Interest Rate Models:
from Parametric Statistics to Infinite Dimensional Stochastic Analysis

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Motivation (*where it all started*)

- September 99, Cont's talk in Ascona
- Winter 99, Cont visits Princeton
- Graduate Seminar
- Filipovic's PhD

Tutorial Prerequisites / Objectives

- Prerequisites

1. Mathematical: Some Probability, including Wiener Process and Stochastic Calculus
2. Professional Experience: NONE

- Objectives

1. Read Professional & Mathematical Publications
2. Dive into Mathematical Research Problems

Tutorial Contents

Chapter 1

The Term Structure of Interest Rates:

A Crash Course

The Time Value of Money

One dollar is worth more now than later

Simplest possible fixed income instrument.

- Cash flow: ONE SINGLE PAYMENT (principal or nominal X)
- At a given date in the future (maturity date) say n years from now
- Present value:

$$P(X, n) = \frac{1}{(1+r)^n} X$$

Discount Bond

$$P(X,n) = \frac{1}{(1+r)^n} X$$

Present value of a nominal amount X due in n years time.

- discount bond or zero coupon bond

  only cash exchange at the end of the life of the instrument

- r is called the (yearly) discount rate or spot interest rate

Treasury Bills

Securities issues by the US government with a time to maturity of one year or less.

NO coupon payments

Example

- Investor buys a $\$100,000$ 13-week T-bill at a 6% yield
- Investor pays $98,500 at the inception of the contract
- Investor receives the nominal value $\$100,000$ at maturity 13 weeks later

Computations

$13 = 52/4$ weeks = one quarter, and 6% is an annual rate, so

discount is $100,000 \times .06/4 = 1,500$

Terminology

Rates, yields, spreads, $\ldots$ are usually quoted in basis points. There are 100 basis points in one percentage point.

Auctions

Maturities of Treasury Bill issues:

- 13 weeks: *three-month bills*– Auctioned off every Monday
- 26 weeks: *six-month bills*– Auctioned off every Monday
- 52 weeks: *one-year bills*– Auctioned off every Month

Accurate only at inception!

The Wall Street Journal

Treasury Sets Offering Of About $22 Billion In Short-Term Bills

Dow Jones Newswires

WASHINGTON—The Treasury Department plans to raise $4.98 billion in new cash with the sale on Tuesday of about $22 billion in short-term bills to redeem $17.02 billion in maturing bills.

The offering will be divided between $12 billion 13-week and $10 billion 26-week bills maturing on March 29, 2001, and June 28, 2001, respectively. The Cusip number for the three-month bills is 912795FZ9. The Cusip number for the six-month bills is 912795GN5.

Noncompetitive tenders for the bills, available in minimum $1,000 denominations, must be received by noon EST Tuesday at the Treasury or at Federal Reserve banks or branches. Competitive tenders for the bills must be received by 1 p.m, EST.

TREASURY BILLS

MATURITY	DAYS TO MAT.	BID	ASKED	CHG.	ASKED YLD.
Dec 28 '00	6	4.80	4.72	−0.62	4.79
Jan 04 '01	13	4.77	4.69	−0.60	4.76
Jan 11 '01	20	4.74	4.66	−0.58	4.74
Jan 18 '01	27	4.80	4.72	−0.57	4.80
Jan 25 '01	34	4.68	4.64	−0.54	4.73
Feb 01 '01	41	4.99	4.95	−0.53	5.05
Feb 08 '01	48	5.06	5.02	−0.51	5.12
Feb 15 '01	55	5.07	5.03	−0.49	5.14
Feb 22 '01	62	5.18	5.16	−0.47	5.28
Mar 01 '01	69	5.20	5.18	−0.45	5.30
Mar 08 '01	76	5.21	5.19	−0.44	5.32
Mar 15 '01	83	5.21	5.19	−0.42	5.33
Mar 22 '01	**90**	**5.25**	**5.24**	**−0.40**	**5.38**
Mar 29 '01	97	5.24	5.22	−0.38	5.37
Mar 29 '01	97	5.26	5.25	....	5.40
Apr 05 '01	104	5.26	5.24	−0.36	5.39
Apr 12 '01	111	5.27	5.25	−0.35	5.41
Apr 19 '01	118	5.29	5.27	−0.33	5.44
Apr 26 '01	125	5.30	5.28	−0.31	5.45
May 03 '01	132	5.31	5.29	−0.29	5.47
May 10 '01	139	5.32	5.30	−0.28	5.49
May 17 '01	146	5.34	5.32	−0.26	5.51
May 24 '01	153	5.35	5.33	−0.24	5.53
May 31 '01	160	5.36	5.34	−0.22	5.55
Jun 07 '01	167	5.39	5.37	−0.20	5.58
Jun 14 '01	174	5.39	5.37	−0.19	5.59
Jun 21 '01	**181**	**5.40**	**5.39**	**−0.17**	**5.62**
Jun 28 '01	188	5.38	5.37	....	5.60
Aug 30 '01	251	5.26	5.24	−0.15	5.47
Nov 29 '01	342	5.08	5.07	−0.13	5.33

