Interest Rate Models:

from Parametric Statistics to Infinite Dimensional Stochastic Analysis

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Chapter 7

Infinite Dimensional HJM Models

Generalized HJM Models

- F Hilbert space of functions on $\mathcal{X} = [0, x^*)$
- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual assumptions
- Ito process $\{f_t;\ t \geq 0\}$ with values in F (will be more specific later)
- Prices of the zero-coupon (discount) bonds:

$$P_t(x) = e^{-\int_0^x f_t(y)dy}, \qquad t \geq 0,\ x \in \mathcal{X}$$

- Notation:
  - $I(f) =' f$ for the anti-derivative of a function f which vanishes at 0, (i.e. $[I(f)](x) = I_x f = \int_0^x f(y)dy$)

$$P_t = e^{-I(f_t)} = e^{-' f_t}.$$

Short and Long Interest Rates

Existence of a short and a long interest rates depend upon the choice of the space F

Assumption

- For every $x \in \mathcal{X}$, the evaluation functional:

$$\delta_x : F \ni f \hookrightarrow \delta_x(f) = f(x)$$

  is in the dual space F^* (OK for our last two examples)

- Short interest rate r_t

$$r_t = f_t(0) = -\frac{d}{dx} P_t \Big|_{x=0}$$

- Money market account:

$$B_t = e^{\int_0^t r_s ds}$$

(traded asset that pays the floating interest rate r_t continuously compounded)

- We use B_t as a numeraire, i.e. the unit in which the prices of all the other assets are expressed.

- Prices expressed in units of the numeraire (discounted prices) are denoted with a tilde

$$\tilde{P}_t(x) = B_t^{-1} P_t(x) = e^{-\int_0^t r_s ds - \int_0^x f_t(y) dy}$$

- Long interest rate $\ell_t = f_t(x^*)$

Generalized Portfolios and Trading Strategies

Bjork, Di Masi, Kabanov,Runggaldier

Assumption $\tilde{F}$ (real separable) Hilbert space s.t. $f \in F \Longrightarrow e^{-I(f)} \in \tilde{F}$

- portfolio or trading strategy (ϕ, η) with
  - $\phi = \{\phi_t; t \geq 0\}$ $\tilde{F}^*$ - valued predictable process weakly bounded
  - $\eta = \{\eta_t; t \geq 0\}$ $\mathbb{R}$ - valued predictable and integrable process
- value of the portfolio at time t

$$V_t(\phi, \eta) =< \phi_t, P_t > +\eta_t B_t$$

$< \cdot, \cdot >$ duality between $\tilde{F}^*$ and $\tilde{F}$, typically $< \phi_t, P_t >= \int_0^{x^*} P_t(x)\phi_t(dx)$

- Trading strategy admissible if

$$V_t(\phi, \eta) \geq -M$$

a.s. for all $t > 0$ for some $M > 0$

- Trading strategy self-financing if

$$V_t(\phi, \eta) = V_0(\phi, \eta) + \int_0^t < \phi_s, dP_s > + \int_0^t \eta_s dB_s$$

or in differential form:

$$dV_t(\phi, \eta) = < \phi_t, dP_t > + \eta_t dB_t$$

Equivalently (after discounting)

$$d\tilde{V}_t(\phi, \eta) = < \phi_t, d\tilde{P}_t >$$

Examples of Claims

$$\xi = \int \Phi_t(P_t)\nu(dt)$$

and finite linear combinations

- Any T' - zero coupon bond $\xi = P(0, T') = P_0(T')$

$$\nu(dt) = \delta_0(dt) \quad \text{and} \quad \Phi_t(f) = f(T')$$

- Any plain vanilla European call with maturity T_1 and strike K on a bond with maturity $T_2 > T_1$

$$\xi = (P(T_1, T_2) - K)^+ = (P_{T_1}(T_2 - T_1) - K)^+$$

so

$$\nu(dt) = \delta_{T_1}(dt) \quad \text{and} \quad \Phi_t(f) = (f(T_2 - T_1) - K)^+$$

Generalized HJM Models

- Existence of a strongly continuous semigroup $\{S_t;\, t \geq 0\}$ of bounded operators on F, the infinitesimal generator of which, say A, satisfies $Af = f'$ for a core in $\mathcal{D}(A)$
  - OK if one can define S_t as the left shift

$$[S_t f](x) = f(x+t)$$

  - OK in last two spaces F, even when $x^* < \infty$

- SPDE for the forward curve

$$f_t = S_t f_0 + \int_0^t [Af_s + \alpha_s] ds + \int_0^t \sigma_s dW_s$$

where

  - $\{W_t; t \geq 0\}$ Wiener process in a (real separable) Banach space E
  - $\{\alpha_t; t \geq 0\}$ F-valued adapted process, $\int_0^t \|\alpha_s\|_F ds < \infty$ a.s.
  - $\{\sigma_t; t \geq 0\}$ adapted process with values in the space of bounded linear operators from E into F, $\int_0^t \|\sigma_s\|_{HS}^2 ds < \infty$ a.s.

