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ESSENTIALS of the ARBITRAGE THEORY

Part I. Basic notions and theorems of the “Arbitrage Theory”
Part II. Martingale measures and their constructions
Part III. Quickest detection of the appearing of the arbitrage possibilities

Appendix I. A. N. Shiryaev, A. S. Cherny. “Vector Stochastic Integrals and the Fundamental Theory of Asset Pricing”.
Appendix II. J. Kallsen, A. N. Shiryaev. “The Cumulant Process and Esscher’s Change of Measure”.

Lectures
in IPAM (Institute for Pure
and Applied Mathematics),
UCLA; 3–4–5 January 2001

Introduction

From the economical point of view stochastic "Theory of Arbitrage" should give criteria for existence of the arbitrage possibilities, i.e. the possibilities of getting of the non-risky profit on the financial market where prices have random character.

A remarkable achievement of the modern stochastic financial theory is understanding that the problems of economical character admit a clear mathematical realization in the form that the arbitrage-free financial market's prices should have very special probability structure of the "martingale type" with respect to the so-called "martingale measure".

We shall consider the basic notions and facts of the "Arbitrage Theory" in the "Chapter I" of our lectures considering the both cases of the discrete and continuous time together. In particular we intend to explain that kind of the novelties, difficulties and pecularities we have for case of the financial models acting in continuous time.

In the "Chapter II" we consider some question of the construction of the "martingale measures" including the method of the "Esscher's transforms".

In the "Chapter III" several statistical problems of the "Arbitrage Theory" will be presented.

We assume that the observable process of the prices may change its probability characteristics in an unknown random time. This time we interpret as a time of the appearing of an arbitrage possibility. We consider several different formulations of the problem of the "quickest detection of the appearing of an arbitrage". From point of view of the mathematical statistics these problems relate with so-called "change-point" and "disorder" problems of the statistics of the stochastic processes.

Part I.
Basic notions and theorems of the "Arbitrage Theory"

We begin with the necessary probabilistic concepts and several models of the dynamics of market prices.

1. Taking the probabilistic approach we shall assume that all our consideration are carried out with respect to some probability space

$$(\Omega, \mathscr{F}, \mathsf{P})$$

satisfying to the Kolmogorov's axioms: Ω is the space of *elementary events* ω ("market situations"), $\mathscr{F}$ is some σ-algebra of *subsets* of Ω (the set of "observable market events"), P is a *probability* or *probability measure* on $\mathscr{F}$.

Time and dynamics are essential parts of the stochastic financial theory. For that reason it seems worthwhile to assume that on $(\Omega, \mathscr{F}, \mathsf{P})$ we have an additional structure – the *flow* $\mathbb{F} = (\mathscr{F}_t)_{t \geqslant 0}$ of sub-γ-algebras such that $\mathscr{F}_0 \subseteq \mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F}$, $s \leqslant t$. In a case of the continuous time $t \in [0, \infty)$ or $t \in [0, T)$. For a case of the discrete time $t = n \in \{0, 1, \dots\}$ or $t = n \in \{0, 1, \dots, N\}$.

The point in introducing this flow $\mathbb{F} = (\mathscr{F}_t)_{t \geqslant 0}$ which is also called a *filtration*, becomes clear once one have accepted the following interpretation:

$$\mathscr{F}_t \text{ is the set of events observable through time } t.$$

We can express it otherwise by saying that $\mathscr{F}_t$ is the "information" on the market situation that is available to an observer up to time t inclusive.

Thus, we assume that our underlying probabilistic model is a *filtered probability space*

$$\big(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathsf{P}\big)$$

which is also called a *stochastic basis.*

2. Stochastic model of the financial market. Discrete time. Regarding $\mathscr{F}_n$ as the information that had been accessible to observation through time n, it is natural to assume that *prices* $X = (X_n^1, \dots, X_n^d)_{n\geqslant 0}$ are such that

$$X_n^i - \mathscr{F}_n\text{-measurable}$$

or that (using a more descriptive language), prices of the stochastic model of the financial market are formed on the basis of the developments observable on the market up to time n.

There are the two most common methods for the description of the prices $S = (S_n)_{n\geqslant 0}$ where $S_n = X_n^i$ for some $i = 1, \dots, d$.

The *first* method, which is similar to the formula for "*compound interest*" uses the representation

$$S_n = S_0\, e^{H_n}, \tag{1}$$

where $H_n = h_0 + h_1 + \dots + h_n$ with $h_0 = 0$ and the random variables $h_n = h_n(\omega)$, $n \geqslant 0$, are $\mathscr{F}_n$-measurable. Hence

$$H_n = \log \frac{S_n}{S_0}$$

and the "logarithmic returns" can be evaluated by the formula

$$h_n = \log \frac{S_n}{S_{n-1}} = \log\left(1 + \frac{\Delta S_n}{S_{n-1}}\right),$$

where $\Delta S_n = S_n - S_{n-1}$.

If we set

$$\widehat{h}_n = \frac{\Delta S_n}{S_{n-1}}\,, \quad \widehat{H}_n = \widehat{h}_1 + \dots + \widehat{h}_n, \qquad n \geqslant 1,$$

then we can rewrite (1) as

$$S_n = S_0 \prod_{1\leqslant k\leqslant n} (1 + \widehat{h}_k) \tag{2}$$

or, equivalently, as

$$S_n = S_0 \prod_{1\leqslant k\leqslant n} (1 + \Delta\widehat{H}_k) = S_0\, e^{\widehat{H}_n} \prod_{1\leqslant k\leqslant n} (1 + \Delta\widehat{H}_k)\, e^{-\Delta\widehat{H}_k}. \tag{3}$$

The representation (2) (or (3) is just the *second* method of the description of prices. It is equivalent to the wellknown "*simple interest*" formula for a bank account.

Let $\mathscr{E}(\widehat{H})_n$ be the expression on the right-hand side of (3):

$$\mathscr{E}(\widehat{H})_n = e^{\widehat{H}_n} \prod_{1\leqslant k\leqslant n} (1+\Delta\widehat{H}_k)\, e^{-\Delta\widehat{H}_k}. \tag{4}$$

We call the stochastic sequence

$$\mathscr{E}(\widehat{H}) = \bigl(\mathscr{E}(\widehat{H})_n\bigr)_{n\geqslant 0}, \qquad \mathscr{E}(\widehat{H})_0 = 1, \tag{5}$$

defined by this expression (4) the *stochastic exponential* generated by the variables $\widehat{H} = (\widehat{H}_n)_{n\geqslant 0}$, $\widehat{H}_0 = 1$, or the Doléans exponential.

Thus, we can say that the first method for the description of prices uses the *usual* exponential

$$S_n = S_0\, e^{H_n}$$

while the second method involves the *stochastic* exponential:

$$S_n = S_0\, \mathscr{E}(\widehat{H})_n, \tag{6}$$

where

$$\widehat{H}_n = \sum_{1\leqslant k\leqslant n} (e^{\Delta H_k} - 1)$$

or

$$\boxed{\widehat{H}_n = H_n + \sum_{1\leqslant k\leqslant n} (e^{\Delta H_k} - 1 - \Delta H_k).} \tag{7}$$

It is also clear from (1) and (2) that

$$H_n = \sum_{1\leqslant k\leqslant n} \log(1+\Delta\widehat{H}_k) \tag{8}$$

or

$$\boxed{H_n = \widehat{H}_n + \sum_{1\leqslant k\leqslant n} \bigl(\log(1+\Delta\widehat{H}_k) - \Delta\widehat{H}_k\bigr).} \tag{9}$$

3. Stochastic model of the financial market. Continuous time. In this case we assume that prices $X = (X_t^1, \dots, X_t^d)_{t \geqslant 0}$ are such that

$$X_t^i - \mathscr{F}_t\text{-measurable.}$$

In the sequel we shall assume also that every component $S_t = X_t^i$, $i = 1, \dots, t$ is a *semimartingale*, i.e. it is a stochastic process which admits a representation (not necessarily in a unique way)

$$S_t = S_0 + A_t + M_t, \tag{10}$$

where $A = (A_t)_{t \geqslant 0}$ is a process of *bounded variation* $(\int_0^t |dA_s| < \infty,\ t \geqslant 0)$ and $M = (M_t)_{t \geqslant 0}$ is a *local martingale* $(M \in \mathscr{M}_{\mathrm{loc}})$. We recall that $M \in \mathscr{M}_{\mathrm{loc}}$ if there exists a sequence of stopping times $(\tau_k)_{k \geqslant 1}$ such that $\tau_k \uparrow \infty$, (P-a.s.), $k \to \infty$, and each "stopped" sequence

$$M^{\tau_k} = (M_{t \wedge \tau_k})_{t \geqslant 0}$$

is a martingale.