Excerpts from WSJ December 22nd, 2000

Local (Tucson) Paper

TREASURY ISSUES

- The Federal Reserve will auction 13-week and 26-week Treasury bills Tuesday. Last week's 13-week discount rate was 5.7 percent. The 26-week discount rate was 5.5 percent.
- The next auction of the 52-week bill is not yet scheduled. The last auction was Nov. 28. The discount rate was 5.71 percent.
- The next auction of the two-year note is scheduled for Jan. 24. The last auction was Wednesday. The coupon rate was 5.125 percent with a yield of 5.13 percent.
- The next auction of the five-year note is not yet scheduled. The last auction of the five-year note was Nov. 7. The coupon rate was 5.75 percent with a yield of 5.87 percent.
- The next auction of the 30-year bond is not yet scheduled. The last auction of the 30-year bond was Aug. 10. The coupon rate was 6.25 percent with a yield of 5.697 percent.

The Department of the Treasury Bureau of the Public Debt home page address is: **www.publicdebt.treas.gov/**

For a packet of information and forms to invest in the various instruments, call (800) 722-7398 from 9 a.m. to 5:15 p.m. PST Monday through Friday. Such forms also can be obtained by writing the Bureau of the Public Debt, Department A, Washington, DC, 20239-1000.

Excerpts from a local Tucson paper on December 31st, 2000

The Discount Factor

The present value of any future cashflow can be computed by multiplying its nominal value by the appropriate value of the discount factor

$$d_{t,m} = \delta(t,T) = \frac{1}{(1+r_{t,m})^m}$$

- current time t
- time to maturity m
- maturity date $T = t + m$
- $r_{t,m}$ yearly interest rate in force at time t for this time to maturity.

It assumed (implicitly) that the time to maturity $T-t$ is a whole number m of years.

$$\log(1+r_{t,m}) = -\frac{1}{m}\log d_{t,m}$$

using $\log(1+x) \sim x$ when x is small

$$r_{t,m} \sim -\frac{1}{m}\log d_{t,m}$$

equality if we use continuous compounding.

Natural generalization to continuous time models with continuous compounding of the interest.

$$d(t,T) = e^{-(T-t)r(t,T)}.$$

Coupon Bearing Bonds

Regular stream of future cash flows

- Payment Amounts $C_1, C_2, \cdots, C_m,$
- at times $T_1, T_2, \cdots, T_m,$
- Terminal payment X at the maturity date T_m.

