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# ESSENTIALS of the ARBITRAGE THEORY

Part I. Basic notions and theorems of the "Arbitrage Theory"
Part II. Martingale measures and their constructions
Part III. Quickest detection of the appearing of the arbitrage possibilities

Appendix I. A. N. Shiryaev, A. S. Cherny. "Vector Stochastic Integrals and the Fundamental Theory of Asset Pricing".
Appendix II. J. Kallsen, A. N. Shiryaev. "The Cumulant Process and Esscher's Change of Measure".

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## Part II.
## Martingale measures and their constructions

**1.** The "First" and the "Second" fundamental theorems show clearly how "martingale measures" are important for solving the problem "Arbitrage or No Arbitrage".

It is reasonable to start our discussion of the construction of the martingale measures that are (locally) absolutely continuous or equivalent to the original basic measure $\mathsf{P}$ involved in the definition of the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\geqslant 0}, \mathsf{P})$

with a discrete (with respect to time) version a result established by I. Girsanov for processes of diffusion type, which became the prototype for a number of results for martingales, local martingales, semimartingales and so on.

Set $h_n = -\mu_n + \sigma_n \varepsilon_n$ where $\mu_n$ and $\sigma_n$ are $\mathscr{F}_{n-1}$-measurable $\mathscr{F}_0 = \{\varnothing, \Omega\}$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots)$ is a sequence of $\mathscr{F}_n$-measurable *iid* random variables, $\varepsilon_n \sim \mathscr{N}(0,1)$, $\sigma_n > 0$.

From these assumptions it follows that

$$\mathrm{Law}(h_n \mid \mathscr{F}_{n-1}, \mathsf{P}) = \mathscr{N}(\mu_n, \sigma_n^2)$$

which allows one to call $h = (h_n)$ a conditionally Gaussian sequence (with respect to $\mathsf{P}$) with (conditional) expectation

$$\mathsf{E}(h_n \mid \mathscr{F}_{n-1}) = \mu_n$$

and variance

$$\mathsf{D}(h_n \mid \mathscr{F}_{n-1}) = \sigma_n^2.$$

Setting $H_n = \sum\limits_{k=1}^{n} h_k$, $A_n = \sum\limits_{k=1}^{n} \mu_k$, $M_n = \sum\limits_{k=1}^{n} \sigma_k \varepsilon_k$ we get

$$\Delta H_n = -\mu_n \Delta + \sigma_n \Delta W_n \tag{1}$$

where $\Delta x_n = x_n - x_{n-1}$, $\Delta = 1$, $\Delta W_n = \varepsilon_n$, which one can regard as a discrete counterpart to the stochastic differential

$$dH_t = -\mu_t \, dt + \sigma_t \, dW_t \tag{2}$$

of some Itô process $H = (H_t)_{t \geqslant 0}$ generated by a Wiener process $W = (W_t)_{t \geqslant 0}$ with the local drift $\mu = (\mu_t)_{t \geqslant 0}$ and the local volatility $\sigma = (\sigma_t)_{t \geqslant 0}$.

Our construction of the measure $\widetilde{\mathsf{P}}$ is based on the sequence of (positive) random variables

$$Z_n = \exp\biggl\{ \sum_{k=1}^{n} \frac{\mu_k}{\sigma_k} \varepsilon_k - \frac{1}{2} \sum_{k=1}^{n} \Bigl( \frac{\mu_k}{\sigma_k} \Bigr)^2 \biggr\}. \tag{3}$$

LEMMA 1. 1) *The sequence $Z = (Z_n)_{n \geqslant 1}$ is $\mathsf{P}$-martingale with $\mathsf{E} Z_n = 1$, $n \geqslant 1$.*

2) *Let $\mathscr{F} = \bigvee \mathscr{F}_n$ and assume that*

$$\mathsf{E} \exp\biggl\{ \frac{1}{2} \sum_{k=1}^{\infty} \Bigl( \frac{\mu_k}{\sigma_k} \Bigr)^2 \biggr\} < \infty \qquad (\textit{the “Novikov condition”}). \tag{4}$$