We remind that

a) Each local martingale $M = (M_t)_{t \geqslant 0}$ satisfying the condition

$$\mathsf{E} \sup_{s \leqslant t} M_s^- < \infty, \qquad t \geqslant 0,$$

is a supermartingale $(\mathsf{E}|M_t| < \infty,\ \mathsf{E}(M_t \mid \mathscr{F}_s) \leqslant M_s,\ s \leqslant t)$;

b) Each local martingale $M = (M_t)_{t \geqslant 0}$ with

$$\mathsf{E} \sup_{s \leqslant t} |M_s| < \infty, \qquad t \geqslant 0,$$

is a martingale;

c) Each local martingale $M = (M_t)_{t \geqslant 0}$ with

$$\mathsf{E} \sup_{t \geqslant 0} |M_t| < \infty$$

is a uniformly integrable $\left(\sup_{t \geqslant 0} \mathsf{E}\big(|M_t|\, I(|M_t| > C)\big) \to 0,\ C \to \infty\right)$ martingale.

It is clear that

$$\boxed{\mathscr{M} \subseteq \mathscr{M}_{\mathrm{loc}}}$$

where $\mathscr{M}$ is the class of martingales.

Remark 1. For the case of continuous time we assume that trajectories $t \rightsquigarrow S_t(\omega)$, $\omega \in \Omega$, are right-continuous with limits from the left. In French literature a process of this type is called un processus *càdlàg* – Continu À Droite avec des Limites À Gauche.

REMARK 2. For the case of the *discrete* time

$$\boxed{\mathscr{M}_{\text{loc}} = \mathscr{M}T = G\mathscr{M},} \tag{11}$$

where $\mathscr{M}T$ is the class of *martingale transforms* ($M \in \mathscr{M}T$ if there exists a martingale $m = (m_n)_{n\geqslant 0}$ and a sequence $\gamma = (\gamma_n)_{n\geqslant 0}$ such that γ_n – $\mathscr{F}_{n-1}$-measurable, $\mathscr{F}_{-1} = \mathscr{F}_0$, and

$$M_n = M_0 + \sum_{1\leqslant k\leqslant n} \gamma_k \Delta m_k,$$

and $G\mathscr{M}$ is the class of a *generalized martingale* ($M \in G\mathscr{M}$ if $\mathsf{E}|M_0| < \infty$, $\mathsf{E}(|M_n| \mid \mathscr{F}_{n-1}) < \infty$, $\mathsf{E}(M_n \mid \mathscr{F}_{n-1}) = M_{n-1}$ (P-a.s.)).

REMARK 3. For the case of the discrete time every sequence $S = (S_n)_{n\geqslant 0}$ is a semimartingale.

Really, if $\mathsf{E}|S_n| < \infty$, $n \geqslant 0$, it follows from *Doob's decomposition*:

$$S_n = S_0 + A_n + M_n, \tag{12}$$

where

$$A_n = \sum_{1\leqslant k\leqslant n} \mathsf{E}\big(\Delta X_k \mid \mathscr{F}_{k-1}\big), \quad A_0 = 0,$$
$$M_n = \sum_{1\leqslant k\leqslant n} \mathsf{E}\big(\Delta X_k - \mathsf{E}(\Delta X_k \mid \mathscr{F}_{k-1})\big), \quad M_0 = 0.$$

(Note that A_n – $\mathscr{F}_{n-1}$-measurable; every sequence $A = (A_n)_{n\geqslant 0}$, $A_0 = 0$, with that property is called *predictable*.)

In general cases ($\mathsf{E}|S_n| \leqslant \infty$, $n \geqslant 0$) we take $a > 0$ and put

$$S_n = S_0 + S_n^{(\leqslant a)} + S_n^{(>a)}$$

with

$$S_n^{(\leqslant a)} = \sum_{1\leqslant k\leqslant n} \Delta S_k\, I(|\Delta S_k| \leqslant a), \quad S_0^{(\leqslant a)} = 0,$$
$$S_n^{(>a)} = \sum_{1\leqslant k\leqslant n} \Delta S_k\, I(|\Delta S_k| > a), \quad S_0^{(>a)} = 0,$$

To $S^{(\leqslant a)} = (S_n^{(\leqslant a)})_{n\geqslant 0}$ we may apply the Doob decomposition:

$$S_n^{(\leqslant a)} = A_n^{(\leqslant a)} + M_n^{(\leqslant a)}.$$

So,

$$S_n = S_0 + \big(A_n^{(\leqslant a)} + S_n^{(>a)}\big) + M_n^{(\leqslant a)},$$

where $M^{(\leqslant a)} = (M_n^{(\leqslant a)})_{n\geqslant 1}$ is a martingale. And the sequence $(A_n^{(\leqslant a)} + S_n^{(>a)})_{n\geqslant 1}$ has, apparently, bounded variation.

REMARK 4. For case of the continuous time we have no the property $\mathscr{M}_{\text{loc}} = \mathscr{M}T = G\mathscr{M}$: the class $G\mathscr{M}$ is not defined and generally

$$\boxed{\mathscr{M}_{\text{loc}} \subseteq \mathscr{M}T} \tag{13}$$

(see below).

4. Investment Portfolio on (B, S)-market. Discrete time. We assume that the securities market under consideration operates in the conditions of "uncertainty" that can be described in the probabilistic framework in terms of a filtered probability space

$$\big(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n \geqslant 0}, \mathsf{P}\big).$$

We shall assume that a vector of prices $X = (X^0, X^1, \dots, X^d)$ has the following (B, S)-form:

$$X^0 = B,$$

where $B = (B) \geqslant 0$ is a *bank account* (a "risk-free" asset) and

$$(X^1, \dots, X^d) = (S^1, \dots, S^d)$$

where $S = (S^1, \dots, S^d)$ is a d-dimensional *stock* ("risk" assets), $S^i = (S^i) \geqslant 0$.

DEFINITION 1. A (predictable) stochastic sequence $\pi = (\beta, \gamma)$ where $\beta = (\beta_n)_{n \geqslant 0}$ and $\gamma = (\gamma_n^1, \dots, \gamma_n^d)_{n \geqslant 0}$ with $\mathscr{F}_{n-1}$-measurable β_n and γ_n^i for all $n \geqslant 0$ ($\mathscr{F}_{-1} = \mathscr{F}_0$) and $i = 1, \dots, d$ is called an *investment portfolio* on (B, S)-market.

If $d = 1$ then we shall write γ_n and S_n in place of γ_n^1 and S_n^1.

We want now emphasize several important points.

The variables $\beta_n = \beta_n(\omega)$ and $\gamma_n^i = \gamma_n^i(\omega)$ can be positive, equal to zero, or even negative which means that the investor can *borrow* from the bank account or *sell* stock *short*.

The assumption of $\mathscr{F}_{n-1}$-measurability means that the variables $\beta_n(\omega)$ and $\gamma_n^i(\omega)$ describing the financial position of the investor at time n (the amount he has on the bank account, the stock in his possession) are determined by the information available at time $n-1$, not n (the "tomorrow" position of β_n, γ_n^i is completely defined by the "today" situation).

To emphasize the dynamics of an investment portfolio one often uses the term "investment strategy" instead.

DEFINITION 2. The *value* of an investment portfolio π acting on the (B, S)-market is the stochastic sequence

$$X^\pi = (X_n^\pi)_{n \geqslant 0}$$

where

$$X_n^\pi = \beta_n B_n + \sum_{i=1}^d \gamma_n^i S_n^i. \tag{14}$$

To avoid lengthy formulas we shall use "coordinate-free" notation in what follows, denoting the scalar product

$$(\gamma_n, S_n) = \sum_{i=1}^{d} \gamma_n^i S_n^i$$

by $\gamma_n S_n$ and writting

$$X_n^\pi = \pi_n X_n \tag{15}$$

instead of

$$X_n^\pi = \beta_n B_n + \gamma_n S_n. \tag{16}$$

For two arbitrary sequences $a = (a_n)_{n \geqslant 0}$ and $b = (b_n)_{n \geqslant 0}$ we have for $n \geqslant 1$

$$\Delta(a_n b_n) = a_n \Delta b_n + b_{n-1} \Delta a_n$$

where $\Delta x_n = x_n - x_{n-1}$.

