Interest Rate Models:
from Parametric Statistics to Infinite Dimensional Stochastic Analysis

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Chapter 2

Practical Estimation of the Term Structure

Functional Spaces for Yield/Forward Curves

Yield Curve $\Longleftrightarrow$ Forward Curve

Requirements for a Space F of Forward Curves $x \hookrightarrow f(x)$

- All f's defined on the same interval $[0, x^*)$ with $x^* = \infty$ possibly
- Evaluation functionals make sense:
  - $f(0)$ short rate
  - $f(x^*)$ long rate
- Reasonable implementation of the (left) shift semigroup $\{S_t\}_{t\geq 0}$

$$[S_t f](x) = f(x + t)$$

Examples

1. $F = L^2([0,1])$ proposed by Cont et al. for the fluctuations of the yield curve around a (straight) baseline joining the short and the long interest rates (random strings)

2. In his analysis of the invariant measures (last lecture) for a finitely many factor HJM model in Musiela's notation, Vargiolu uses for F one of the Sobolev's spaces:

$$H^1_\gamma = \{f \in L^2(\mathbb{R}_+, e^{-\gamma x}dx);\ f' \in L^2(\mathbb{R}_+, e^{-\gamma x}dx)\}$$

equipped with the norm:

$$\|f\|^2_\gamma = \int_0^\infty |f(x)|^2 e^{-\gamma x}dx + \int_0^\infty |f'(x)|^2 e^{-\gamma x}dx$$

where $\gamma \geq 0$

- $F = H^1_0$ **too small**: does not contain any non zero constant function, or any function converging toward a non-zero limit at ∞.

- H^1_γ **too large**: its elements and their derivatives can be very large at infinity

3. Space $F = H_w$ introduced and used by Filipovic

$$H_w = \{f \in L^1_{loc}(\mathbb{R}_+, dx);\ f \text{ is absolutely continuous and } \int_0^\infty f'(x)^2 w(x)dx < +\infty\}$$

weight function w is

- nondecreasing continuously differentiable function from $\mathbb{R}_+$ onto $[1, \infty)$
- $w^{-1/3}$ is integrable.

H_w is a Hilbert space for the norm:

$$\|f\|^2 = |f(0)|^2 + \int_0^\infty |f'(x)|^2 w(x)dx$$

4. Modified version of H_w with $x^* < \infty$

we rarely encounter bonds with time to maturity greater than 30 years.

The Effective Dimension of the Space of Yield Curves

Suspicion that the actual (observed) yield/forward curves are restricted to a manifold of small dimension (Nelson-Siegel and the Nelson-Siegel-Svansson families for example)

Principal Component Analysis (PCA)

Assumptions: the bonds used are

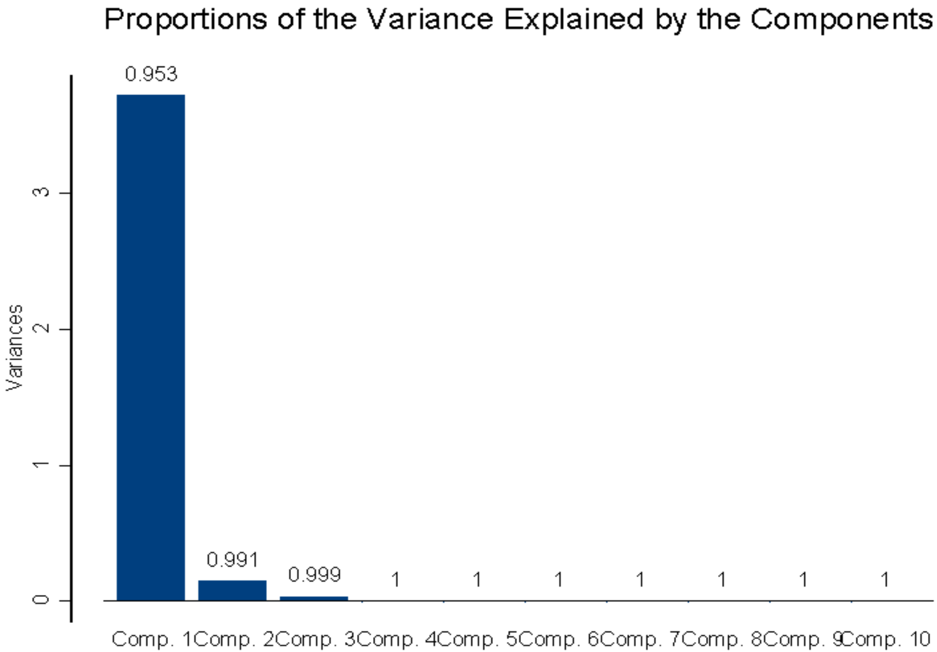
- default free
- no embedded options
- no call or convertibility features
- we ignore the effects of taxes and transaction costs