*Then $Z = (Z_n)_{n \geqslant 1}$ is a uniformly integrable martingale with limit ($\mathsf{P}$-a.s.) $Z_\infty = \lim Z_n$ such that*

$$Z_\infty = \exp\biggl\{ \sum_{k=1}^{\infty} \Bigl( \frac{\mu_k}{\sigma_k} \Bigr) \varepsilon_k - \frac{1}{2} \sum_{k=1}^{\infty} \Bigl( \frac{\mu_k}{\sigma_k} \Bigr)^2 \biggr\} \tag{5}$$

*and $Z_n = \mathsf{E}(Z_\infty \mid \mathscr{F}_n)$, $\mathsf{E} Z_\infty = 1$.*

LEMMA 2. 1) *Let* $h=(h_n)_{n\leqslant N}$ *be a conditionally Gaussian sequence such that*

$$\operatorname{Law}(h_n \mid \mathscr{F}_{n-1}; \mathsf{P}) = \mathscr{N}(\mu_n, \sigma_n^2), \qquad n \leqslant N.$$

*Let* $\mathscr{F}_N = \mathscr{F}$ *and* $\widetilde{\mathsf{P}}_N$ *be the measure defined by formula* $\widetilde{\mathsf{P}}_N(d\omega) = Z_N(\omega)\,\mathsf{P}(d\omega)$.

*Then the sequence* $h=(h_n)_{n\leqslant N}$ *is conditionally Gaussian with respect to* $\widetilde{\mathsf{P}}_N$ *and*

$$\operatorname{Law}(h_n \mid \mathscr{F}_{n-1}, \widetilde{\mathsf{P}}_N) = \mathscr{N}(0, \sigma_n^2), \qquad n \leqslant N.$$

If $H_n = \sum_{k=1}^{n} h_k$ then one may find a (new) sequence of the *iid* random variables $\widetilde{\varepsilon} = (\widetilde{\varepsilon}_n)_{n\geqslant 1}$, $\widetilde{\varepsilon}_n \sim \mathscr{N}(0,1)$, such that ($\widetilde{\mathsf{P}}_N$-a.s.)

$$H_n = \sum_{k=1}^{n} \sigma_k \widetilde{\varepsilon}_k, \qquad n \leqslant N,$$

so, with respect to the measure $\widetilde{\mathsf{P}}_N$ the sequence $H=(H_n)_{n\leqslant N}$ is a local martingale (= a martingale transform).

2) If $\sigma_n^2$, $n \leqslant N$, are independent on $\omega$, then $h=(h_n)_{n\leqslant N}$ is a sequence of independent Gaussian random variables with respect to $\widetilde{\mathsf{P}}_N$:

$$\operatorname{Law}(h_n \mid \mathscr{F}_{n-1}; \widetilde{\mathsf{P}}_N) = \mathscr{N}(0, \sigma_n^2), \qquad n \leqslant N.$$

3) If $\mathscr{F} = \bigvee \mathscr{F}_n$ and $\mathsf{E} Z_\infty = 1$ (i.e. $Z = (Z_n)_{n\geqslant 1}$ is a uniformly integrable martingale) then the sequence $h=(h_n)_{n\geqslant 1}$ is the conditionally Gaussian with respect to $\widetilde{\mathsf{P}}_\infty$ where $\widetilde{\mathsf{P}}_\infty(d\omega) = Z_\infty(\omega)\,\mathsf{P}(d\omega)$ and the sequence $H=(H_n)_{n\geqslant 1}$ is a $\widetilde{\mathsf{P}}_\infty$-local martingale.

**2.** Very similar formulations one may done for the case of continuous time. Namely, suppose that the process $H=(H_t)_{t\geqslant 0}$ is an Itô process with

$$dH_t = -\mu_t\,dt + \sigma_t\,dW_t, \qquad \sigma_t > 0, \quad H_0 = 0, \tag{6}$$

where $W=(W_t)_{t\geqslant 0}$ is a $\mathsf{P}$-Wiener process (a Brownian motion).

We define

$$Z_t = \exp\biggl\{\int_0^t \Bigl(\frac{\mu_u}{\sigma_u}\Bigr) dW_u - \frac{1}{2}\int_0^t \Bigl(\frac{\mu_u}{\sigma_u}\Bigr)^2 du\biggr\}, \tag{7}$$

and put $\widetilde{\mathsf{P}}_T(d\omega) = Z_T\,\mathsf{P}(d\omega)$.