Applying this formula to the right-hand side of (16) we see that

$$\Delta X_n^\pi = (\beta_n \Delta B_n + \gamma_n \Delta S_n) + (B_{n-1} \Delta \beta_n + S_{n-1} \Delta \gamma_n). \tag{17}$$

This shows that changes ($\Delta X_n^\pi = X_n^\pi - X_{n-1}^\pi$) in the value of a portfolio are, in general, sums of two components: changes of the state of a unit bank account and of stock prices ($\beta_n \Delta B_n + \gamma_n \Delta S_n$) and changes in the portfolio composition ($B_{n-1} \Delta \beta_n + S_{n-1} \Delta \gamma_n$).

It is reasonable to assume that *real* changes of this value are due to the increments ΔB_n and ΔS_n (and not to $\Delta \beta_n$ and $\Delta \gamma_n$).

Thus, we conclude that the *capital gains* on the investment portfolio π are described by the sequence $G^\pi = (G_n^\pi)_{n \geqslant 0}$ where $G_0^\pi = 0$ and

$$G_n^\pi = \sum_{k=1}^{n} (\beta_k \Delta B_k + \gamma_k \Delta S_k). \tag{18}$$

Hence the *value* of the portfolio at time n ("*capital*" at time n) is

$$X_n^\pi = X_0^\pi + G_n^\pi$$

or

$$X_n^\pi = X_0^\pi + \sum_{k=1}^{n} (\beta_k \Delta B_k + \gamma_k \Delta S_k) \tag{19}$$

which brings as in a natural way to the following

Definition 3. We say that an investment portfolio π is *self-financing* ($\pi \in SF$) if its value $X^\pi = (X_n^\pi)_{n\geqslant 0}$ can be represented as (19).

That is, self-financing here is equivalent to the following condition describing "admissible" portfolios π:

$$B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n = 0. \tag{20}$$

It is perfectly clear that in the operations with the portfolio π one would better reduce the number of different assets in it or, at least, simplify their structure. In accordance with one possible approach, "if we a priori have $B_n > 0$, $n \geqslant 1$, then we can set $B_n \equiv 1$". This is a consequence of the following observation.

Along with the (B,S)-market we can consider a new market $(\widetilde{B},\widetilde{S})$ where

$$\widetilde{B} = (\widetilde{B}_n)_{n\geqslant 0} \quad \text{with} \quad \widetilde{B}_n = 1$$

and

$$\widetilde{S} = (\widetilde{S}_n)_{n\geqslant 0} \quad \text{with} \quad \widetilde{S}_n = \frac{S_n}{B_n}\,.$$

Then the value $\widetilde{X}^\pi = (\widetilde{X}_n^\pi)_{n\geqslant 0}$ of the portfolio $\pi = (\beta,\gamma)$ is as follows:

$$\widetilde{X}_n^\pi = \beta_n\widetilde{B}_n + \gamma_n\widetilde{S}_n = \beta_n + \gamma_n\widetilde{S}_n = \frac{1}{B_n}(\beta_n B_n + \gamma_n S_n) = \frac{X_n^\pi}{B_n}\,. \tag{21}$$

In addition, if π is self-financing in the (B,S)-market, then it has this property also on the $(\widetilde{B},\widetilde{S})$-market:

$$\widetilde{B}_{n-1}\Delta\beta_n + \widetilde{S}_{n-1}\Delta\gamma_n = \frac{1}{B_{n-1}}(B_{n-1}\Delta\beta_n + S_{n-1}\Delta\gamma_n) = 0.$$

Since $\Delta\widetilde{B}_n \equiv 0$, it follows by (19) that

$$\widetilde{X}_n^\pi = \widetilde{X}_0^\pi + \sum_{k=1}^n \gamma_k \Delta S_k \tag{22}$$

for $\pi \in SF$, or, more explicity,

$$\widetilde{X}_n^\pi = \widetilde{X}_0^\pi + \sum_{k=1}^n \Bigl[\sum_{i=1}^d \gamma_k^i \Delta\widetilde{S}_k^i\Bigr], \qquad \widetilde{S}_k^i = \frac{S_k^i}{B_k}\,.$$

Thus, from (21) and (22) we see that the discounted value $\dfrac{X^\pi}{B} = \left(\dfrac{X_n^\pi}{B_n}\right)_{n\geqslant 0}$ of $\pi \in SF$ satisfies the relation

$$\boxed{\Delta\left(\frac{X_n^\pi}{B_n}\right) = \gamma_n\Delta\left(\frac{S_n}{B_n}\right)} \tag{23}$$

which, for all its simplicity, plays a key role in many calculations based on the concept of "arbitrage-free market". Note that (23) is in a certain sense more consistent from the financial point of view than the equality

$$\Delta X_n^\pi = \beta_n \Delta B_n + \gamma_n \Delta S_n. \tag{24}$$

Indeed, comparing prices one is more interested in their *relative* values, than in the absolute ones. This explain why in the sequel we consider (for the case of continuous time too) the discounted variable $\widetilde{B} = \dfrac{B}{B} \equiv 1$ and $\widetilde{S} = \left(\dfrac{S}{B}\right)$ in place of B and S. For simplicity let's assume now that for our initial (B,S)-market $B \equiv 1$ and so in this case $(\widetilde{B}, \widetilde{S}) = (B,S)$.

The above-discussed evolution of the capital X^π in a (B,S)-market relates to the case when there are no "inflows or outflows" of funds and the "transaction costs" are negligible. Of course, we can think of others schemes, where the change of capital ΔX_n^π does not proceed in accordance with (24), but has a more complicated form, where shareholder dividents, consumption, transaction costs, etc. are taken into account. (See, for example, [ESF; Chapter V, § 1a]).

5. Arbitrage and Absence of Arbitrage. Discrete time. In a few words, the "absence of arbitrage" on a market means that this market is "fair", "rational", and one can make no "riskless" profit there.

For formal definitions we shall assume that we have a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\geqslant 0}, \mathsf{P})$ and a (B,S)-market on this space formed by $d+1$ assets:

$$\text{a bank account } B = (B_n)_{n\geqslant 0}$$

with $\mathscr{F}_{n-1}$-measurable B_n, $B_n > 0$ and

$$\text{a } d\text{-dimensional risk asset } S = (S^1, \dots, S^d)$$

where $S^i = (S_n^i)_{n\geqslant 0}$ and S_n^i are positive and $\mathscr{F}_n$-measurable.

We fix now some $N \geqslant 1$ and we are interested in the value X_N^π of one or another self-financing strategy $\pi \in SF$ at this "terminal" instant N.

DEFINITION 4. We say that a self-financing strategy π brings about an opportunity for arbitrage (at time N) if, for starting capital

$$X_0^\pi = 0 \tag{25}$$

we have

$$X_N^\pi \geqslant 0 \quad (\mathsf{P}\text{-a.s.}) \tag{26}$$

and $X_N^\pi > 0$ with positive P-probability i.e.

$$\mathsf{P}(X_N^\pi > 0) > 0. \tag{27}$$

Let SF_{arb} be the class of self-financing strategies with opportunities for arbitrage.

If $\pi \in SF_{\text{arb}}$ and $X_0^\pi = 0$, then

$$\mathsf{P}(X_N^\pi \geqslant 0) = 1 \implies \mathsf{P}(X_N^\pi > 0) > 0. \tag{28}$$

DEFINITION 5. We say that there exist no opportunities for arbitrage on a (B,S)-market or that the market is arbitrage-free if $SF_{\mathrm{arb}} = \varnothing$. In other words, if the starting capital X_0^π of a strategy π is zero, then

$$\mathsf{P}(X_N^\pi \geqslant 0) = 1 \implies \mathsf{P}(X_N^\pi = 0) = 1. \tag{29}$$

The following remarkable result, which, due to its importance, is called the *First fundamental arbitrage (or asset pricing) theorem.*

THEOREM A. *Assume that*

$$(B,S)\text{-}market$$

on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_n)_{n\geqslant N}, \mathsf{P})$ *is formed by a bank account* $B = (B_n)_{n\geqslant N}$, $B_n > 0$, *and finitely many assets* $S = (S^1, \dots, S^d)$, $S^i = (S_n^i)$.