Bond Dynamics

- First notice that:

$$I_u \circ S_\alpha = I_{u+\alpha} - I_\alpha$$

as seen from:

$$\begin{aligned} I_u \circ S_\alpha g = \int_0^u (S_\alpha g)(s)ds &= \int_0^u g(s+\alpha)ds \\ &= \int_\alpha^{u+\alpha} g(s)ds = (I_{u+\alpha} - I_\alpha)g \end{aligned}$$

- Using $-\log P(t,T) = I_{T-t} f_t$ and

- Applying I_{T-t} to both sides of the SPDE

$$
\begin{aligned}
-\log P(t,T) &= I_{T-t}S_t f_0 + \int_0^t I_{T-t}S_{t-s}\alpha(s)ds + \int_0^t I_{T-t}S_{t-s}\sigma(s)dW_s \\
&= I_T f_0 - I_t f_0 + \int_0^t I_{T-s}\alpha(s)ds - \int_0^t I_{t-s}\alpha(s)ds \\
&+ \int_0^t I_{T-s}\sigma(s)dW_s - \int_0^t I_{t-s}\sigma(s)dW_s
\end{aligned}
$$

Using

- $\log P(0,t) = I_t f_0$ and
- $\int_0^t f_s(0)ds = I_t f_0 + \int_0^t I_{t-s}\alpha(s)ds + \int_0^t I_{t-s}\sigma(s)dW_s$

we get (Fubini)

$$
\log P(t,T) = \log P(0,t) + \int_0^t \left(f_s(0) - I_{T-s}\alpha(s)\right) ds - \int_0^t I_{T-s}\sigma(s)dW_s
$$

Using Ito's formula with the exponential function

$$\begin{aligned} P(t,T) &= P(0,T) \\ &\quad + \int_0^t P(s,T)\left(f_s(0) - I_{T-s}\alpha(s) + \frac{1}{2}\|I_{T-s}\sigma(s)\|^2_{K*}\right)ds \\ &\quad - \int_0^t P(s,T) I_{T-s}\sigma(s)dW_s \end{aligned}$$

[HJM condition] The discounted bond price $\tilde{P}(t,T)$, $0 \leq t \leq T$ is a martingale under $\mathbb{P}$ if

$$\alpha_s(x) = \langle I_x\sigma_s, \delta_x\sigma_s\rangle_{H*}$$

Review Cont et al here

Replication / Hedging

- Go to risk neutral world
- Given a claim ξ form the martingale

$$M_t = \mathbb{E}^{\mathbb{Q}}\{\xi|\mathcal{F}_t\}$$

- MRT gives

$$M_t = M_0 + \int_0^t \psi_t dW_t$$

for some Ito integrand with values in H_W^*

- Need conditions on ξ so that I can multiply ψ_t by P_t^{-1} and still be able to apply σ_t^{*-1}
- if $\mu_t = \sigma_t^{*-1}[P_t^{-1}\psi_t]$ is admissible DONE because

$$M_t = M_0 + \int_0^t < \mu_s, dP_s >$$

- POSSIBLE
- NOT DONE: controlling the maturities in the μ portfolio by the maturities in ξ !!!!!!!

Markovian Models

Assumption

$$\alpha_t(\omega) = \alpha_t(f_t(\omega)), \qquad \text{and} \qquad \sigma_t(\omega) = \sigma_t(f_t(\omega)).$$

for some deterministic functions α and σ on $[0, \infty) \times F$.

The SPDE reads

$$df_t = [Af_t + \alpha_t(f_t)]dt + \sigma_t(f_t)dW_t$$

and its evolution form reads:

$$f_t = S_t f_0 + \int_0^t S_{t-s}\alpha_s(f_s)\,ds + \int_0^t S_{t-s}\sigma_s(f_s)dW_s$$

and the usual Lipschitz conditions guarantee existence and uniqueness of the solution of this evolution form.