If

$$\mathsf{E}\exp\biggl\{\frac{1}{2}\int_0^T \Bigl(\frac{\mu_u}{\sigma_u}\Bigr)^2 du\biggr\} < \infty \tag{8}$$

then the process $(H_t)_{t\leqslant T}$ is a $\widetilde{\mathsf{P}}_T$-local martingale such that ($\widetilde{\mathsf{P}}_T$-a.s.)

$$H_t = \int_0^t \sigma_u\,d\widetilde{W}_u, \qquad t \leqslant T, \tag{9}$$

where $(\widetilde{W}_u)_{u\leqslant T}$ is a $\widetilde{\mathsf{P}}_T$-Wiener process.

If

$$\mathsf{E}\exp\left\{\frac{1}{2}\int_0^\infty\left(\frac{\mu_u}{\sigma_u}\right)^2du\right\}<\infty \qquad \text{(“Novikov condition”)} \tag{10}$$

then $Z=(Z_t)_{t\geqslant 0}$ will be uniformly integrable martingale $\Big($i.e. equivalent to the property $\mathsf{E}Z_\infty=1$ where $Z_\infty=\lim\limits_{t\to\infty}Z_t\Big)$.

Let's give a proof of the “Novikov condition” for case of the continuous time. (The case of the discrete time can be considered by the similar way and, in fact, can be obtained from the result of the continuous time.) At that, using Dambis, Dubins–Schwartz theorem ([KY]) this problem can be reformulated by the following way:

Let

$$Z_t(\lambda)=e^{\lambda B_t-\frac{\lambda^2}{2}t} \tag{11}$$

where $\lambda\in\mathbb{R}$, $B=(B_t)$ is a Brownian motion.

We claim that the “*Novikov condition*”

$$\mathsf{E}e^{\frac{\lambda^2}{2}\tau}<\infty \tag{12}$$

*implies that*

$$\mathsf{E}Z_\tau(\lambda)=1. \tag{13}$$

For proof that (12)⇒(13) let's assume firstly that instead (12) we have a little bit more stronger condition

$$\mathsf{E}e^{\frac{1+\varepsilon}{2}\lambda^2\tau}<\infty \tag{14}$$

for some $\varepsilon>0$. Under such condition one may prove (R. Liptser and A. Shiryaev) that $\mathsf{E}Z_\tau(\lambda)=1$. The proof is simple. Really, it is sufficient to show that for some $\delta>0$

$$\sup_{t\geqslant 0}\mathsf{E}\big(Z_{t\wedge\tau}(\lambda)\big)^{1+\delta}<\infty. \tag{15}$$

Taking for simplicity $\lambda=1$ we have

$$\big(Z_t(1)\big)^{1+\delta}=\psi_t^{(1)}\psi_t^{(2)}$$

with

$$\psi_t^{(1)}=e^{(1+\delta)B_t-\frac{p(1+\delta)^2}{2}t},$$
$$\psi_t^{(2)}=e^{(\frac{p(1+\delta)^2}{2}-\frac{1+\delta}{2})t}$$

and $p = 1+\varepsilon$, $q = \dfrac{1+\varepsilon}{\varepsilon}$. By Hölder inequality

$$\mathsf{E}\big(\varepsilon(1)_t\big)^{1+\delta} = \big(\mathsf{E}(\psi_t^{(1)})^p\big)^{1/p}\big(\mathsf{E}(\psi_t^{(2)})^q\big)^{1/q} = \big(\mathsf{E}(\psi_t^{(2)})^q\big)^{1/q}$$

because $\mathsf{E}(\psi_t^{(1)})^p = 1$.