Assume also that this market operates at the instants $n = 0, 1, \dots, N < \infty$, $\mathscr{F}_0 = \{\varnothing, \Omega\}$ *and* $\mathscr{F}_N = \mathscr{F}$.

Then this (B,S)-market is arbitrage-free if and only if there exists (at least one) measure $\widetilde{\mathsf{P}}$ (a "martingale" measure) equivalent to the measure P such that the d-dimensional discounted sequence

$$\frac{S}{B} = \left(\frac{S_n}{B_n}\right)_{n\leqslant N}$$

is a $\widetilde{\mathsf{P}}$-martingale, i.e.

$$\mathsf{E}_{\widetilde{\mathsf{P}}}\left|\frac{S_n^i}{B_n}\right| < \infty$$

for all $i = 1, \dots, d$ and $n = 0, 1, \dots, N$ and

$$\mathsf{E}_{\widetilde{\mathsf{P}}}\left(\frac{S_n^i}{B_n} \,\Big|\, \mathscr{F}_{n-1}\right) = \frac{S_{n-1}^i}{B_{n-1}} \quad (\widetilde{\mathsf{P}}\text{-}a.s.)$$

for $n = 1, \dots, N$.

Let $M(\mathsf{P})$ and $M_{\mathrm{loc}}(\mathsf{P})$ be the sets of all probability measures $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that the discounted prices $\dfrac{S}{B} = \left(\dfrac{S_n}{B_n}\right)_{n\leqslant N}$ are martingales and local martingales, respectively, relative to these measures.

We can formulate Theorem A as follows: *the conditions*

(i) *a (B,S)-market is arbitrage-free*

and

(ii) *the set of martingale measures $M(\mathsf{P})$ is not empty $(M(\mathsf{P}) \neq \varnothing)$*

are equivalent.

Theorem A* below is a natural generalization of this version of the Theorem A: it provides several useful equivalent characterizations of an arbitrage-free market and clears up the structure of the set of martingale measures.

First of all we need some new notation.

Let $\mathsf{Q} = \mathsf{Q}(dx)$ be a probability measure in $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ and let

$K(\mathsf{Q})$ be the *topological support* of Q (the smallest closed set carrying Q);
$L(\mathsf{Q})$ be the *closed convex hull* of $K(\mathsf{Q})$;
$H(\mathsf{Q})$ be the *smallest affine hyperplane* containing $K(\mathsf{Q})$;
$L^0(\mathsf{Q})$ be the *relative interior* of $L(\mathsf{Q})$ (in the topology of the hyperplane $H(\mathsf{Q})$).

Let $\mathsf{Q}_n(\omega, \cdot)$ be the regular conditional distributions

$$\mathsf{P}\left(\Delta\left(\frac{S_n}{B_n}\right) \in \cdot \;\middle|\; \mathscr{F}_{n-1}\right), \qquad 1 \leqslant n \leqslant N.$$

THEOREM A* (an extended version of the First fundamental theorem). *Assume that the conditions of Theorem* A *are satisfied. Then the following assertions are equivalent*:

a) *a (B,S)-market is arbitrage-free*;
b) $M(\mathsf{P}) \neq \varnothing$;
c) $M_{\mathrm{loc}}(\mathsf{P}) \neq \varnothing$;
d) $0 \in L^0(\mathsf{Q}_n(\omega, \cdot))$ *for each* $n \in \{1, 2, \dots, N\}$ *and* P*-almost all* $\omega \in \Omega$.

We have already mentioned that the assumption of the absence of arbitrage has a clear economic meaning; this is a desirable property of a market to be "rational", "efficient", "fair". The value of the Theorem A (which is due to J.M. Harrison, D.M. Kreps, S.R. Pliska in the case of finite Ω and to R.C. Dalang, A. Morton and W. Willinger for arbitrary Ω) is that it shows a way to analytic calculations relevant to transactions of financial assets in these "arbitrage-free markets". (This is why it is called the *First fundamental asset pricing theorem.*) See more details and proofs in [ESF; Chapter V].

6. Investment Portfolio on (B,S)-market. Continuous time. We consider now a financial (B,S)-market of $d+1$ assets $(B,S) = (B, S^1, \dots, S^d)$ that operates in uncertain conditions of the stochastic character described by a filtered probability space (stochastic basis) $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathsf{P})$ where $(\mathscr{F}_t)_{t\geqslant 0}$ is the flow of incoming "information".

Our main assumption about the assets $S = (S^1, \dots, S^d)$ is that they are positive semimartingales. As it was explained before we may assume that a bank account $B = (B_t)_{t\geqslant 0}$ is a such that $B_t \equiv 1$.

By analogy with the discrete-time case we call a *portfolio* an arbitrary predictable $(d+1)$-dimensional process $\pi = (\beta, \gamma^1, \dots, \gamma^d)$ where $\beta = (\beta_t)_{t\geqslant 0}$ and $\gamma^i = (\gamma^i_t)_{t\geqslant 0}$, $i = 1, \dots, d$, and introduce the value (capital) or the value process $X^\pi = (X^\pi_t)_{t\geqslant 0}$ where

$$X^\pi_t = \beta_t B_t + \sum_{i=1}^d \gamma^i_t S^i_t. \tag{30}$$

REMARK. Let $\mathscr{P}$ be the smallest σ-algebra in the space $\mathbb{R}_+ \times \Omega$ such that if a measurable function $Y = (Y(t,\omega))_{t\geqslant 0,\,\omega\in\Omega}$ is $\mathscr{F}_t$-measurable for each $t \geqslant 0$ and its t-trajectories (for each fixed $\omega \in \Omega$) are left-continuous then the map $(t,\omega) \rightsquigarrow Y(t,\omega)$ generated by this function is $\mathscr{P}$-measurable. We call the σ-algebra $\mathscr{P}$ in $\mathbb{R}_t \times \Omega$ the *σ-algebra of predictable sets* and we say that a stochastic process $X = (X_t(\omega))_{t\geqslant 0}$ defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathsf{P})$ is *predictable* if the map $(t,\omega) \rightsquigarrow X(t,\omega)$ $(= X_t(\omega))$ is $\mathscr{P}$-measurable.

We recall that in the discrete-time case we say that a portfolio $\pi = (\beta,\gamma)$, $\gamma = (\gamma^1,\dots,\gamma^d)$ on (B,S)-market is *self-financing* ($\pi \in SF$) if for each $n \geqslant 1$ we have

$$X_n^\pi = X_0^\pi + \sum_{k=1}^n (\beta_k \Delta B_k + \gamma_k \Delta S_k) \tag{31}$$

or, if $B_k \equiv 1$,

$$X_n^\pi = X_0^\pi + \sum_{k=1}^n \sum_{i=1}^d \gamma_k^i \Delta S_k^i = X_0^\pi + \sum_{i=1}^d \sum_{k=1}^n \gamma_k^i \Delta S_k^i. \tag{32}$$

In the same way, a reasonable definition of a self-financing portfolio or self-financing strategy π (we write here also $\pi \in SF$) in the continuous-time case could be the equality

$$X_t^\pi = X_0^\pi + \sum_{i=1}^d \int_0^t \gamma_u^i \, dS_u^i \tag{33}$$

where $(\gamma^i \cdot S^i) = \int_0^t \gamma_u^t \, dS_u^i$ is the *stochastic integral* of the process $\gamma^i = (\gamma_u^i)_{u\geqslant 0}$ with respect to the semimartingale $S^i = (S_u^i)_{u\geqslant 0}$.

It turn out, however, that definition (33) of the self-financing portfolio π for the general theory of arbitrage is *not sufficient*. What's the matter?