PCA of the Yield Curve

- Data on the US yield curve as provided by `BIS`
- 1352 succesive trading days starting January 3rd 1995
- yields on the US T-Bills for times to maturity

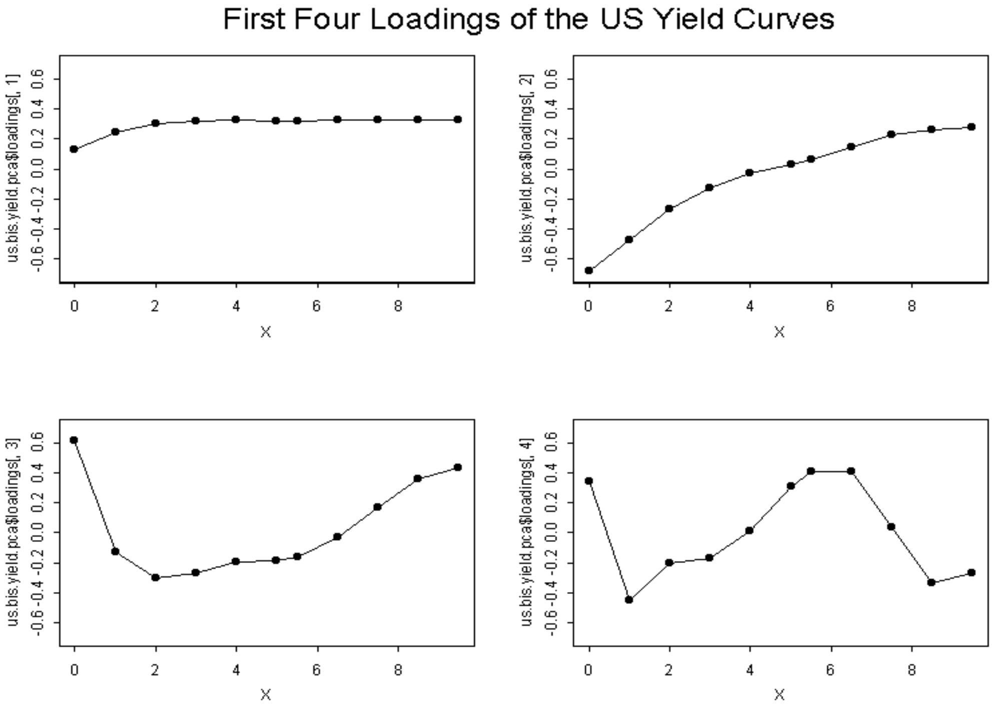
$$x = 0, 1, 2, 3, 4, 5, 5.5, 6.5, 7.5, 8.5, 9.5 \quad \text{months.}$$

Variation Explained



Proportions of the variance explained by the components of the PCA of the daily changes in the US yield.

