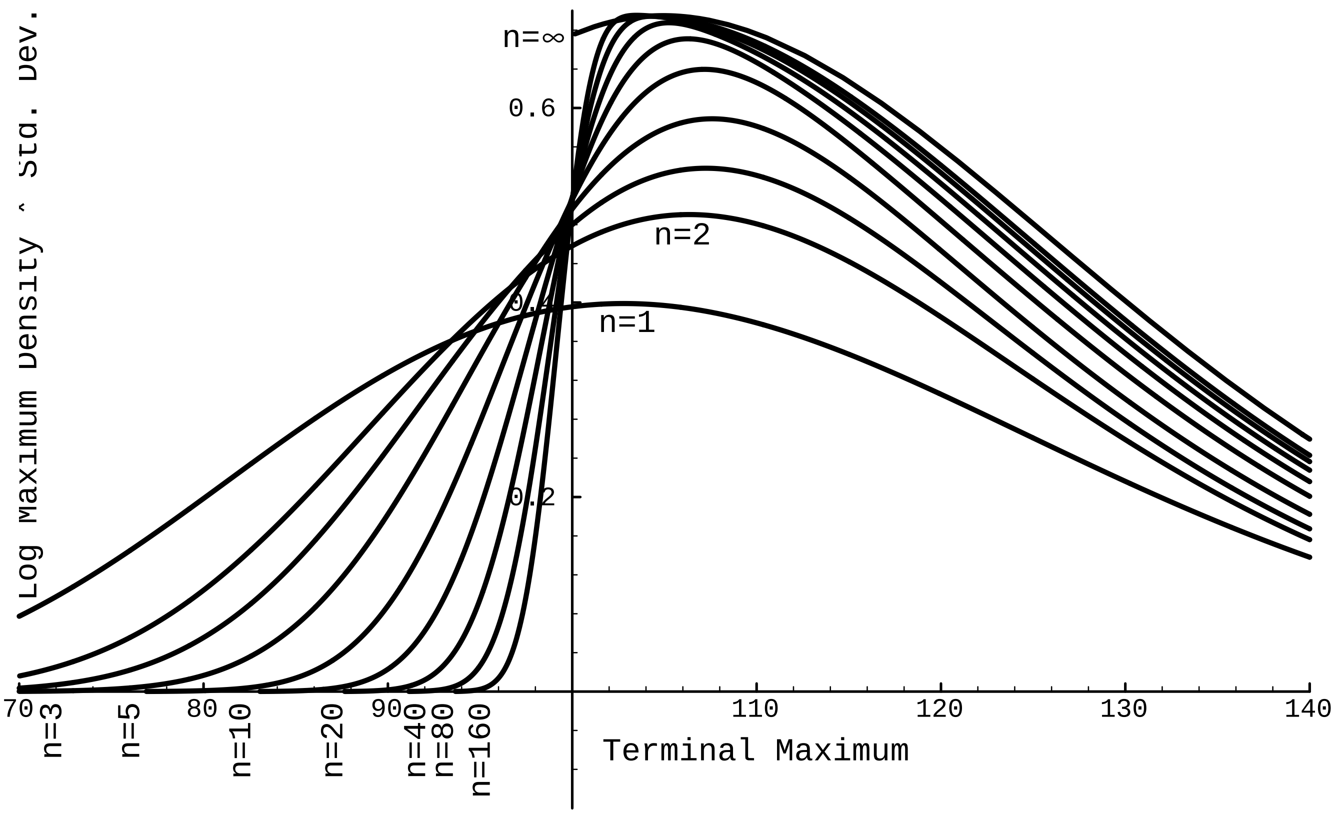
n	Trinom	Extrap	M	Trap Rule	2-pt Extrap
			64	$6.4\underline{7}7127220$	
200	$6.56\underline{8}45$		128	$6.5\underline{5}0227368$	$6.574\underline{5}940846$
400	$6.57\underline{1}40$	$6.5743\underline{5}$	256	$6.56\underline{8}341830$	$6.5743\underline{7}99838$
800	$6.57\underline{2}88$	6.57436	512	$6.57\underline{2}860675$	$6.574366\underline{9}564$
1600	$6.573\underline{6}2$	6.57436	1024	$6.573\underline{9}89779$	$6.5743661\underline{4}75$
3200	$6.573\underline{9}9$	6.57436	2048	$6.5742\underline{7}2018$	$6.57436609\underline{7}1$
6400	$6.574\underline{1}8$	6.57437	4096	$6.5743\underline{4}2575$	$6.574366093\underline{9}$
			8192	$6.57436\underline{0}214$	6.5743660937
			16384	$6.57436\underline{4}624$	6.5743660937

5.2. Lookback Options (continued)

- Effects of Number of Sampling Points
- Choose parameters from Broadie, Glasserman, and Kou (1997)
  $LP(S = 100, t_n = 0.5, \sigma = 0.3, r = 0.1, d = 0.0)$