Let's write (32) in the following scalar-product form ($B_k \equiv 1$)

$$X_n^\pi = X_0^\pi + \sum_{k=1}^n (\gamma_k, \Delta S_k). \tag{34}$$

Then for the *continuous-time* case it is natural to have the expression of the type

$$X_t^\pi = X_0^\pi + \int_0^t (\gamma_u, dS_u) \tag{35}$$

where $\int_0^t (\gamma_u, dS_u)$ is a "stochastic integral of the vector-process $\gamma = (\gamma_u)_{u\geqslant 0}$ with respect to the vector-semimartingale $S = (S_u)_{u\geqslant 0}$". Which should be defined as a limit of some integrals of "simple" processes $\gamma(n) = (\gamma_u(n))_{u\geqslant 0}$ approximating $\gamma = (\gamma_u)_{u\geqslant 0}$ in some suitable sense. It is clear that in contrast to the "component-wise" definition (33), which does not take into account the possible "interference" of the semimartingales involved, our proposed approach, in principle, can extend the class of vector-valued processes $\gamma = (\gamma^1, \dots, \gamma^d)$ because it takes into account this inference when we defined $\int_0^t (\gamma_u, dS_u)$ through limit procedure (in L^2-sense, say) from

$$\int_0^t (\gamma_u(n), dS_u) \equiv \sum_{i=1}^d \int_0^t \gamma_u^i \, dS_u^i.$$

In order to get a feeling how to define the stochastic integral $\gamma \cdot S = \int_0^\cdot (\gamma_u, dS_u)$ let's assume that S is a vector-valued martingale $M = (M^1, \dots, M^d)$.

Let $[M^i, M^j]$ be the *quadratic covariance* of M^i and M^j:

$$[M^i, M^j]_t = M_t^i M_t^j - M_0^i M_0^j - \int_0^t M_{u-}^i \, dM_u^j - \int_0^t M_{u-}^j \, dM_u^i. \tag{36}$$

Clearly, we can find a nondecreasing adapted (to the flow $\mathbb{F} = (\mathscr{F}_t)_{t \geqslant 0}$) process $C = (C_t)_{t \geqslant 0}$ and adapted processes $m^{ij} = (m_t^{ij})_{t \geqslant 0}$ such that

$$[M^i, M^j]_t = \int_0^t m_u^{ij} \, dC_u. \tag{37}$$

Suppose that the norm

$$\|\gamma\| \equiv \mathsf{E}\left[\int_0^\infty \left(\sum_{i,j=1}^d \gamma_u^i m_u^{ij} \gamma_u^j\right) dC_u\right]^{1/2} < \infty. \tag{38}$$

Then we way find a sequence $\gamma(n) = (\gamma^1(n), \dots, \gamma^d(n))$ of the predictable processes such that $\|\gamma(n) - \gamma\| \to 0$, $n \to \infty$.

It is well known that for any local martingale $X = (X_t)_{t \geqslant 0} \in \mathscr{H}^1$ $\Big($i.e. $\mathsf{E} \sup\limits_{u \geqslant 0} |X_u| < \infty\Big)$ with $X_0 = 0$ one may find constants c_1 and c_2 such that

$$c_1 \mathsf{E}[X, X]_\infty^{1/2} \leqslant \mathsf{E} \sup_{u \geqslant 0} |X_u| \leqslant c_2 \mathsf{E}[X, X]_\infty^{1/2}. \tag{39}$$

From (38) it follows that $\|\gamma(n) - \gamma(m)\| \to 0$, $n, m \to \infty$. So, if

$$\gamma(n) \cdot S = \int_0^\cdot (\gamma_u(n), dS_u) \overset{\text{def}}{=} \sum_{i=1}^d \int_0^\cdot \gamma_u^i(n) \, dS_u^i$$

then from (39)

$$\mathsf{E} \sup_{u \geqslant 0} |\gamma(n) \cdot S - \gamma(m) \cdot S| \to 0, \qquad n, m \to \infty. \tag{40}$$

So, the sequence $(\gamma(n) \cdot S)_{n \geqslant 1}$ is fundamental in $\mathscr{H}^1$ and if $\|\gamma\| < \infty$ then there exists a process, denoted by $\gamma \cdot S$ which is $\mathscr{H}^1$-limit of the processes $\gamma(n) \cdot S$, $n \geqslant 1$.

In these considerations we assumed that M is a vector-valued *martingale*. By localization we may extend the given definition of the stochastic integrals $\gamma \cdot M$ (for processes γ with condition $\|\gamma\| < \infty$) to the case then M is a vector-valued *local martingale* ($M \in \mathscr{M}^d_{\text{loc}}$) and γ is a predictable process such that

$$\mathsf{E}\left[\int_0^{\tau_k} \sum_{i,j=1}^d (\gamma^i_u m^{ij}_u \gamma^j_u)\, dC_u\right]^{1/2} < \infty \tag{41}$$

where (τ_k) is a localizing sequence of the stopping times for M. (In this case we write that $\gamma \in L^1_{\text{loc}}(M)$.)

To get a definition of the stochastic integral $\gamma \cdot S$ for a semimartingale

$$S = S_0 + A + M \tag{42}$$

we need now, after giving the definition for $\gamma \cdot M$, to define the stochastic integral $\gamma \cdot A$ for a process $A = (A_t)_{t \geqslant 0}$ of finite variation ($A \in \mathscr{V}^d$).

There exist a nondecreasing adapted process $C = (C_u)_{u \geqslant 0}$ and adapted processes $a^i = (a^i_u)_{u \geqslant 0}$, $i = 1, \dots, d$, such that for any $t \geqslant 0$

$$A^i_t = \int_0^t a^i_u\, dC_u. \tag{43}$$

We define $L_{\text{var}}(A)$ as the space of predictable processes $\gamma = (\gamma^1, \dots, \gamma^d)$ such that for any $t \geqslant 0$

$$\int_0^t \left|\sum_{i=1}^d \gamma^i_u a^i_u\right| dC_u < \infty \quad (\mathsf{P}\text{-a.s.}). \tag{44}$$

Obviously, $L_{\text{var}}(A)$ does not depend on the choice of C and a^i.

For $\gamma \in L_{\text{var}}(A)$ the stochastic integral $(\gamma \cdot A)_t$ is defined by

$$(\gamma \cdot A)_t = \int_0^t \left(\sum_{i=1}^d \gamma^i_u a^i_u\right) dC_u \tag{45}$$

where integral $\int_0^t (\,\cdot\,)\, dC_u$ is the Lebesgue–Stieltjes integral.

We have so far defined two stochastic integrals: $\gamma \cdot M$ and $\gamma \cdot A$ for a local martingale $M = (M^1, \dots, M^d)$ and a process of bounded variation $A = (A^1, \dots, A^d)$. At the moment, it is not clear that these two definition coincide for a process that belongs both to $\mathscr{M}^d_{\text{loc}}$ and $\mathscr{V}^d$. Therefore, in order to avoid ambiguities, we will write $(LM)\gamma \cdot M$ for the integral with respect to a local martingale $M \in \mathscr{M}^d_{\text{loc}}$ and $(LS)\gamma \cdot A$ for the Lebesgue–Stieltjes integral defined above.

Now assume that $S = (S^1, \dots, S^d)$ is a semimartingale. We say that a predictable process $\gamma = (\gamma^1, \dots, \gamma^d)$ is S-integrable ($\gamma \in L(S)$) if there exists a decomposition $S = S_0 + A + M$ with $A \in \mathscr{V}^d$, $M \in \mathscr{M}^d_{\text{loc}}$ such that γ satisfies to the both conditions (41) and (44). In this case, the stochastic integral $\gamma \cdot S$ is defined by

$$\gamma \cdot S = (LS)\gamma \cdot A + (LM)\gamma \cdot M. \tag{46}$$

REMARKS. a) The space $L(S)$ includes all the *locally bounded* predictable processes. So, for such processes γ the integral $\gamma \cdot S$ is well-defined.

b) Suppose that $Z \in \mathscr{V}^d \cap \mathscr{M}^d_{\mathrm{loc}}$ and for γ the conditions (41) and (44) hold $(\gamma \in L^1_{\mathrm{loc}}(Z) \cap L_{\mathrm{var}}(Z))$ then $(LS)\gamma \cdot Z = (LM)\gamma \cdot Z$.

