Interest Rate Models:
from Parametric Statistics to Infinite Dimensional Stochastic Analysis

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Chapter 3

First Stochastic Models of the Term Structure

Factor Models for the Term Structure

- At each time t, the price $P(t,T)$ of a zero coupon bond with maturity T and nominal value $\$1$ is a random variable
- Probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- Maturity date T satisfies $t \leq T < +\infty$.
- Filtration $\{\mathcal{F}_t\}_{t\geq 0}$ defines history (the elements of $\mathcal{F}_t$ are the events prior to time t)
- All processes adapted to this filtration
- The prices $\{P(t,T);\, 0 \leq t \leq T\}$ of the bond with maturity T form a semi-martingale

- In fact for each T, $\{P(t,T);\, 0 \le t \le T\}$ assumed to be an Ito process:

$$dP(t,T) = P(t,T)[\mu^{(P)}(t,T)dt + \sum_{j=1}^{d} \sigma^{(P,j)}(t,T)d\xi_t^{(j)}$$

- For each $T > 0$, the (scalar) processes

$$\{\mu^{(P)}(t,T) : \, 0 \le t \le T\} \quad \text{and} \quad \{\sigma^{(P,j)}(t,T) : \, 0 \le t \le T\}$$

  – are adapted
  – satisfy appropriate integrability conditions

- $\xi_t^{(1)}, \ldots \xi_t^{(d)}$ are d independent $\mathcal{F}_t$-Wiener processes

NB: d Wiener processes driving a continuum of SDE's

(one for each T !!!!!)

- Term structure strongly depends on the state of the economy
- State of the economy at time t given by a (finite dimensional) random vector S_t
- $\{S_t;\ t \geq 0\}$ is the solution of a SDE driven by the Wiener processes $\xi_t^{(1)}, \ldots \xi_t^{(d)}$
- The coefficients of the SDE for P satisfy:

$$\mu^{(P)}(t,T) = \mu^{(P)}(S_t,t,T) \qquad \text{and} \qquad \sigma^{(P,j)}(t,T) = \sigma^{(P,j)}(S_t,t,T)$$

where:

$$(S,t,T) \hookrightarrow \mu^{(P)}(S,t,T) \qquad \text{and} \qquad (S,t,T) \hookrightarrow \sigma^{(P,j)}(S,t,T)$$

are deterministic functions

Short Rate Models

One-factor, the single factor being the short interest rate (overnight rate, 13-week T-Bill rate, $\ldots$.)

- Use standard notation r_t for the single factor S_t
- WALOG assume that we have a single Wiener process W
- We assume $\mathcal{F}_t = \mathcal{F}_t^{(W)}$
- We assume r_t is the solution of a SDE:

$$dr_t = \mu^{(r)}(t, r_t)\, dt + \sigma^{(r)}(t, r_t)\, dW_t$$

$$(t,r) \hookrightarrow \mu^{(r)}(t,r) \quad \text{and} \quad (t,r) \hookrightarrow \sigma^{(r)}(t,r)$$

real valued (deterministic) functions such that existence and uniqueness of a strong solution hold

- $\{r_t\}$ is Markovian
- Not supported by data!!!(Ait-Sahalia)

Pricing

- Price zero coupon riskless bonds to start with
- Treat them as derivatives & use abstract form of Black-Scholes theory
- r_t plays the role of the underlying risky asset
- Money market account $\{B_t;\ t \geq 0\}$ defined by:

$$\begin{cases} dB_t &= r_t B_t \, dt \\ B_0 &= 1 \end{cases}$$

Solution:

$$B_t = e^{\int_0^t r_s \, ds}$$

(still called the *risk free asset*)

- The only tradable asset is the money market account B_t
- Impossible to form portfolios which can replicate interesting contingent claims
- INCOMPLETE MARKET, PLENTY of martingale measures!!
- any equivalent measure $\mathbb{Q} \sim \mathbb{P}$ is an equivalent martingale measure !!
- Brownian filtration + Martingale Representation Theorem

choice of $\mathbb{Q}$ $\iff$ choice of an Ito integrand $\{K_t;\, t \geq 0\}$

(Market Price of Risk)