Take $\delta$ such that

$$\delta(1+\delta) \leqslant \frac{\varepsilon^2}{(1+\varepsilon)(1+2\varepsilon)}\,.$$

Then $(\psi_\tau^{(2)})^q \leqslant e^{(\frac{1}{2}+\varepsilon)\tau}$ and so

$$\sup_{t\geqslant 0} \mathsf{E}(Z_{\tau\wedge t}^{(1)})^{1+\delta} \leqslant \mathsf{E}e^{(\frac{1}{2}+\varepsilon)(\tau\wedge t)} \leqslant \mathsf{E}e^{(\frac{1}{2}+\varepsilon)\tau} < \infty.$$

From this it follows that the martingale $(Z_{t\wedge\tau}(1))_{t\geqslant 0}$ is uniformly integrable and as a corollary

$$\mathsf{E}Z_\tau(1) = 1.$$

The same is true for any $\lambda \in \mathbb{R}$: $\mathsf{E}Z_\tau(\lambda) = 1$.

Now suppose that $\mathsf{E}e^{\frac{\lambda^2}{2}\tau} < \infty$. Let's put $\lambda_\varepsilon = (1-\varepsilon)\lambda$ with $0 < \varepsilon < 1$ then we find that

$$\mathsf{E}e^{\frac{1+\varepsilon}{2}\lambda_\varepsilon^2\tau} = \mathsf{E}e^{\frac{(1+\varepsilon)(1-\varepsilon)^2\lambda^2}{2}\tau} \leqslant \mathsf{E}e^{\frac{\lambda^2}{2}\tau} < \infty.$$

From this and previous consideration we get

$$\mathsf{E}Z_\tau(\lambda_\varepsilon) = 1.$$

Then applying the Hölder inequality with $1/p = 1-\varepsilon$, $1/q = \varepsilon$ we find that

$$\begin{aligned} 1 &= \mathsf{E}Z_\tau(\lambda_\varepsilon) = \mathsf{E}Z_\tau\big((1-\varepsilon)\lambda\big) \\ &= \mathsf{E}e^{\lambda(1-\varepsilon)B_\tau - \frac{\lambda^2}{2}(1-\varepsilon)^2\tau} = \mathsf{E}e^{(1-\varepsilon)(\lambda B_\tau - \frac{\lambda^2}{2}\tau)}\cdot e^{\frac{(1-\varepsilon)\varepsilon\lambda^2\tau}{2}} \\ &\leqslant \big(\mathsf{E}Z_\tau(\lambda)\big)^{1-\varepsilon}\big(\mathsf{E}e^{(1-\varepsilon)\frac{\lambda^2}{2}\tau}\big)^{\varepsilon} \leqslant \big(\mathsf{E}Z_\tau(\lambda)\big)^{1-\varepsilon}\big(\mathsf{E}e^{\frac{\lambda^2}{2}\tau}\big)^{\varepsilon}. \end{aligned} \tag{16}$$

If $\mathsf{E}e^{\frac{\lambda^2}{2}\tau} < \infty$ then from (16) we get with limit passage $\varepsilon \downarrow 0$ that

$$1 \leqslant \mathsf{E}Z_\tau(\lambda).$$

But $\mathsf{E}Z_\tau(\lambda) \leqslant 1$ because $(Z_{t\wedge\tau}(\lambda))_{t\geqslant 0}$ is a nonnegative supermartingale with $Z_0(\lambda) = 1$. So,

$$\mathsf{E}e^{\frac{\lambda^2}{2}\tau} < \infty \implies \mathsf{E}Z_\tau(\lambda) = 1. \tag{17}$$

It is interesting to note that from (16) it is easy to see that instead the condition $\mathsf{E}e^{\frac{\lambda^2}{2}\tau} < \infty$ it is sufficient to assume only that

$$\lim_{\varepsilon\downarrow 0} \varepsilon \log \mathsf{E}e^{\frac{(1-\varepsilon)\lambda^2}{2}\tau} = 0. \tag{18}$$

Indeed, from this condition it follows that for sufficiently small $\varepsilon > 0$

$$\mathsf{E}e^{\frac{1+\varepsilon}{2}\lambda_\varepsilon^2\tau} = \mathsf{E}e^{\frac{(1+\varepsilon)(1-\varepsilon)^2}{2}\lambda^2\tau} = \mathsf{E}e^{\frac{(1-\varepsilon)(1-\varepsilon^2)}{2}\lambda^2\tau} < \infty.$$

Thus, again $1 = \mathsf{E}Z_\tau(\lambda_\varepsilon)$ and the inequalities in (16) hold and by (18) we get that $\mathsf{E}Z_\tau(\lambda) = 1$.

