Interest Rate Models:

from Parametric Statistics to Infinite Dimensional Stochastic Analysis

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Chapter 6

Infinite Dimensional Ornstein Uhlenbeck Processes

One Dimensional O-U Processes

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- $\{X_t\}$ has a unique invariant measure, say $\mu = N(0, \Sigma)$ n-dimensional Gaussian measure

$$\mu(dx) = \frac{1}{Z(\Sigma)} e^{-<\Sigma^{-1}x,x>/2} dx,$$

$Z(\Sigma)$ being a normalizing constant

$$\Sigma = \int_0^\infty e^{-sA} B e^{-sA}\, ds.$$

Notice that the fact that Σ solves:

$$\Sigma A + A\Sigma = B$$

is the crucial property of the matrix Σ

- If $X_0 \sim \mu$, $\{X_t\}_{t\geq 0}$ is stationary mean zero (Markov) Gaussian process with autocovariance tensor:

$$\Gamma(s,t) = \mathbb{E}_\mu\{X_s X_t\} = \begin{cases} \Sigma e^{-(t-s)A} & \text{whenever} \quad 0 \leq s \leq t \\ e^{-(s-t)A}\Sigma & \text{whenever} \quad 0 \leq t \leq s. \end{cases}$$

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- If $x, y \in \mathbb{R}^n$ we have:

$$\begin{aligned} < x, \Gamma(s,t) y >&= \mathbb{E}_\mu\{< x, X_s >< y, X_t >\} \\ &=< \sqrt{\Sigma} x, \sqrt{\Sigma} e^{-|t-s|A} y > \\ &=< x, e^{-|t-s|A} y >_\Sigma \end{aligned}$$

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- Generated by the Dirichlet form of the measure μ, i.e. to the quadratic form:

$$Q(f,g) = \int_{\mathbb{R}^n} < \nabla f(x), \overline{\nabla g(x)} >_B \ \mu_\Sigma(dx)$$

defined on the subspace $\mathcal{Q}$ of $L^2(\mathbb{R}^n, \mu_\Sigma(dx))$ comprising the absolutely continuous functions whose first derivatives (in the sense of distributions) are still in the space $L^2(\mathbb{R}^n, \mu_\Sigma(dx))$.

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- The $d \times n$ dispersion matrix B is replaced by a map σ from H or E into the state space F of f_t replacing $\mathbb{R}^n$
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- The role played by $\sqrt{B}dW_t$ is now played by a H-valued (cylindrical) Wiener process with covariance given by the operator B. The appropriate mathematical object is a linear function, say W_B, from the tensor product $L^2([0,\infty),dt)\hat{\otimes}_2 H_B$ into a Gaussian subspace of $L^2(\Omega,\mathcal{F},\mathbb{P})$ where $(\Omega,\mathcal{F},\mathbb{P})$ is the complete probability space we work with. If $x \in H_B$ and $t \geq 0$, then $W_B(\mathbf{1}_{[0,t)}(\cdot)x)$ should play the same role as $< x, W_t >$ in the finite dimensional case.

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- SPDE when A is a partial differential operator
- Integral form

$$f_t = f_0 - \int_0^t Af_s ds + \sigma W(t)$$

- Weak Form: for $f^* \in H^*$

$$< f^*, f_t >=< f^*, f_0 > - \int_0^t < A^* f^*, f_s > ds + < \sigma^* f^*, W(t) >$$

which makes sense when f^* belongs to the domain $\mathcal{D}(A^*)$ of the adjoint operator A^* and the domain $\mathcal{D}(\sigma^*)$ of the adjoint operator σ^*. Very seldom tractable

- Evolution Form (variation of constant formula):

$$f_t = e^{tA} f_0 + \int_0^t e^{(t-s)S} \sigma dW(s)$$

This requires that the exponentials of operators exist, i.e. that A generates a semigroup of operators.

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  - A is the infinitesimal generator of a C_0-semigroup $\{e^{tA}; t \geq 0\}$ of bounded operators on F.
  - The domain of A,

$$\mathcal{D}(A) = \{f \in F;\ \lim_{t \to 0} \frac{e^{tA} f - f}{t} \text{ exists.}\},$$

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E^* & \hookrightarrow & H_W^* & & \\
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- The variance/covariance operator $B : E \to F$ is a bounded linear operator (thus it has a restriction to H_W and this restriction is Hilbert-Schmidt when F is a Hilbert space.)

  Note one could take B from H_W only if we start with a cylindrical Brownian motion.

Formally the Ornstein Uhlenbeck process satisfies the stochastic differential equation

$$dX_t = AX_t dt + BdW_t$$

or in integral form

$$X_t = X_0 + \int_0^t AX_s ds + \int_0^t BdW_s.$$