X is called nominal value or face value or principal value

Bond price at time t:

$$B(t) = \sum_{t \leq T_j} C_j d(t, T_j) + X d(t, T_m)$$

More on Coupon Bonds

- Coupon payments C_j's are made at regular time intervals.
- Quoted as (annual) percentage c of the face value X of the bond, i.e. $C_j = cX$
- Frequency of six months for Treasuries
- Possibly another periodicity

  Notation n_y for the number of coupon payments per year

$r_1, r_2, \cdots, r_m$ the interest rates for the m periods ending with the coupon payments $T_1, T_2, \cdots, T_m$

$$
\begin{aligned}
B(t) &= \frac{C_1}{1+r_1/n_y} + \frac{C_2}{(1+r_2/n_y)^2} + \cdots + \frac{C_m}{(1+r_m/n_y)^m} + \frac{X}{(1+r_m/n_y)^m} \\
&= \frac{cX}{n_y(1+r_1/n_y)} + \frac{cX}{n_y(1+r_2/n_y)^2} + \cdots + \frac{cX}{n_y(1+r_m/n_y)^m} + \frac{X}{(1+r_m/n_y)^m}
\end{aligned}
$$

bond price equations

Treasury Notes

- Treasury securities with time to maturity ranging from 1 to 10 years at the time of sale
- Unlike bills, they have coupons every six months
- Auctioned on a regular cycle. The Fed acts as agent for the Treasury, awarding competitive bids in decreasing order of price, highest prices first
- Smallest denomination $\$5,000$ for notes with two to three years to maturity at the time of issue
- Smallest denomination $\$1,000$ for notes with four or more years to maturity at the time of issue

Treasury Bonds

- Treasury securities with more than 10 years to maturity at the time of sale
- Sold at auctions
- Bare coupons

Few differences between Treasury notes and bonds

More Wall Street Journal Excerpts

GOVT. BOND & NOTES

ATE	MATURITY MO/YR	BID	ASKED	CHG.	ASKED YLD.
4 5/8	Dec 00n	99:29	99:31	+ 1	5.78
5 1/2	Dec 00n	99:29	99:31	....	6.61
4 1/2	Jan 01n	99:28	99:30	+ 2	4.99
5 1/4	Jan 01n	99:30	100:00	+ 2	5.14
5 3/8	Feb 01n	99:31	100:01	+ 3	5.07
7 3/4	Feb 01n	100:10	100:12	− 3	5.08
11 3/4	Feb 01	100:31	101:01	+ 3	4.61
5	Feb 01n	99:28	99:30	+ 2	5.25
[illegible]	Feb 01n	100:00	100:02	+ 2	5.20
[illegible]	Nov 26	[illegible]	[illegible]	+ 8	[illegible]
6 5/8	Feb 27	114:30	115:02	+ 8	5.53
6 3/8	Aug 27	111:20	111:24	+ 7	5.53
6 1/8	Nov 27	108:10	108:12	+ 7	5.52
3 5/8	Apr 28i	98:19	98:20	+ 3	3.71
5 1/2	Aug 28	99:26	99:28	+ 6	5.51
5 1/4	Nov 28	96:12	96:14	+ 5	5.50
5 1/4	Feb 29	96:14	96:16	+ 5	5.50
3 7/8	Apr 29i	103:01	103:02	+ 3	3.70
6 1/8	Aug 29	109:13	109:14	+ 4	5.47
6 1/4	May 30	112:07	112:08	+ 5	5.41

Wall Street Journal Treasury notes and bond quotes on December 22, 2000

Remarks

- ”n” is used when the instrument is a T-Note
- None of the decimal parts happen to be greater than 31 (compare with T-Bills)
- Prices of Treasury notes and bonds are quoted in percentage points and 32nds of a percentage point

$$99.28 \text{ is actually } 99 + 28/32 = 99.875$$

which represents

$$\$998,750$$

per million of dollars of nominal amount

STRIPS

A coupon bearing bond is a composite instrument comprising a zero coupon bond with maturity T_m and a set of zero coupon bonds whose maturity dates T_j and face values cX/n_y

The principal and the interest components (of eligible issues) can be traded separately under the Treasury STRIPS program

