Particle Methods in Filtering & Applications in Finance

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Partially Observed Stochastic Systems

- State of the system X_n

$$X_{n+1} = F(X_n, V_n)$$

- $F(x, v)$ is a known function of x and v
- $\{V_n\}_n$ white noise (so X_n is Markov)
- Observation Y_n

$$Y_n = H(X_n, W_n)$$

- $\{W_n\}_n$ white noise
- $H(x, v)$ is a known function of x and w
  Often additive(for math. proofs) $H(x, w) = h(x) + w$
- $\{V_n\}_n$ and $\{W_n\}_n$ assumed independent (for math. proofs)

The Optimal Filter

Goal

Estimate, at each time n, the state X_n from all the observations $\mathbf{Y}_n = \{Y_n, Y_{n-1}, \cdots, Y_0\}$ up to that time.

Optimal solution (in the least squares sense)

conditional distribution of X_n given all the observations $\mathbf{Y}_n$

$$\pi_n(\,\cdot\,|\mathbf{Y}_n) = \pi_n(\,\cdot\,|\{Y_n, Y_{n-1}, \cdots, Y_0\})$$

Stettner's Equation

$$\pi_{n+1} = \phi_n(\pi_n, Y_{n+1})$$

Dynamical system in the (infinite dimensional) space of probability measures

Conditionally Gaussian Case Kalman-Bucy

The Particle Approximation

$$\pi_n(dx) = \mathbb{P}\{X_n \in dx | \mathbf{Y}_n\} \approx \frac{1}{m} \sum_{j=}^{m} \delta_{f_n^j}(dx)$$

Need two kinds of particles:

- those used to simulate $\mathbb{P}\{X_n | \mathbf{Y}_n\}$

$$f_n^1, .., f_n^m$$

- those used to simulate $\mathbb{P}\{X_{n+1} | \mathbf{Y}_n\}$

$$p_{n+1}^1, \dots, p_{n+1}^m$$

One Step Ahead Prediction

- Assume
$$f_n^1, \cdots, f_n^m$$
form a random sample from the distribution $\pi_n = \mathbb{P}\{X_n|\mathbf{Y}_n\}$
- Assume $v_n^1, .., v_n^m$ are m independent realizations of the noise V_n

Then
$$p_{n+1}^j = F(f_n^j, v_n^j)$$
gives a random sample
$$p_{n+1}^1, \cdots, p_{n+1}^m$$
from the conditionaldistribution $\mathbb{P}\{X_{n+1}|\mathbf{Y}_n\}$.

Filtering, or *Updating*

- Assume

$$p^1_{n+1}, \cdots, p^m_{n+1}$$

sample from the conditional distribution $\mathbb{P}\{X_{n+1}|\mathbf{Y}_n\}$

- Given a new observation y_{n+1}
  - $\alpha^j_{n+1} = \mathbb{P}\{(Y_{n+1} = y_{n+1}|p^j_n)$ likelihood of each particle p^j_{n+1}
  - $\alpha^j_{n+1} = r(G(y_{n+1}, p^j_{n+1}))|\frac{\partial G}{\partial Y_n}(G(y_{n+1}, p^j_{n+1}))|$

$$
\begin{aligned}
\mathbb{P}\{X_{n+1} = p^i_{n+1} | \mathbf{Y}_{n+1}\} &= \frac{\mathbb{P}\{X_{n+1} = p^i_{n+1}, Y_{n+1} | \mathbf{Y}_n)}{\mathbb{P}\{Y_{n+1} | \mathbf{Y}_n)} \\
&= \frac{\mathbb{P}\{Y_{n+1} | p^i_{n+1})\mathbb{P}\{X_{n+1} = p^i_{n+1} | \mathbf{Y}_n)}{\sum_{j=1}^m \mathbb{P}\{Y_{n+1} | p^j_{n+1}\}\mathbb{P}\{X_{n+1} = p^j_{n+1} | \mathbf{Y}_n)} \\
&= \frac{\alpha^i_{n+1} \cdot \frac{1}{m}}{\frac{1}{m}\sum_{j=1}^m \alpha^j_{n+1}} \\
&= \frac{\alpha^i_{n+1}}{\sum_{j=1}^m \alpha^j_{n+1}}
\end{aligned}
$$

r (observation) white (additive) noise density

Given this computation we define:

$$f^j_{n+1} = \begin{cases} p^1_{n+1} & \text{with probability} \frac{\alpha^1_{n+1}}{\alpha^1_{n+1}+\ldots+\alpha^m_{n+1}} \\ \vdots & \\ p^m_{n+1} & \text{with probability} \frac{\alpha^m_{n+1}}{\alpha^1_{n+1}+\ldots+\alpha^m_{n+1}} \end{cases}$$