Loadings



From left to right and top to bottom, sequential plots of the first four US yield loadings

Swap Contracts

- Traded publicly since 1981
- The most popular fixed income derivatives
- Liquid instruments used to hedge interest rate risk of fixed income portfolios

Contract Elements

Two parties exchange (swap) some specified cash flows at agreed upon times

Interest Rate Swaps

- counterparty A, agrees to make interest payments determined by an instrument P_A (say, a 30 year US Treasury bond rate)
- counterparty B, agrees to make interest payments determined by another instrument P_B (say, the London Interbank Offer Rate – LIBOR)
- equal principals (on which interest payments are computed)
- same payment schedules (quartely, semi-annualy, $\ldots$)

Example

plain vanilla contract involves a fixed interest rate and the 3 or 6 months LIBOR rate.

A Price Formula for a Plain Vanilla Swap

- X the common principal
- R the fixed interest rate on which the swap is written
- $T_1, T_2, \cdots, T_m$ the dates after the current date t, of the payments are scheduled
- $r(t,T)$ variable interest rate

Present value of the cashflow

$$P_{swap} = X \sum_{j=1}^{m} (T_j - T_{j-1})(r(t,T_{j-1}) - R)d(t,T_j)$$

with the convention $T_0 = t$.

Remark

If we add a payment of the principal X at time T_m, then the cashflow of the swap becomes identical to the cashflows generated by a portfolio long a (fixed rate) coupon bearing bond and short a floating rate bond with the same face value

Simple algebraic manipulations

$$P_{swap}(t,T) = X\left(1 - [P(t,T_m) + R\sum_{j=1}^{m}(T_j - T_{j-1})P(t,T_j)]\right)$$

The Swap Rate Curve

On any given day t, the swap rate $R_{swap}(t,T)$ with maturity $T = T_m$ is the unique value of the fixed rate r, which makes the swap price equal to 0

i.e.the value of the fixed interest rate for which the counter-parties will agree to enter the swap contract without paying or receiving a premium

Solve

$$P_{swap}(t,T) = 0$$

for r. This gives:

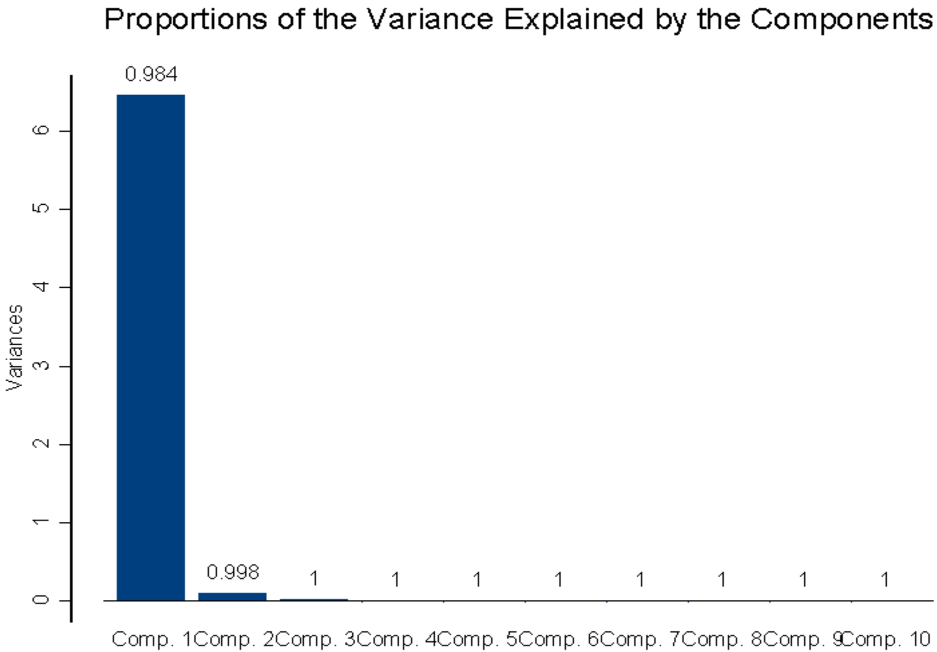
$$R_{swap}(t,T_m) = \frac{1 - P(t,T_m)}{\sum +j = 1^m (T_j - T_{j-1}) P(t,T_j)}.$$