Separate Trading of Registered Interest and Principal Securities

Still More Wall Street Journal Excerpts

U.S. TREASURY STRIPS

MATURITY		TYPE	BID	ASKED	CHG.	ASKED YLD.
Feb	01	ci	99:13	99:14	+ 2	4.18
Feb	01	np	99:07	99:07	+ 3	5.72
May	01	ci	97:29	97:29	+ 3	5.53
May	01	np	97:28	97:28	+ 3	5.61
Aug	01	ci	96:22	96:22	+ 4	5.33
Aug	01	np	96:16	96:17	+ 4	5.53
Nov	01	ci	95:12	95:12	+ 5	5.41
Nov	01	np	95:11	95:12	+ 8	5.43
Feb	02	ci	94:20	94:21	+ 5	4.89
May	02	ci	93:08	93:09	+ 5	5.08
May	02	np	92:29	92:30	+ 5	5.36
Aug	02	ci	92:03	92:04	+ 6	5.07
Aug	02	np	91:25	91:26	+ 6	5.27
Nov	02	ci	91:07	91:08	+ 6	4.92
Feb	03	ci	89:25	89:26	+ 7	5.09
Feb	03	np	89:19	89:21	+ 7	5.17
May	03	ci	88:28	88:30	+ 7	4.98
Jul	03	ci	88:04	88:06	+ 8	4.98
Aug	03	ci	87:22	87:24	+ 8	5.01

Wall Street Journal Treasury STRIPS quotes on December 22, 2000

Accrued Interests

- Formulae implicitly assumed that t was the time of a coupon payment
- Bond price jumps by cX/n_y at the times T_j of the coupon payments
- Smooth out the discontinuities by including *accrued interest* earned by the bond holder since the time of the last coupon payment
- If last coupon payment (before the present time t) was on T_n, then the accrued interest:

$$AI(T_n, t) = \frac{t - T_n}{T_{n+1} - T_n} \frac{cX}{n_y},$$

Clean Prices

transaction price = clean price + the accrued interest.

$$CP(t, T_m) = P_{X,C}(t, T_m) - AI(t, T_n)$$

if $T_n \leq t < T_{n+1}$

The Spot (Zero Coupon) Yield Curve

$$d_{t,m} = \frac{1}{(1+r_{t,m})^m} \qquad \text{or} \qquad r_{t,m} = \left(\frac{1}{d_{t,m}^{1/m}} - 1\right)$$

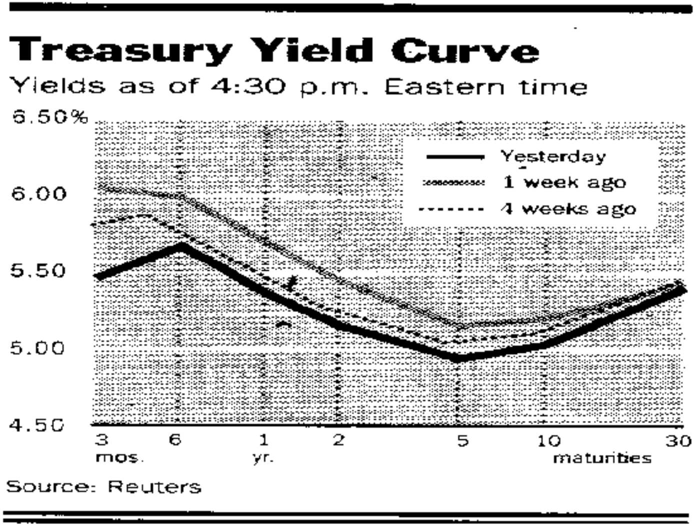
- $r_{t,m}$ is called the *zero coupon yield*

  as the yield to maturity on a zero coupon bond

- The sequence of spot rates $\{r_{t,j};\ j = 1, \cdots, m\}$ where m is a distant maturity is called the *term structure of (spot) interest rate* or the *zero coupon yield curve* at time t

- It is usually plotted against the time to maturity $x_j = T_j - t$ in years.

Wall Street Journal Yield Curves



Source: Wall Street Journal, December 22, 2000.

Wall Street Journal Treasury yield curves on December 22, 2000. Notice the non uniform time scale on the horizontal axis.