These particles form a random sample of the conditional distribution $\pi_{n+1} = \mathbb{P}\{X_{n+1}|\mathbf{Y}_{n+1}\}$.

Algorithm Summary

Kitagawa (1994)

1. Initialization: generate an initial random sample of m particles $f_1^0, \cdots, f_0^m$.
2. For each time step n, we repeat the following process:
   - Generate independent particles v_n^j from the distribution of the system noise
   - Generate the particles p_{n+1}^j using the formula $p_{n+1}^j = F(f_n^j, v_n^j)$
   - Given a new observation, compute the likelihood α_{n+1}^j
   - Resample the p_{n+1}^j to produce the f_{n+1}^j's

Proved Results For fixed n particle approximation converges toward π_n. (Del Moral, Guillonnet, Lyons, Crisan, $\cdots$)

In Car Navigation Systems

The data

- Static data: representation of the network of roads and streets on which the vehicles travel
- Dynamic data: sequences of time stamped estimates of the position of the GPS receiver GPS tracks

The Street Network

A Schematic of the Streets of Princeton

An Example of a Track

Beginning Part of a Track Illustrating some of the Pitfalls

Animation Example

Z. Peng (C)

A. Bibas (matlab)

Stochastic Volatility Estimation

$$dS_t = S_t(\mu dt + \sigma_t dW_t),$$

where σ_t satisfies

$$d\sigma_t = -\lambda(\sigma_t - \sigma_0)dt + r d\tilde{W}_t.$$

Descretization:

$$X_{t+\Delta t} = \frac{S_{t+\Delta t}}{S_t} = (1 + \mu\Delta t) + \sigma_t \sqrt{\Delta t}\epsilon_{t+\Delta t}$$

and

$$\sigma_{t+\Delta t} \sim \mathcal{N}\left(\sigma_0 + e^{-\lambda\Delta t}(\sigma_t - \sigma_0), \sqrt{\frac{r^2}{2\lambda}(1 - e^{-2\lambda\Delta t})}\right).$$

Perfect Observation

State Equation

$$\begin{pmatrix} X_{t+\Delta t} \\ \sigma_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} 1 + \mu dt \\ \sigma_0 + e^{-\lambda \Delta t}(\sigma_t - \sigma_0) \end{pmatrix} + \begin{pmatrix} \sigma_t \sqrt{\Delta t} \epsilon_{t+\Delta t} \\ \sqrt{\frac{r^2}{2\lambda}(1 - e^{\lambda \Delta t})} \tilde{\epsilon}_{t+\Delta t} \end{pmatrix}$$

Observation Equation

$$Y_t = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} X_t \\ \sigma_t \end{pmatrix}$$

Including the Parameters in the State Space Model

State Equation

$$\begin{pmatrix} X_{t+\Delta t} \\ \sigma_{t+\Delta t} \\ \lambda_{t+\Delta t} \\ c_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} 1+\mu dt \\ \sigma_0 + e^{-\lambda_t \Delta t}(\sigma_t - \sigma_0) \\ \lambda_t \\ c_t \end{pmatrix} + \begin{pmatrix} \sigma_t \sqrt{\Delta t}\epsilon_{t+\Delta t} \\ \sqrt{c_t(1-e^{\lambda_t \Delta t})}\tilde{\epsilon}_{t+\Delta t} \\ 0 \\ 0 \end{pmatrix}$$

Observation Equation

$$Y_t = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_t \\ \sigma_t \\ \lambda_{t+\Delta t} \\ c_{t+\Delta t} \end{bmatrix}$$