7. Properties of the stochastic integral $\gamma \cdot S$**.** In the discrete-time case every process $X = (X_n)_{n \geqslant 0}$ with

$$X_n = X_0 + \sum_{k=1}^{n} \gamma_k \Delta M_k, \tag{47}$$

where $\gamma = (\gamma_k)_{k \geqslant 1}$ is a predictable sequence and $M = (M_k)_{k \geqslant 0}$ is a martingale (ot a local martingale), is, by definition, a martingale transform ($X \in \mathscr{M}T$) and by (11) $X \in \mathscr{M}_{\mathrm{loc}}$, i.e. X is a local martingale (for any predictable finite-valued sequence $\gamma = (\gamma_k)_{k \geqslant 1}$). Does exist the corresponding result for the continuous-time case?

Let's begin with the following definition.

DEFINITION 6. A d-dimensional process X is a martingale transform if there exist $M \in \mathscr{M}^d_{\mathrm{loc}}$ and d-dimensional process $\gamma = (\gamma^1, \dots, \gamma^d)$ such that $\gamma^i \in L(M^i)$ and

$$X^i = X^i_0 + \gamma^i \cdot M^i, \qquad i = 1, \dots, d.$$

The class of d-dimensional (resp: one-dimensional) martingale transforms will be denoted by $\mathscr{M}T^d$ (resp: $\mathscr{M}T$).

The following **example** given by M. Emery shows that in the continuous-time case (even in the one-dimensional setting) the situation is different from the discrete-time case and the class $\mathscr{M}T$ can be *strictly* larger than the class $\mathscr{M}_{\mathrm{loc}}$.

Let σ be an exponentially distributed random variable with $\mathsf{P}(\sigma > t) = e^{-t}$. Let η be independent from σ with $\mathsf{P}(\eta = \pm 1) = \frac{1}{2}$. Set

$$X_t = \begin{cases} 0 & \text{if } t < \sigma, \\ \eta/\sigma & \text{if } t \geqslant \sigma, \end{cases}$$

and $\mathscr{F} = \sigma(\sigma, \eta)$, $\mathscr{F}_t = \mathscr{F}^X_t$.

Then $X \in \mathscr{M}T$ and $X \notin \mathscr{M}_{\mathrm{loc}}$. Indeed, set $\gamma_t = 1/t$,

$$M_t = \begin{cases} 0 & \text{if } t < \sigma, \\ \eta & \text{if } t \geqslant \sigma. \end{cases}$$

Then $M \in \mathscr{M}_{\mathrm{loc}}$, $\gamma \in L_{\mathrm{var}}(M)$ and $\gamma \cdot M = X$. Thus, $X \in \mathscr{M}T$.

Furthemore, for any $(\mathscr{F}_t)$-stopping time T with $\mathsf{P}(T > 0) > 0$ we have $\mathsf{E}|X_T| = \infty$. Therefore, $X \notin \mathscr{M}_{\mathrm{loc}}$.

It is interesting to note that in this example there exists a measure $\widetilde{\mathsf{P}} \sim \mathsf{P}$ such that X is a local $\widetilde{\mathsf{P}}$-martingale (and even a martingale). However, one may construct a martingale transform that is not a local martingale with respect to any $\widetilde{\mathsf{P}} \sim \mathsf{P}$ (F. Delbaen and W. Schachermayer).

The above example shows that a martingale transform need not be a local martingale. But, under some additional conditions, a martingale transform is a local martingale:

a) if $M \in \mathscr{M}^d_{\rm loc}$ and a predictable process γ is locally bounded, then the process X defined as $X^i = \gamma^i \cdot M^i$, $i = 1, \dots, d$, belongs to $\mathscr{M}^d_{\rm loc}$;

b) if $X \in \mathscr{M}T$ and X is bounded from below, then $X \in \mathscr{M}_{\rm loc}$.

The notion of the martingale transform admits various definitions. For example, F. Delbaen and W. Shachermayer introduced the notion of the "sigma-martingale" which in fact coincides with the notion of the martingale transform defined above. The following statement shows that various possible definitions of the martingale transform are the same:

Let $X = (X^1, \dots, X^d)$ is a d-dimensional semimartingale. The following conditions are equivalent:

(a) $X \in \mathscr{M}T^d$;

(b) there exist a d-dimensional martingale $N = (N^1, \dots, N^d)$ with $N^i \in \mathscr{H}^1$ $(i = 1, \dots, d)$ and a strictly positive one-dimensional process K such that $K \in L(N^i)$ for all $i = 1, \dots, d$ and

$$X^i = X^i_0 + K \cdot N^i, \qquad i = 1, \dots, d.$$

8. Investment self-financing portfolio and the "First fundamental theorem" (sufficien conditions). Continuous time. In this section we consider (B, S)-market with the bank account $B = (B_t)_{t \geqslant 0}$ and the stock $S = (S^1, \dots, S^d)$. We shall assume that $B_t \equiv 1$ and S is d-dimensional non-negative semimartingale. By $\pi = (\beta, \gamma)$, $\gamma = (\gamma^1, \dots, \gamma^d)$, we denote a portfolio assuming that

$$\gamma \in L(S)$$

that makes possible to define the stochastic integral $\gamma \cdot S$.

The value of the portfolio $\pi = (\beta, \gamma)$ is defined by the expression

$$X^\pi_t = \beta_t B_t + \gamma_t S_t \quad \left(= \beta_t + \sum_{i=1}^d \gamma^i_t S^i_t \right). \tag{48}$$

We recall that in the discrete-time case a portfolio $\pi = (\beta, \gamma)$ is self-financing $(\pi \in SF)$ if for each $n \geqslant 1$

$$X^\pi_n = X^\pi_0 + \sum_{k=1}^n \gamma_k \Delta S_k \quad \left(= X^\pi_0 + \sum_{k=1}^n \sum_{i=1}^d \gamma^i_k \Delta S^i_k \right). \tag{49}$$

In the same way, a reasonable definition of a *self-financing portfolio* or a self-financing strategy π in the continuous-time case could be the equality

$$X^\pi_t = X^\pi_0 + (\gamma \cdot S)_t \tag{50}$$

where $(\gamma \cdot S)_t$ is the stochastic integral (denoted also as $\int_0^t (\gamma_u, dS_u)$) by an anology with (49) which is well-defined by our permanent assumption that $\gamma \in L(S)$.

We proceed now to main definitions related to the absence of arbitrage (NA – No Arbitrage) in semimartingale models of the (B, S)-market.

The following concept (cf. with the definition in the discrete-time case) is, in effect, classical.

Definition 7. We say that the property NA holds at time T if for each strategy $\pi \in SF \cap L(S)$ with $X_0^\pi = 0$ we have

$$\mathsf{P}(X_T^\pi \geqslant 0) = 1 \implies \mathsf{P}(X_T^\pi = 0) = 1. \tag{51}$$

For the case of the discrete time we know that the property NA is equivalent to the property of existence a martingale measure. However, for the continuous time the situation is more complicated and first of all by purely technical reasons related with the fact that stochastic integrals $\gamma \cdot S$ one may defined only for a special class of the predictable processes γ, namely for $\gamma \in L(S)$. For the case of the continuous time the property $\mathscr{M}_{\text{loc}} = \mathscr{M}T$, which has been used in the proof of the "First fundamental theorem" in the discrete-time case, already does *not* hold (in fact, $\mathscr{M}_{\text{loc}} \subseteq \mathscr{M}T$) that makes the corresponding formulations much more complicated.

Let's introduce several classes of admissible strategies; their role will be completely revealed by our discussions of "martingale criteria" of the absence of the opportunities for arbitrage.

Definition 8. For each $a \geqslant 0$ we set

$$SF_a = \{\pi \in SF : X_t^\pi \geqslant -a,\ t \in [0,T]\},$$
$$SF_+ = \{\pi \in SF : X_t^\pi \geqslant -a \text{ for some } a \geqslant 0,\ t \in [0,T]\} \quad \Bigl(= \bigcup_{a \geqslant 0} SF_a\Bigr).$$

The meaning of "a-admissibility $X_t^\pi \geqslant -a, t \in T$" is perfectly clear: the quantity $a \geqslant 0$ is a bound (resulting from economic considerations) on the possible losses in the process of the implementation of the strategy π.

We shall denote SF^0 the subclass of the strategies π from SF for which $X_0^\pi = 0$. The corresponding notion SF_a^0 and SF_+^0 we use for strategies π from SF_a and SF_+ satisfying to the property $X_0^\pi = 0$.