$\{f_t;\ t \geq 0\}$ is a Markov process with state space F

Gaussian Markov Models

- Deterministic drift and volatility

$$df_t = [Af_t + \alpha_t]dt + \sigma_t dW_t$$

or in evolution form:

$$f_t = S_t f_0 + \int_0^t S_{t-s}\alpha_s\,ds + \int_0^t S_{t-s}\sigma_s dW_s$$

- Not an equation !!
- $\{f_t\}_{t\geq 0}$ Gaussian process (Ornstein Uhlenbeck process) in F
- Kennedy analyzed Gaussian forward models

Ergodicity Properties of the Markovian Evolutions

Notation:

- μ_t for distribution of f_t
- γ_t for distribution of $\int_0^t S_{t-s}\sigma_s dW_s$
- $\tilde{\alpha}_t = \int_0^t S_{t-s}\alpha_s\, ds$

The evolution equation implies:

$$\mu_t = (S_t\mu_0) * (\tilde{\alpha}_t + \gamma_t)$$

In order to control:

$$\lim_{t\to\infty} \mu_t$$

we address separately the three following problems:

- Does the limit $\tilde{\alpha} = \lim_{t\to\infty} \tilde{\alpha}_t$ exist?
- Does the limit $\gamma_\infty = \lim_{t\to\infty} \gamma_t$ exist as a probability measure on F?
- Does the limit $\nu_\infty = \lim_{t\to\infty} S_t\mu_0$ exist as a probability measure on F?

In the best case scenario (YES – YES – YES):

$$\mu_\infty = \nu * (\tilde{\alpha} + \gamma_\infty)$$

We have an invariant measure obtained as a

mixture of shifts (by elements of the support of ν) of the Gaussian measure γ_∞ shifted to have mean $\tilde{\alpha}$

Analysis of $\tilde{\alpha}$

Assumption

$\alpha_t \equiv \alpha \in F$ satisfies $\alpha(x^*) = 0$.

$\tilde{\alpha}$ is the antiderivative of α which vanishes at x^*

$$\tilde{\alpha}(x) = \int_x^{x^*} \alpha(y) dy$$

Analysis of γ_∞

γ_t is a mean zero Gaussian measure with covariance

$$\begin{aligned}\Gamma_t(x,y) &= \mathbb{E}\left\{< \delta_x, \int_0^t S_{t-s}\sigma_s dW_s >< \delta_x, \int_0^t S_{t-s}\sigma_s dW_s >\right\}\\ &= \int_0^t < \sigma_s^* S_{t-s}^* \delta_x, \sigma_s^* S_{t-s}^* \delta_y > ds\end{aligned}$$

Assumption $\sigma_t \equiv \sigma$

Covariance of γ_∞ given by $\Gamma_\infty(x,y)$ equal to the integral of the kernel $\Sigma(x,y)$ of the operator $\sigma^*\sigma$ along the part of the ray $\{(x+t, y+t); t \geq 0\}$ contained in $[0,x^*) \times [0,x^*)$ (draw a picture !!!)

Analysis of ν

- In cases iii) and iv) the space F is s.t.

$$\lim_{t\to\infty} S_t f = f(x^*)\mathbf{1}$$

($S_t f$ converges toward the constant function equal to $f(x^*)$)

ν is the "marginal" of μ_0 for x^*. It is a measure concentrated on the constant functions ($\sim \mathbb{R}$)

If we draw a curve f at random from distribution μ_∞, ν choose $f(x^*)$ for us !!!

- In case ii) (Vargiolu) $\{S_t\}_t$ has many non-trivial invariant measures (periodic functions ...)

Asymptotic Behavior

- For large times t (ergodic theorem) we expect f_t to look like an element of f of F drawn at random according to the distribution μ_∞
- Such a random sample f is obtained by:
  - Choose a level for $f(x^*)$ at random according to ν
  - Shift $\tilde{\alpha}$ to give it this value at x^*
  - Perturb this candidate for f by a random element of F generated according to the distribution γ_∞

Principal Component Analysis of the Forward Curves

If we choose a Generalized HJM model for the historical dynamics of the forward curve, then the diagonalization of the covariance operator of γ_∞ should fit the empirical results of the PCA, i.e.

- The eigenvalues of this covariance operator should decay at the same rate as the (empirical) proportions of the variance explained by the principal components
- The eigenfunctions corresponding to the largest eigenvalues should look like the main loadings of the PCA