Noisy Observations

- σ_t unobserved state
- X_t observation

State Evolution

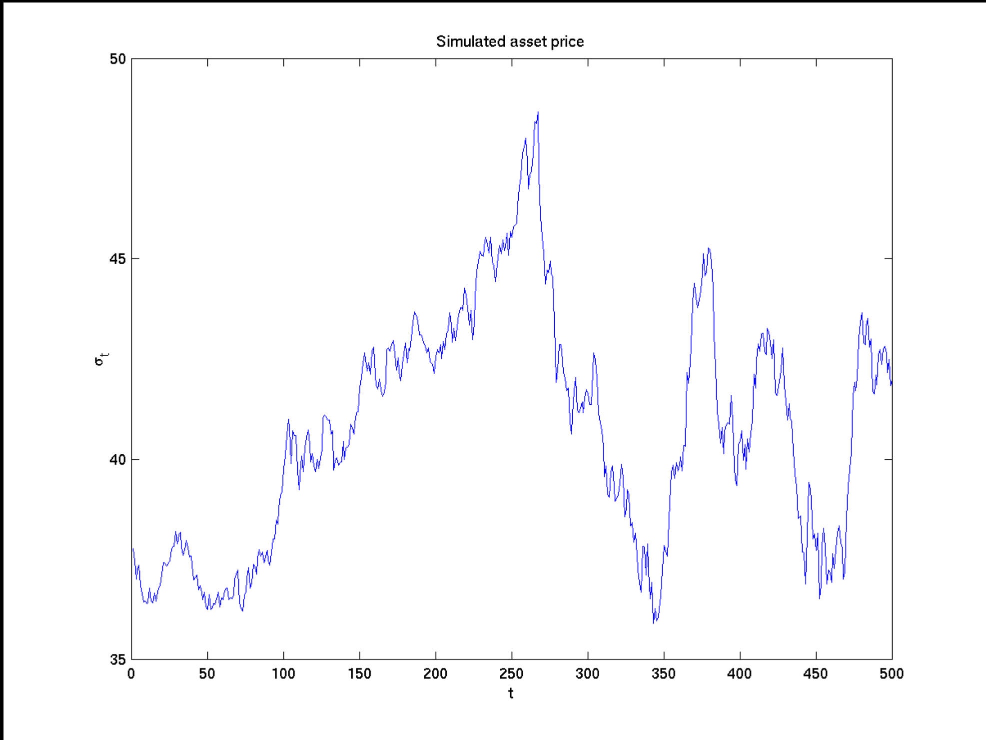
$$\begin{pmatrix} \sigma_{t+\Delta t} \\ \lambda_{t+\Delta t} \\ c_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \sigma_0 + e^{-\lambda_t \Delta t}(\sigma_t - \sigma_0) \\ \lambda_t \\ c_t \end{pmatrix} + \begin{pmatrix} \sqrt{c_t(1 - e^{\lambda_t \Delta t})}\tilde{\epsilon}_{t+\Delta t} \\ 0 \\ 0 \end{pmatrix}$$