For a discussion of various issues of arbitrage in semimartingale models it can be useful to introduce several classes of $\mathscr{F}_T$-measurable "test" pay-off functions $\psi_T = \psi_T(\omega)$ ("subgains") that can be *majorized* by the returns ("gains")

$$(\gamma \cdot S)_T = \int_0^T \gamma_u \, dS_u$$

of a strategy $\pi = (\beta, \gamma) \in L(S)$ in one of the classes of admissible strategies SF_a and SF_+.

Definition 9. For $a \geqslant 0$ let

$$\Psi_a^0(S) = \{\psi_T \in L_\infty(\Omega, \mathscr{F}_T, \mathsf{P}) : \psi_T \leqslant (\gamma \cdot S)_T \text{ for some } \pi = (\beta,\gamma) \in SF_a^0 \cap L(S)\}$$

and also

$$\Psi_+^0(S) = \{\psi_T \in L_\infty(\Omega, \mathscr{F}_T, \mathsf{P}) : \psi_T \leqslant (\gamma \cdot S)_T \text{ for some } \pi = (\beta,\gamma) \in SF_+^0 \cap L(S)\}$$
$$\Bigl(= \bigcup_{a \geqslant 0} \Psi_a^0(S)\Bigr).$$

DEFINITION 10. We say that the property NA_a holds for some $a \geqslant 0$ if for $\psi_T \in \Psi_a^0(S)$

$$\mathsf{P}(\psi_T \geqslant 0) = 1 \implies \mathsf{P}(\psi_T = 0) = 1 \tag{52}$$

(compare with (51)), or, equivalently,

$$\Psi_a^0(S) \cap L_\infty^+(\Omega, \mathscr{F}_T, \mathsf{P}) = \{0\}$$

where $L_\infty^+(\Omega, \mathscr{F}_T, \mathsf{P})$ is the subset of nonnegative random variable in

$$L_\infty(\Omega, \mathscr{F}_T, \mathsf{P}) = \Big\{\psi = \psi(\omega) : \psi - \mathscr{F}_T\text{-measurable and } \|\psi\| < \infty \text{ where } \|\psi\|_\infty \equiv \operatorname*{ess\,sup}_\omega |\psi(\omega)| = \inf\{C < \infty : \mathsf{P}(|\psi| > C) = 0\}\Big\}.$$

We say also that the property NA_+ holds if for $\psi_T \in \Psi_+^0(S)$ we have the implication (52).

The next theorem, which describes some *sufficient* conditions of the absence of arbitrage, is probably the most useful result of the "Arbitrage Theory" in semimartingales models from the standpoint of financial mathematics and engineering.

THEOREM 1 (a sufficient condition of the "First fundamental theorem"). *Suppose that $\widetilde{\mathsf{P}}$ is a probability measure on $(\Omega, \mathscr{F}_T)$ such that $\widetilde{\mathsf{P}} \sim \mathsf{P}$ and $S = (S^1, \dots, S^d)$ is a local martingale ($S \in \mathscr{M}_{\mathrm{loc}}^d(\widetilde{\mathsf{P}})$). In other words let's assume that we have the property* ELMM (*Existence of a Local Martingale Measure*).

Then

$$\begin{aligned} \mathrm{ELMM} &\implies \mathrm{NA}_a, \quad a \geqslant 0, \\ \mathrm{ELMM} &\implies \mathrm{NA}_+. \end{aligned} \tag{53}$$

PROOF. Let π be a strategy from $SF_a^0 \cap L(S)$. By the property b) from Section 7

$$\gamma \cdot S \in \mathscr{M}_{\mathrm{loc}}(\widetilde{\mathsf{P}}). \tag{54}$$

Because $(\gamma \cdot S)_t \geqslant -a$, $t \leqslant T$, the process

$$\gamma \cdot S = (\gamma \cdot S)_{t \leqslant T}$$

is a $\widetilde{\mathsf{P}}$-supermartingale with $(\gamma \cdot S)_0 = 0$. So,

$$\widetilde{\mathsf{E}}(\gamma \cdot S)_T \leqslant \widetilde{\mathsf{E}}(\gamma \cdot S)_0 = 0, \tag{55}$$

where $\widetilde{\mathsf{E}}$ is expectation with respect to the measure $\widetilde{\mathsf{P}}$. If we assume that $X_T^\pi = (\gamma \cdot S)_T \geqslant 0$ (P-a.s.) then because $\widetilde{\mathsf{P}} \sim \mathsf{P}$ we have

$$0 \leqslant \widetilde{\mathsf{E}}(\gamma \cdot S)_T.$$

So, $(\gamma \cdot S)_T = 0$ ($\widetilde{\mathsf{P}}$- and P-a.s.).

Now assume that $\psi_T \in \Psi_a^0(S)$. Then

$$\psi_T \leqslant (\gamma \cdot S)_T \tag{56}$$

for some $\pi \in SF_a^0 \cap L(S)$.

By (56)

$$\widetilde{\mathsf{E}}\psi_T \leqslant \widetilde{\mathsf{E}}(\gamma \cdot S)_T \leqslant 0. \tag{57}$$

So, if $\mathsf{P}(\psi_T \geqslant 0) = 1$ then $\widetilde{\mathsf{P}}(\psi_T \geqslant 0) = 1$ and (57) implies that $\psi_T = 0$ ($\widetilde{\mathsf{P}}$- and P-a.s.). In other words, ELMM $\Rightarrow$ NA_a and, as a corollary, we get ELMM $\Rightarrow$ NA_+.

This last statement "ELMM $\Rightarrow$ NA_+" can be a little bit extended by the following way. The space $L_\infty(\Omega, \mathscr{F}_T, \mathsf{P})$ is a Banach space with respect to the norm $\|\psi\|_\infty = \operatorname*{ess\,sup}_\omega |\psi(\omega)|$. We shall denote the closure of the spaces $\Psi_a^0(S)$ and $\Psi_+^0(S)$ with respect to the norm $\|\cdot\|_\infty$ by $\overline{\Psi}_a^0(S)$ and $\overline{\Psi}_+^0(S)$.

Definition 11. We say that the property $\overline{\mathrm{NA}_+}$ holds if

$$\overline{\Psi}_+^0(S) \cap L_\infty^+(\Omega, \mathscr{F}_T, \mathsf{P}) = \{0\}. \tag{58}$$

Theorem 2 (one more sufficient condition of the "First fundamental theorem"). *Suppose that condition of Theorem* 1 *hold. Then*

$$ELMM \implies \overline{NA}_+. \tag{59}$$

Proof. Assume that $\psi_T \in \overline{\Psi}_+^0(S)$ and $\psi_T \geqslant 0$. Then there exists sequence of functions $(\psi_T^k)_{k\geqslant 1}$ in $\Psi_+^0(S)$ such that

$$\|\psi_T - \psi_T^k\|_\infty \equiv \operatorname*{ess\,sup}_\omega |\psi_T(\omega) - \psi_T^k(\omega)| \leqslant \frac{1}{k} \to 0, \qquad k \to \infty, \tag{60}$$

and moreover

$$\psi_T^k \leqslant (\gamma^k \cdot S)_T \tag{61}$$

for $\pi^k \in SF_{a_k}^0$ with some $a_k \geqslant 0$. By (60), (61) we obtain

$$-\frac{1}{k} \leqslant (\gamma^k \cdot S)_T. \tag{62}$$

So, using Fatou's lemma, we see that

$$0 \leqslant \widetilde{\mathsf{E}}\psi_T = \widetilde{\mathsf{E}} \lim_k \psi_T^k = \widetilde{\mathsf{E}} \varliminf_k \psi_T^k \leqslant \widetilde{\mathsf{E}} \varliminf_k (\gamma^k \cdot S)_T \leqslant \varliminf_k \widetilde{\mathsf{E}}(\gamma^k \cdot S)_T \leqslant 0 \tag{63}$$

where the last inequality is a consequence of the $\widetilde{\mathsf{P}}$-supermartingale property of the stochastic integrals $(\gamma^k \cdot S)_t$, $t \leqslant T$ (see the property b) in Section 7).

Remark. The property $\overline{\mathrm{NA}}_+$, which is a refinement of NA_+, is consistently used by F. Delbaen and W. Schachermayer, who call it the NFLVR-property: No Free Lunch with Vanishing Risk.