- Play Girsanov's game, i.e. set

$$\tilde{W}_t = W_t + \int_0^t K_s\, ds$$

$$\tilde{\mu}_t = \mu(t, r_t) - \sigma(t, r_t) K_t, \qquad \text{and} \qquad \tilde{\sigma}_t = \sigma(t, r_t)$$

then:

$$dr_t = \tilde{\mu}_t\, dt + \tilde{\sigma}_t\, dW_t$$

under $\mathbb{Q}$

- The dynamics of r_t under $\mathbb{P}$ are not enough to price the bonds $P(t,T)$ even if we impose the no-arbitrage condition
- For pricing purposes, models of the short interest rate r_t will have to specify either
  - Dynamics under $\mathbb{P}$
  - Market Price of Risk

  or equivalently:
  - Dynamics under $\mathbb{Q}$

CALIBRATION TO MARKET PRICES

Specific Models

All the classical/popular models can be recast in one single form of the risk neutral dynamics:

$$dr_t = (\alpha_t + \beta_t r_t)\,dt + \sigma_t r_t^\gamma\,dW_t$$

where $\gamma > 0$ and

$$t \hookrightarrow \alpha_t \quad \text{and} \quad t \hookrightarrow \beta_t \quad \text{and} \quad t \hookrightarrow \sigma_t$$

are deterministic functions of t

- If $\gamma \neq 0$, hope for r_t to remain positive
- If $\beta_t > 0$, r_t "mean-reverts" to α_t/β_t
- Coefficients not Lipschitz when $0 < \gamma < 1$

The Vasicek Model

- $\alpha_t \equiv \alpha$, $\beta_t \equiv \beta$, $\sigma_t \equiv \sigma$
- $\gamma = 0$

$$dr_t = (\alpha - \beta r_t)\, dt \;+\; \sigma\, dW_t$$

- mean reverting to the level α/β
- σ/β governs the fluctuations
- Gaussian so explicit formulae
- Gaussian so possible negative values
  - happens rarely if the parameters cooperate
  - how about real interest rates?

The Cox-Ingersoll-Ross (CIR) Model

- $\alpha_t \equiv \alpha$, $\beta_t \equiv \beta$, $\sigma_t \equiv \sigma$ as in Vasicek Model
- $\gamma = 1/2$

$$dr_t = (\alpha - \beta r_t)\, dt \, + \, \sigma \sqrt{r_t}\, dW_t$$

- mean reverting to the level α/β
- σ/β governs the fluctuations
- Non-Gaussian process, remains positive for all times

Other Frequently Used Models

- the **Dothan model**
  $\alpha_t \equiv 0$, $\beta_t \equiv -\beta$ and $\sigma_t \equiv \sigma$ constant while $\gamma = 1$.

$$dr_t = \beta r_t \, dt \; + \; \sigma r_t \, dW_t$$

- the **Black-Derman-Toy model**
  $\alpha_t \equiv 0$ and $\gamma = 1$.

$$dr_t = -\beta r_t \, dt \; + \; \sigma_t r_t \, dW_t$$

- the **Ho-Lee model**
  $\beta_t \equiv 0$ and $\sigma_t \equiv \sigma$ constant while $\gamma = 0$.

$$dr_t = \alpha_t \, dt \; + \; \sigma \, dW_t$$

PDE's and Numerical Computations

Assume that in a risk neutral world (i.e. under a given equivalent martingale measure)

$$dr_t = \mu(t, r_t)\, dt + \sigma(t, r_t)\, dW_t$$

Arbitrage pricing paradigm the price at time t of any contingent claim ξ with maturity T is given by

$$V_t = \mathbb{E}^{\mathbb{Q}}\{\xi e^{-\int_t^T r_s ds} | \mathcal{F}_t\}$$

For a zero coupon bond with maturity T:

$$P(t, T) = \mathbb{E}^{\mathbb{Q}}\{e^{-\int_t^T r_s ds} | \mathcal{F}_t\}$$

r_t is Markovian

$$\begin{aligned} V_t &= \mathbb{E}^{\mathbb{Q}}\{f(r_T) e^{-\int_t^T r_s ds} | r_s,\ 0 \le s \le t\} \\ &= \mathbb{E}^{\mathbb{Q}}\{f(r_T) e^{-\int_t^T r_s ds} | r_t\} \end{aligned}$$

so V_t is a deterministic function of t and r_t. Set:

$$F_r(t,T) = \mathbb{E}^{\mathbb{Q}}\{f(r_T)e^{-\int_t^T r_s ds}|r_t = r\}$$

then $V_t = F_{r_t}(t,T)$. Feynman-Kac formula

> The no-arbitrage price at time t of any contingent claim ξ of the form $\xi = f(r_T)$ with maturity $T > t$ is of the form $F(t,r_t)$ where F is a solution of the parabolic equation:
>
> $$\frac{\partial F}{\partial t}(t,r)+\mu^{(r)}(t,r)\frac{\partial F}{\partial r}(t,r)+\frac{1}{2}\sigma^{(r)}(t,r)^2\frac{\partial^2 F}{\partial r}(t,r)-rF(t,r) = 0$$
>
> with the terminal condition $F(T,r) \equiv f(r)$.