Observation Equation

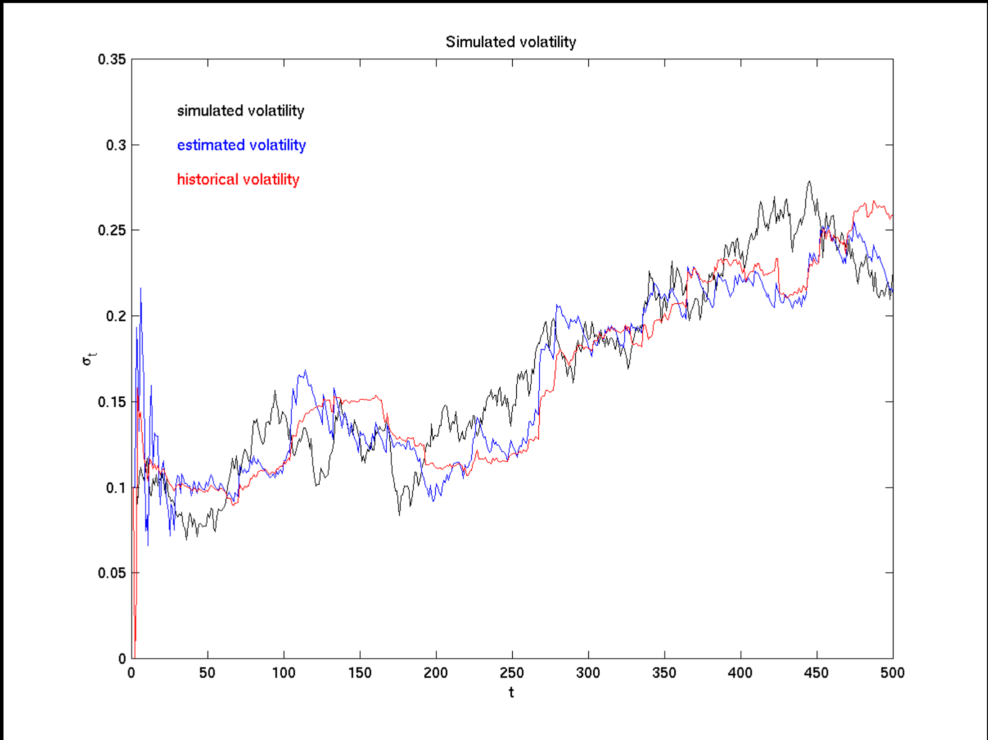
$$X_{t+\Delta t} = \frac{S_{t+\Delta t}}{S_t} = (1 + \mu \Delta t) + \sigma_t \sqrt{\Delta t} \epsilon_{t+\Delta t}$$

Regime Switching Stochastic Volatility

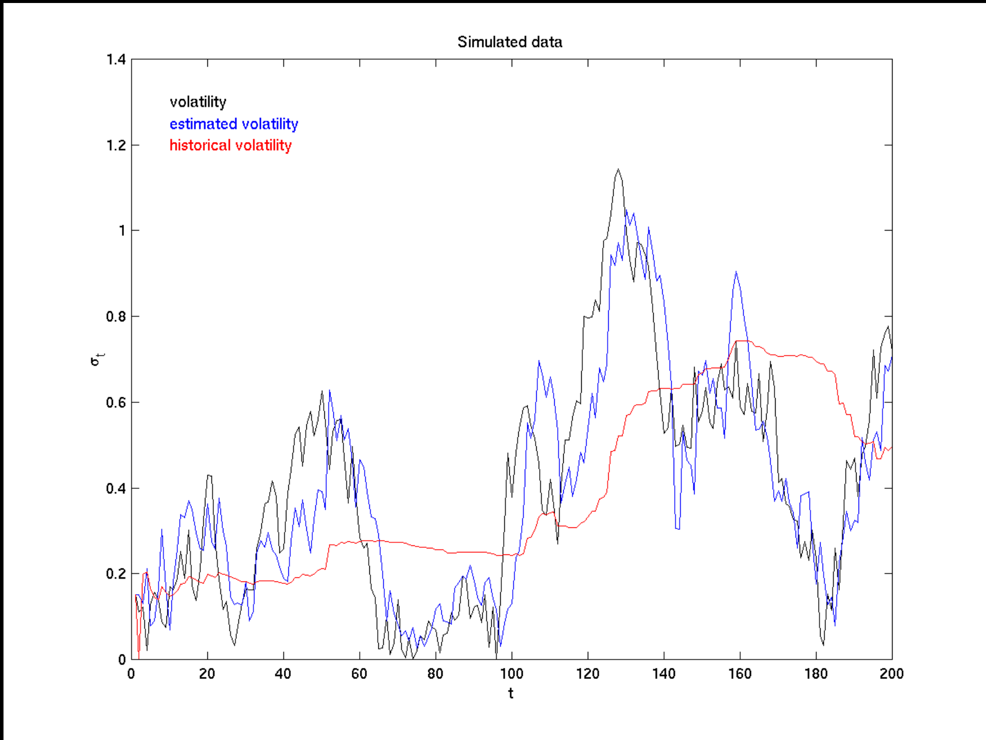
A. Papavasiliou $\Delta t = 0.004$, $\mu = 0.006$, $\lambda = 2$, $c = 0.5$, $\sigma_0 = 0.1$ & $\sigma_0 = 0.1$



Volatility


Simulated volatility
simulated volatility
estimated volatility
historical volatility
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
50
100
150
200
250
300
350
400
450
500
t
σt

Another Example



Still Another Example