**3.** We intend now to prove the statements of the Lemma 2.

By Bayes's formula ($\widetilde{\mathsf{P}}_N$-a.s.)

$$
\begin{aligned}
\mathsf{E}_{\widetilde{\mathsf{P}}_N}(e^{i\lambda h_n} \mid \mathscr{F}_{n-1}) &= \mathsf{E}\Big(e^{i\lambda h_n}\frac{Z_n}{Z_{n-1}} \;\Big|\; \mathscr{F}_{n-1}\Big) \\
&= \mathsf{E}\big(e^{(i\lambda\sigma_n+\frac{\mu_n}{\sigma_n})\varepsilon_n - i\lambda\mu_n - \frac{1}{2}(\frac{\mu_n}{\sigma_n})^2} \mid \mathscr{F}_{n-1}\big) \\
&= \mathsf{E}\big(e^{(i\lambda\sigma_n+\frac{\mu_n}{\sigma_n})\varepsilon_n - \frac{1}{2}(i\lambda\sigma_n+\frac{\mu_n}{\sigma_n})^2}\cdot e^{\frac{1}{2}(i\lambda\sigma_n+\frac{\mu_n}{\sigma_n})^2 - i\lambda\mu_n - \frac{1}{2}(\frac{\mu_n}{\sigma_n})^2} \mid \mathscr{F}_{n-1}\big) \\
&= e^{-\frac{\lambda^2\sigma_n^2}{2}}
\end{aligned}
$$

where we use the equality

$$
\mathsf{E}e^{(i\lambda\sigma_n+\frac{\mu_n}{\sigma_n})\varepsilon_n - \frac{1}{2}(i\lambda\sigma_n+\frac{\mu_n}{\sigma_n})^2} = 1
$$

and the fact that the $\sigma_n^2$ are $\mathscr{F}_{n-1}$-measurable. So, we obtain the equality

$$
\mathsf{E}_{\widetilde{\mathsf{P}}_N}(e^{i\lambda h_n} \mid \mathscr{F}_{n-1}) = e^{-\frac{\lambda^2\sigma_n^2}{2}}
$$

which means that the sequence $h = (h_n)$ remains conditionally Gaussian with respect to the new measure $\widetilde{\mathsf{P}}_N$, but has now a trivial "drift" component:

$$
\text{Law}(h_n \mid \mathscr{F}_{n-1}; \widetilde{\mathsf{P}}_N) = \mathscr{N}(0, \sigma_n^2), \qquad n \leqslant N. \tag{19}
$$

One can say that the transitions from $\mathsf{P}$ to the measure $\widetilde{\mathsf{P}}_N$ *eliminates* ("kills") the drift $\mu = (\mu_n)_{n\leqslant N}$ of the sequence $h = (h_n)_{n\leqslant N}$, but *preserves* the conditional variance.

We have already mentioned that from (19) it follows that if $\widetilde{\varepsilon} = (\widetilde{\varepsilon}_n)_{n\leqslant N}$ is a sequence of $\mathscr{F}_n$-measurable random variables $\widetilde{\varepsilon}_n$ with

$$
\text{Law}(\widetilde{\varepsilon}_n \mid \mathscr{F}_{n-1}; \widetilde{\mathsf{P}}_N) = \mathscr{N}(0, 1) \tag{20}
$$

(one can always construct such a sequence, although it may be necessary to enlarge our initial probability space), then

$$
\text{Law}(h_n,\, n \leqslant N \mid \widetilde{\mathsf{P}}_N) = \text{Law}(\sigma_n\widetilde{\varepsilon}_n,\, n \leqslant N \mid \widetilde{\mathsf{P}}_N). \tag{21}
$$

Hence it is clear that the sequence $(h_n)_{n\leqslant N}$ "behaves" as a local martingale-difference $(\sigma_n\widetilde{\varepsilon}_n)_{n\leqslant N}$ with respect to $\widetilde{\mathsf{P}}_N$, while in terms of the original measure $\mathsf{P}$ a property similar to (21) can be expressed as follows:

$$
\text{Law}(h_n - \mu_n,\, n \leqslant N \mid \mathsf{P}) = \text{Law}(\sigma_n\varepsilon_n,\, n \leqslant N \mid \mathsf{P}). \tag{22}
$$

**4.** Now we considere a more-to-earth situation of the (positive!) prices

$$S_n = S_0 e^{H_n}, \tag{23}$$

where $H_n = h_1 + \cdots + h_n$, $n \geqslant 1$, and, in particular,

$$h_n = -\mu_n + \sigma_n \varepsilon_n \tag{24}$$

as in the previous presentation.