PCA of the Swap Rates

- Data from Data Stream
- From May 1998 to March 2000
- swap rates with the payment schedule given by the 15 times

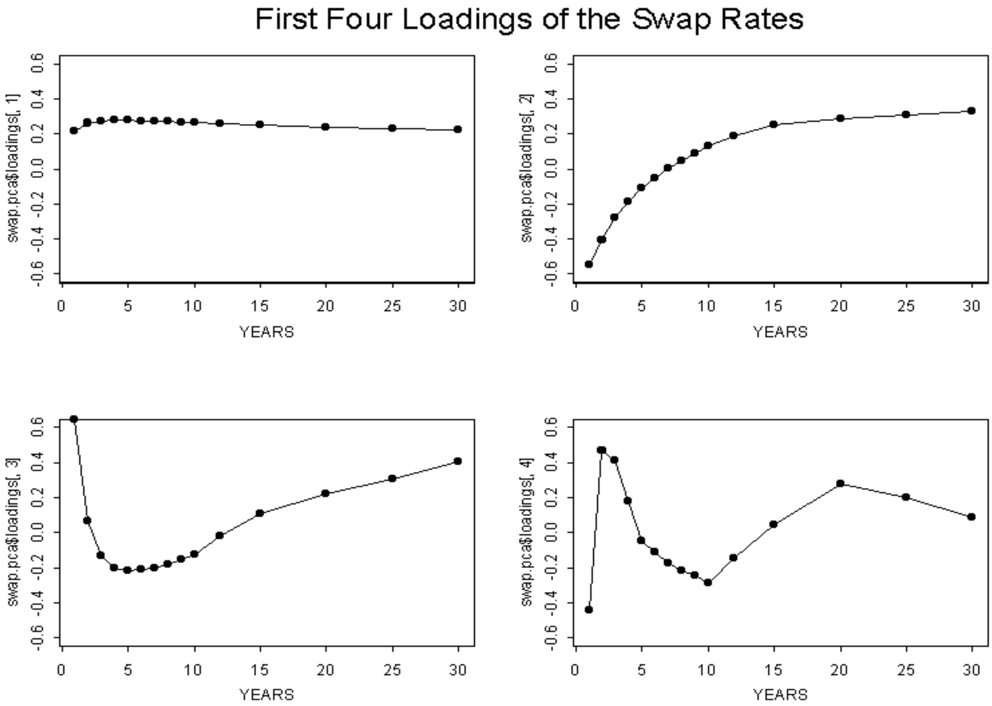
$$x = 1, 2, \cdots, 10, 12, 15, 20, 25, 30 \quad \text{years}$$

Variation Explained



Proportions of the variance explained by the components of the PCA of the daily changes in the swap rates.

Loadings



From left to right and top to bottom, sequential plots of the first four swap rate loadings

Parametric Yield/Forward Curve Estimation

1. Choose a family $\{f(x,\theta)\}_\theta$ parametrized by a finite dimensional parameter θ
2. Given observations $\{f(x_j)\}_{j=1,\cdots,m}$ on a given day, the estimate of the forward curve is

$$x \hookrightarrow f(x,\hat\theta)$$

where

$$\hat\theta = \arg\min_\theta \mathcal{L}_2(\{f(x_j)\}_{j=1,\cdots,m}, f(\,\cdot\,,\theta))$$

for a certain cost function $\mathcal{L}_2$

The Nelson-Siegel Family

- 4-dimensional parameter $\theta = (\theta_1 > 0, \theta_2, \theta_3, \theta_4 > 0)$

$$f_{NS}(x,\theta) = \theta_1 + (\theta_2 + \theta_3 x)e^{-x/\theta_4}$$

- Hump when $\theta_3 > 0$, Dip when $\theta_3 < 0$
- Discount Rate Curve

$$d_{NS}(x,\theta) = \exp\left[-\theta_1 x + [\theta_4(\theta_2 + \theta_3\theta_4) + \theta_3\theta_4 x]e^{-x/\theta_4}\right]$$