Explicit Solutions: Vasicek Model

$$\frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2} + (\alpha - \beta r)\frac{\partial P}{\partial r} - rP + \frac{\partial P}{\partial t} = 0$$

with appropriate boundary condition, so for a zero coupon bond

$$P_r(t,T) = e^{A(T-t)-B(T-t)r}$$

with

$$B(x) = \frac{1}{\alpha}\left(1 - e^{-\alpha x}\right)$$

and

$$A(x) = \frac{[B(x) - x](\alpha\beta - \sigma^2/2)}{\alpha^2} - \frac{\sigma^2 B(x)^2}{4\alpha}$$

Affine Model

Explicit Solutions: CIR Model

$$\frac{1}{2}\sigma^2 r \frac{\partial^2 P}{\partial r^2} + (\alpha - \beta r)\frac{\partial P}{\partial r} - rP + \frac{\partial P}{\partial t} = 0$$

with appropriate boundary condition, so for a zero coupon bond

$$P_r(t,T) = e^{A(T-t)-B(T-t)r}$$

with

$$B(x) = \frac{2(e^{\gamma x} - 1)}{(\gamma + \alpha)(e^{\gamma x} - 1) + 2\gamma}$$

and

$$A(x) = \left(\frac{2\gamma e^{(\alpha+\gamma)x/2}}{(\gamma + \alpha)(e^{\gamma x} - 1) + 2\gamma}\right)^{2\alpha\beta/\sigma^2}$$

with

$$\gamma = \sqrt{\alpha^2 + 2\sigma^2}$$

Rigid Term Structures

TIME DEPENDENT (EVOLUTIONARY) MODELS

- the **Vasicek-Hull-White model** corresponds to the case $\gamma = 0$ and consequently in this case the dynamics of the short term rate are given by the SDE:

$$dr_t = (\alpha - \beta r_t)\, dt \, + \, \sigma_t \, dW_t$$

- **CIR-Hull-White model** corresponds to the case $\gamma = 1/2$ in which case the dynamics of the short term rate are given by the SDE:

$$dr_t = (\alpha - \beta r_t)\, dt \, + \, \sigma_t \sqrt{r_t}\, dW_t.$$

Back to the Multi Factor Models

View terms in the original equation as functions of the maturity T:

$$f(t,\,\cdot\,) = f(0,\,\cdot\,) + \int_0^t \alpha(s,\,\cdot\,)ds + \sum_{j=1}^{d} \int_0^t \sigma^{(j)}(s,\,\cdot\,)d\xi_s^{(j)}$$

or in differential form:

$$df(t,\,\cdot\,) = \alpha(t,\,\cdot\,)dt + \sum_{j=1}^{d} \sigma^{(j)}(t,\,\cdot\,)d\xi_t^{(j)}$$

Screaming for an interpretation as equations for the dynamics of the forward curve.

Unfortunately, for different t's, the $f(t,\,\cdot\,)$ have different domains of definition!!!

The Musiela's Notation

Reparametrize the forward curve by t and the time to maturity $x = T - t$

$$f_t(x) = f(t, t+x), \qquad\qquad t \geq 0,\ x \geq 0,$$

Forward curve at time t:

$$f_t : x \hookrightarrow f_t(x)$$

Already discussed the choice of a space F of functions of x.

$$df_t = [\frac{d}{dx} f_t + \alpha_t] dt + \sum_{j=1}^{d} \sigma_t^{(j)} d\xi_t^{(j)}$$

provided

$$\alpha_t : x \hookrightarrow \alpha_t(x) = \alpha(t, t+x) \quad \text{and} \quad \sigma_t^{(j)} : x \hookrightarrow \sigma_t^{(j)}(x) = \sigma^{(j)}(t, t+x)$$

The HJM prescription appears a time evolution in infinite dimension via a stochastic differential equation in a function space.

$$df_t = [Af_t + \alpha_t]dt + \sum_{j=1}^{d} \sigma_t^{(j)} d\xi_t^{(j)}$$

A is an

- not defined everywhere in F
- possibly unbounded

(differential) operator on F