It we put

$$\widetilde{H}_n = H_n + \sum_{k \leqslant n} (e^{\Delta H_k} - \Delta H_k - 1) \quad \Big( = \sum_{k \leqslant n} (e^{\Delta H_k} - 1) \Big) \tag{25}$$

then we find that

$$S_n = S_0 \mathscr{E}(\widehat{H})_n \tag{26}$$

where the stochastic exponential

$$\mathscr{E}(\widehat{H})_n = e^{\widetilde{H}_n} \prod_{k \leqslant n} (1 + \Delta \widehat{H}_k) e^{-\Delta \widehat{H}_k} \quad \Big( = \sum_{k \leqslant n} (1 + \Delta \widehat{H}_k) \Big). \tag{27}$$

If $\Delta \widehat{H}_k \geqslant -1$ and the sequence $\widehat{H} = (\widehat{H}_k)_{k \geqslant 1}$ is a local martingale then $(\mathscr{E}(\widehat{H})_k)_{k \geqslant 1}$ is also a local martingale but because $\mathscr{E}(\widehat{H})_n \geqslant 0$ for all $n \geqslant 1$ the process $(\mathscr{E}(\widehat{H})_k)_{k \geqslant 1}$ is a martingale, indeed.

Consider now the case (24).

First, it is reasonable to consider the question of the conditions ensuring that the sequence $S = (S_n)_{n \geqslant 1}$ is a martingale with respect to the *original* measure $\mathsf{P}$.

We have already seen that it is sufficient to this end that the sequence $\widehat{H} = (\widehat{H}_n)_{n \geqslant 1}$ with $\Delta \widehat{H}_n = e^{\Delta H_n} - 1$ be a local martingale, i.e. $\mathsf{E}(|\Delta \widehat{H}_n| \mid \mathscr{F}_{n-1}) < \infty$ and $\mathsf{E}(\Delta \widehat{H}_n \mid \mathscr{F}_{n-1}) = 0$, or, equivalently,

$$\mathsf{E}(e^{\Delta H_n} \mid \mathscr{F}_{n-1}) = 1 \quad (\mathsf{P}\text{-a.s.}). \tag{28}$$

Since we assume that $\Delta H_n = -\mu_n + \sigma_n \varepsilon_n$ we can rewrite condition (28) as follows:

$$\mathsf{E}(e^{-\mu_n + \sigma_n \varepsilon_n} \mid \mathscr{F}_{n-1}) = 1 \tag{29}$$

which is equivalent to the relation

$$\mathsf{E}(e^{\sigma_n \varepsilon_n} \mid \mathscr{F}_{n-1}) = e^{\mu_n}.$$

The left-hand side here is equal to $e^{\frac{1}{2}\sigma_n^2}$. Thus, we arrive to the condition

$$\mu_n = \frac{\sigma_n^2}{2}, \qquad n \geqslant 1, \tag{30}$$

ensuring that the logarithmically conditionally Gaussian sequence

$$S_n = S_0 \exp\Big\{\sum_{k=1}^{n}(-\mu_k + \sigma_k \varepsilon_k)\Big\}, \qquad n \geqslant 1, \tag{31}$$

is a martingale with respect to $\mathsf{P}$. Of course, this is what one could expect because the sequence

$$\left(\exp\Big\{\sum_{k=1}^{n}\Big(\sigma_k\varepsilon_k - \frac{\sigma_k^2}{2}\Big)\Big\}\right)_{n\geqslant 1}$$

is a martingale.

We now proceed to the case when (30) fails.