- Probability Density vs. Number of Sampling Points
- Continue with parameters from Broadie, Glasserman, and Kou (1997)
  $LP(S = 100, t_n = 0.5, \sigma = 0.3, r = 0.1, d = 0.0)$



5.3. Puttable Cliquets

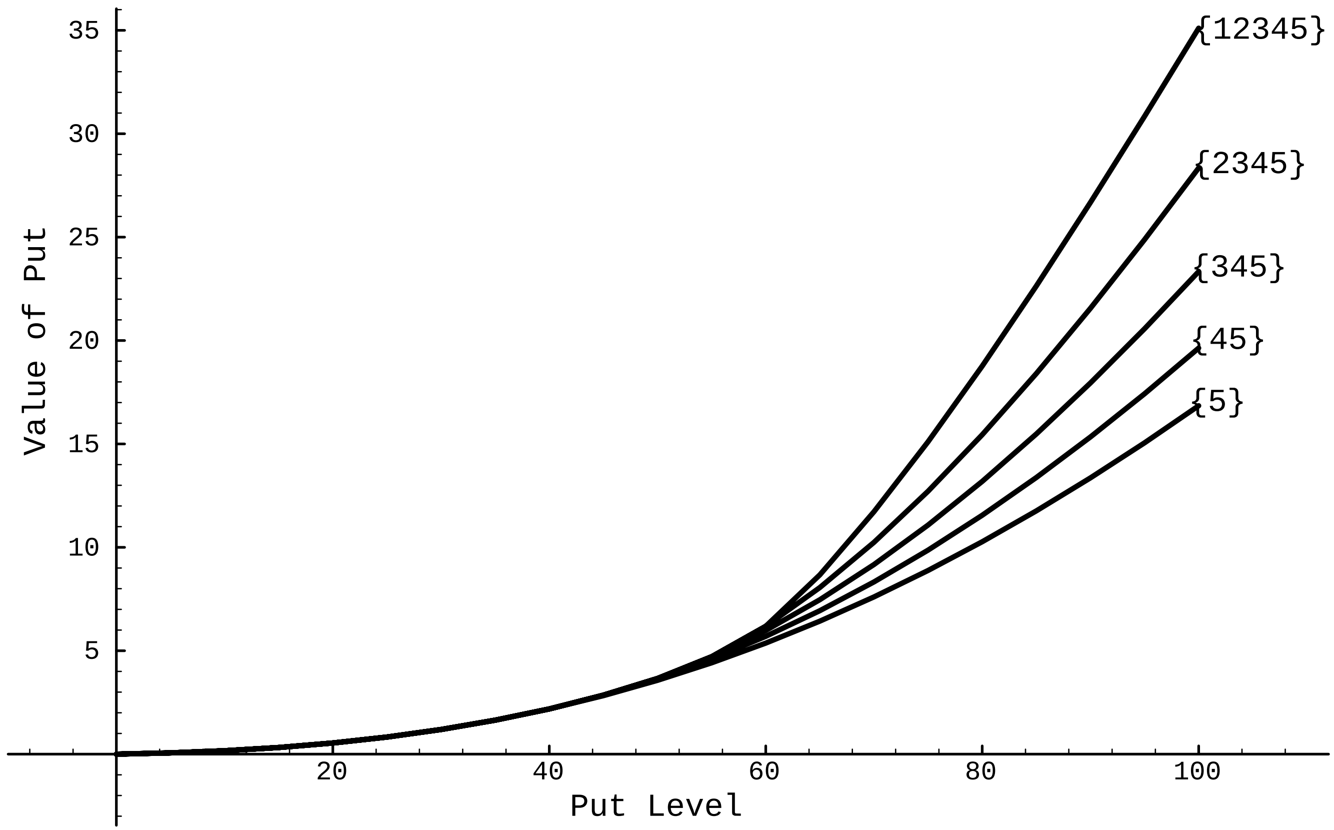
- Choose 5 one year at-the-money optionlets, fixed notional, $S = 100$, $\sigma = 0.3$, $r = 0.1$, $d = 0.0$

  Exact Value of Cliquet Without Put: 56.085

- Explore effects of varying put level and exercise dates
  - For Bermudian options, consider cases of constant and discounted put levels
  - Examine put values

5.3. Puttable Cliquets (continued)

- Put value versus put strike:



6.0. Conclusions

- The convolution method is a powerful, general, elegant technique for the valuation of path-dependent options with discrete sampling and/or exercise points.

- Particularly for moderate numbers of sample points, the approach is superior to available methods.

- Convergence properties are strong, particularly when some attention is devoted to regularizing contact points. Extrapolation techniques are readily applicable.

- Even for regions near barriers, values are extremely accurate.

- The convolution method allows access to probability densities of discretely sampled path "measures" not readily accessible by other means.