- Yield Curve

$$r_{NS}(x,\theta) = -\theta_1 + [\frac{\theta_4(\theta_2 + \theta_3\theta_4)}{x} + \theta_3\theta_4]e^{-x/\theta_4}$$

Used in Finland and Italy

The Swensson Family

- Add an extra exponential to produce a second hump/dip
- 6-dimensional parameter θ, and defined by:

$$f_S(x,\theta) = \theta_1 + (\theta_2 + \theta_3 x)e^{-\theta_4 x} + \theta_5 x e^{-\theta_6 x}$$

- Yield Curve

$$r_S(x,\theta) = \theta_1 - \frac{\theta_1\theta_4}{x}(1 - e^{-\theta_4 x}) + \frac{\theta_2}{\theta_4}\left[\frac{1}{\theta_4 x}(1 - e^{-\theta_4 x}) - e^{-\theta_4 x}\right] + \frac{\theta_5}{\theta_6}\left[\frac{1}{\theta_6 x}(1 - e^{-\theta_6 x}) - e^{-\theta_6 x}\right]$$

Used in many countries including Canada, Germany, France and the UK.

Implementation Issues

- Indirect Observations
- If
  - B_j bond prices available on a given day
  - $B_j(\theta)$ prices one would obtain using formulae above
- Find $\hat{\theta}$ minimizing

$$\mathcal{L}(\theta) = \sum_j w_j |B_j - B_j(\theta)|^2$$

  weights w_j's are function of the duration and the yield to maturity of the j-th bond
- Short maturity bills/bonds filtered out because of liquidity problems (large bid-ask spreads)
- Accrued interests and clean prices used

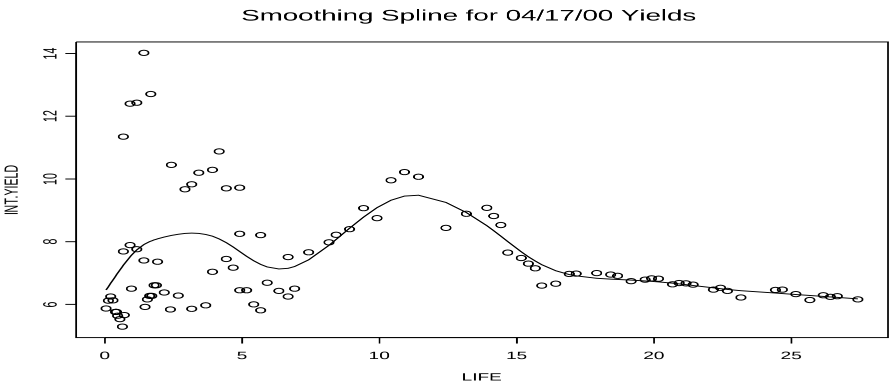
- Difficult optimization: first Nelson-Siegel, then one adds two extra

Non-Parametric Estimation

- Inductive (bootstrap) Empirical Procedure: unsatisfactory algorithm leading to piecewise constant curves
- Smoothing Splines (Central Banks of Japan & US) The resulting forward curve is the graph of the function $x \hookrightarrow \varphi(x)$ minimizing

$$\mathcal{L}(\varphi) = \sum_j w_j |B_j - B_j(\varphi)|^2 + \lambda \int |\varphi''(x)|^2 \, dx$$

Example


Smoothing Spline for 04/17/00 Yields
INT.YIELD
6
8
10
12
14
0
5
10
15
20
25
LIFE