This name can be explained as follows. In the discussion of the absence of arbitrage in its NA_+-version we take for "test" functions ψ only nonnegative functions that are smaller or equal to the "returns" $(\gamma\cdot S)_T$ from the strategy $\pi=(\beta,\gamma)\in SF^0_+\cap L(S)$.

However, in our consideration of the $\overline{\mathrm{NA}}_+$-version of the absece of arbitrage we can take (also nonnegative) "test" functions $\psi_T\in\overline{\Psi}^0_+(S)\cap L^+_\infty(\Omega,\mathscr{F}_T,\mathsf{P})$, e.g., ones that are the *limits* (in the norm $\|\cdot\|_\infty$) of some sequences of elements ψ^k_T, $k\geqslant 1$, of $\Psi^0_t(S)$ which can take, generally speaking, *negative* values (in particular, those can be some integrals $(\gamma^k\cdot S)_T$).

Since $\|\psi^k-\psi\|_\infty\to 0$, $k\to\infty$, we can assume that $\psi^k\geqslant -1/n$ (for all $\omega\in\Omega$) which may be interpreted as *vanishing risk.*

9. Summary of the results on the "First fundamental theorem". For the case of discrete time we say that the "First fundamental theorem" can be formulated, briefly, in the following form

$$\boxed{\mathrm{EMM}\iff\mathrm{ELMM}\iff\mathrm{NA},}\tag{64}$$

where EMM – Existence of a martingale measure and ELMM – Existence of a local martingale measure.

In other words,

$$\begin{aligned}\mathrm{NA}&\iff M(\mathsf{P})\neq\varnothing,\\ \mathrm{NA}&\iff LM(\mathsf{P})\neq\varnothing,\\ \mathrm{NA}&\iff MT(\mathsf{P})\neq\varnothing,\end{aligned}\tag{65}$$

where

$$\begin{aligned}M(\mathsf{P})&=\{\widetilde{\mathsf{P}}\sim\mathsf{P}: S\text{ is a }\widetilde{\mathsf{P}}\text{-martingale}\},\\ LM(\mathsf{P})&=\{\widetilde{\mathsf{P}}\sim\mathsf{P}: S\text{ is a }\widetilde{\mathsf{P}}\text{-local martingale}\},\\ MT(\mathsf{P})&=\{\widetilde{\mathsf{P}}\sim\mathsf{P}: S\text{ is a }\widetilde{\mathsf{P}}\text{-martingale transform}\}.\end{aligned}$$

We have seen already that for the continuous-time case

$$\mathrm{EMM}\implies\mathrm{ELMM}\implies\overline{\mathrm{NA}}_+.\tag{66}$$

These results admit the following improvements (F. Delbaen and W. Schachermayer).

Theorem. 1) *Let $(B,S)=(1,S^1,\dots,S^d)$ be a semimartingale with bounded components. Then*

$$\boxed{\mathrm{EMM}\iff\overline{\mathrm{NA}}_+.}\tag{67}$$

2) *Let* $(B,S)=(1,S^1,\dots,S^d)$ *be a semimartingale with locally bounded components. Then*

$$\boxed{\mathrm{ELMM} \iff \overline{\mathrm{NA}}_+.} \tag{68}$$

3) *In the general semimartingale model* $(B,S)=(1,S^1,\dots,S^d)$

$$\boxed{\mathrm{EMTM} \iff \overline{\mathrm{NA}}_+} \tag{69}$$

where EMTM – *Existence of a martingale transform measure, i.e. there exists a measure* $\widetilde{\mathsf P}\sim\mathsf P$, $\widetilde{\mathsf P}$*-martingale* $N=(N^1,\dots,N^d)$ *and a positive predictable process* $h\in L(N^i)$, $i=1,\dots,d$, *such that* $S^i=S_0^i+h\cdot N^i$ ($\widetilde{\mathsf P}$*-a.s.*).

COROLLARY. *In general semimartingale models of the* $(B,S)=(1,S^1,\dots,S^d)$*-market*

$$\mathrm{EMM} \implies \mathrm{ELMM} \implies \mathrm{EMTM} \iff \overline{\mathrm{NA}}_+.$$

For locally bounded semimartingale case

$$\mathrm{EMM} \implies \mathrm{ELMM} \iff \mathrm{EMTM} \iff \overline{\mathrm{NA}}_+,$$

for bounded semimartingale case

$$\mathrm{EMM} \iff \mathrm{ELMM} \iff \mathrm{EMTM} \iff \overline{\mathrm{NA}}_+.$$

10. The "Second fundamental theorem". Completeness. The results presented below are so important that they may well be called the "Second fundamental arbitrage (or asset pricing) theorem".

Discrete time case. Let $(B,S)=(1,S^1,\dots,S^d)$ is a (B,S)-financial market with $S^i=(S^i_n)_{n\leqslant N}, i=1,\dots,n$, given on the filtered probability space $(\Omega,\mathscr F,(\mathscr F_n)_{n\geqslant N},\mathsf P)$. We assume that $\mathscr F_0$ is $\mathsf P$-trivial and $\mathscr F=\mathscr F_N$.

DEFINITION 12. The model is complete if for each bounded $\mathscr F_N$-measurable function $f=f(\omega)$ there exists a pair (x,H), where $x\in\mathbb R$, $H=(H^1_n,\dots,H^d_n)_{1\leqslant n\leqslant N}$ is a process such that each H^i_n is $\mathscr F_{n-1}$-measurable $(i=1,\dots,d)$ and ($\mathsf P$-a.s.)

$$f=x+\sum_{i=1}^{d}\sum_{k=1}^{N}H^i_k\Delta S^i_k.$$

THEOREM 1 (the "Second fundamental theorem"). *Suppose that the set of the martingale measures* $M(\mathsf P)\neq\varnothing$. *Then the following conditions are equivalent*:

(i) $|M(\mathsf P)|=1$ (*i.e. the set* $M(\mathsf P)$ *contains a single element*);
(ii) $|M_{\mathrm{loc}}(\mathsf P)|=1$;
(iii) *the model is complete.*

Under these conditions $\mathscr F_n=\mathscr F^S_n$, $n\leqslant N$, up to $\mathsf P$-null sets.

REMARK. The proof of the given statement see in [ESF; Chapter V, §§ 4a–4f].

Continuous time case. Let $S = (S^1, \dots, S^d)$ be a semimartingale on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, \mathsf{P})$. We assume that $\mathscr{F}_0$ is P-trivial and $\mathscr{F} = \bigvee_{t\geqslant 0} \mathscr{F}_t$.

DEFINITION 13. The model $(B, S) = (1, S^1, \dots, S^d)$ is *complete* if for each $\mathscr{F}$-measurable bounded function there exists a pair (x, H), where $x \in \mathbb{R}$ and H is a d-dimensional process such that

(i) $H \in L(S)$, i.e. there exists the stochastic integral $H \cdot S$;

(ii) there exists constants a and b such that

$$\mathsf{P}\big(a \leqslant (H \cdot S)_t \leqslant b,\ \forall\, t \geqslant 0\big) = 1;$$

(iii) P-almost surely there exists the limit

$$(H \cdot S)_\infty = \lim_{t\to\infty} (H \cdot S)_t$$

and

$$f = x + (H \cdot S)_\infty \quad (\mathsf{P}\text{-a.s.}).$$

REMARK 1. Condition (ii) may seen a bit unnatural. However, it cannot be eliminated in the Theorem 2 below.

DEFINITION 14. Let $S = (S^1, \dots, S^d)$ be d-dimensional semimartingale. We say that P-local martingale M has the representation property relative to S if there exists S-integrable predictable process $H = (H^1, \dots, H^d)$ such that (P-a.s.)

$$M_t = M_0 + (H \cdot S)_t, \qquad t \geqslant 0.$$

THEOREM 2. *Suppose that $MT(\mathsf{P}) \neq \varnothing$. Then the following conditions are equivalent*:

(i) *the model is complete*;

(ii) $|MT(\mathsf{P})| = 1$;

(iii) *there exists a measure $\widetilde{\mathsf{P}} \in MT(\mathsf{P})$ such that any $\widetilde{\mathsf{P}}$-local martingale M has the representation property relative to S.*

REMARK 2. There are examples of the semimartingales S for which $|MT(\mathsf{P})| = 1$ (and thus, the model is complete), whereas $LM(\mathsf{P}) = \varnothing$ (see Appendix I).