Assume that $n \leqslant N$. We shall construct the required measure $\widetilde{\mathsf{P}}_N$ on $\mathscr{F}_N$ by means of the (conditional) Esscher transformation, in the following form:

$$\widetilde{\mathsf{P}}_N(d\omega) = Z_N(\omega)\,\mathsf{P}(d\omega) \tag{32}$$

with

$$Z_N(\omega) = \prod_{k=1}^{N} z_k(\omega) \tag{33}$$

and

$$z_k(\omega) = \frac{e^{a_k h_k}}{\mathsf{E}(e^{a_k h_k} \mid \mathscr{F}_{k-1})} \tag{34}$$

where we shall choose the $\mathscr{F}_{k-1}$-measurable variables $a_k = a_k(\omega)$ (here $\mathscr{F}_0 = \{\varnothing, \Omega\}$) such that the sequence $(S_n)_{n\leqslant N}$ is a $\widetilde{\mathsf{P}}_N$-martingale.

In our case when $S_n = S_0 e^{H_n}$, this $\widetilde{\mathsf{P}}_N$-martingale property means that

$$\mathsf{E}_{\widetilde{\mathsf{P}}_N}(e^{\Delta H_n} \mid \mathscr{F}_{n-1}) = 1$$

or

$$\mathsf{E}(e^{h_n(a_n+1)} \mid \mathscr{F}_{n-1}) = \mathsf{E}(e^{a_n h_n} \mid \mathscr{F}_{n-1}). \tag{35}$$

Bearing in mind that $h_n = \mu_n + \sigma_n\varepsilon_n$ we see that the equality (35) holds if

$$-\mu_n + \frac{\sigma_n^2}{2} = -a_n\sigma_n^2,$$

i.e.

$$a_n = \frac{\mu_n}{\sigma_n^2} - \frac{1}{2}\,. \tag{36}$$

(If (30) holds for all $n \leqslant N$ then $a_n = 0$ and $Z_N = 1$, i.e. $\widetilde{\mathsf{P}}_N = \mathsf{P}$.)

Choosing $a_n$ in accordance with (36) we obtain

$$\mathsf{E}(e^{a_n h_n} \mid \mathscr{F}_{n-1}) = \exp\left\{-\frac{\mu_n^2}{2\sigma_n^2} + \frac{\sigma_n^2}{8}\right\}.$$

Thus

$$z_n = \frac{e^{a_n h_n}}{\mathsf{E}(e^{a_n h_n} \mid \mathscr{F}_{n-1})} = \exp\left\{-\left(-\frac{\mu_n}{\sigma_n} + \frac{\sigma_n}{2}\right)\varepsilon_n - \frac{1}{2}\left(-\frac{\mu_n}{\sigma_n} + \frac{\sigma_n}{2}\right)^2\right\} \tag{37}$$

and

$$Z_N = \exp\left\{-\sum_{n=1}^{N}\left[\left(-\frac{\mu_n}{\sigma_n} + \frac{\sigma_n}{2}\right)\varepsilon_n + \frac{1}{2}\left(-\frac{\mu_n}{\sigma_n} + \frac{\sigma_n}{2}\right)^2\right]\right\}. \tag{38}$$

Hence the sequence $S = (S_n)_{n\leqslant N}$ with

$$S_n = S_0 e^{H_n}, \qquad H_n = h_1 + \cdots + h_n,$$

and $h_n = \mu_n + \sigma_n\varepsilon_n$ is a $\widetilde{\mathsf{P}}_N$-martingale with $\mathsf{E}_{\widetilde{\mathsf{P}}_N} S_n = S_0$.

**5.** Extensive exposition of the problem of the construction of Martingale measures for the case of the discrete time can be found in [ESF; Chapter V, Section 3] and for the case of the continuous time in [ESF; Chapter VII, Section 3] and in the Appendix II.

**6.** In the two included below appendices (Appendix I: the paper by A. N. Shiryaev and A. S. Cherny and Appendix II: the paper by J. Kallsen and A. N. Shiryaev) readers can find detailed presentations on the problems of the "Arbitrage", "Fundamental Theorems" and "Esscher's Change of Measure".

## Reference

[ESF] Shiryaev A. N. Essentials of Stochastic Finance. World Scientific, Singapore, 1999.

[RY] Revuz D., Yor M. Continuous martingales and Brownian Motion (Third ed.). Springer, Berlin, 1999.