The Par Yield Curve

- A coupon paying bond is said to be priced *at par* if its current market price equals its face (or par) value.
- If a bond price is less than its face value, we say the bond trades *at a discount* its yield is higher than the coupon rate
- If its price is higher than its face value, it is said to trade *at a premium* the yield is lower than the coupon rate.

higher yields correspond to lower prices

- The *par yield* is defined as the yield of a bond priced at par. It is the value of y for which we have the following equality:

$$P = \sum_{j=1}^{m} \frac{yP}{n_y(1+y/n_y)^j} + \frac{P}{(1+y/n_y)^m}.$$

The Par Yield Curve

The Par yield curve is the curve

$$\{y(T_j); T_0 < T_1 < \cdots < T_m\}$$

which describes the coupons $y(T_j)$ required on a (hypothetical) coupon paying bond with maturity T_j for that bond to trade at par.

The Forward Rate Curve

Notation: $f_{t,m}$ rate applicable at time t for the period from the end of the $(m-1)$-th period to the end of the m-th period.

$$
\begin{aligned}
1/d_{t,1} &= 1 + r_{t,1} = 1 + f_{t,1} \\
1/d_{t,2} &= (1 + r_{t,2})^2 = (1 + f_{t,1})(1 + f_{t,2}) \\
\cdots &= \cdots\cdots\cdots \\
1/d_{t,j-1} &= (1 + r_{t,j-1})^{j-1} = (1 + f_{t,1})(1 + f_{t,2}) \cdots (1 + f_{t,j-1}) \\
1/d_{t,j} &= (1 + r_{t,j})^{j} = (1 + f_{t,1})(1 + f_{t,2}) \cdots (1 + f_{t,j-1})(1 + f_{t,j})
\end{aligned}
$$

The Forward Rate Curve (cont)

Computing the ratio of the last two equations gives:

$$\frac{d_{t,j-1}}{d_{t,j}} = 1 + f_{t,j}$$

or equivalently:

$$f_{t,j} = \frac{d_{t,j-1} - d_{t,j}}{d_{t,j}} = -\frac{\Delta d_{t,j}}{d_{t,j}}$$

Rates $f_{t,1}$, $f_{t,2}$, $\cdots$, $f_{t,j}$ implied by the discount factors $d_{t,1}$, $d_{t,2}$, $\cdots$, $d_{t,j}$ are called the **implied forward interest rates**

Continuous Time Analogs

Discount factor (for continuous compounding)

$$x \hookrightarrow d(t,x)$$

where $x = T - t$ time to maturity

Forward rate

$$f(t,x) = -\frac{d'(t,x)}{d(t,x)}$$

Equivalently

$$d(t,x) = e^{-\int_0^x f(t,s)ds}$$

Spot rate/yield

$$r(t,x) = -\frac{1}{x}\int_0^x f(t,s)ds.$$

Equivalently

$$f(t,x) = r(t,x) + xr'(t,x).$$

Extensions

- Tax Issues Complex (ignored here)
- Municipal Bonds: debt securities issued by states, cities, townships, counties, US Territories and their agencies
  - Complex Tax Status
  - Credit Risk with High profile defaults: NY City, Orange County
  - Ratings (S&P and Moody's)
  - Insurance Contracts which pay interest and principal in case of default of the issuer.
  - Quoted as spread over Treasury (in basis points)
- Index Linked Bonds to guarantee real returns and protect the cash flows from inflation.
  - Four types of indexing
    1. *indexed principal bonds* for which both coupons and principal are adjusted for inflation
    2. *indexed coupon bonds* for which only the coupons are adjusted for inflation
    3. *zero coupon bonds* which pay no coupon but for which the principal is adjusted for inflation

4. *indexed annuity bonds* which pay inflation adjusted coupons and no principal on redemption

– Most common index used is the Consumer Price Index (CPI)

– More popular in Europe than in the US.

INFLATION-INDEXED TREASURY SECURITIES

RATE	MAT.	BID/ASKED	CHG.	*YLD.	ACCR. PRIN.
3.625	07/02	100-01/02	+ 1	3.581	1085
3.375	01/07	98-06/07	− 1	3.705	1097
3.625	01/08	99-15/16	+ 3	3.705	1076
3.875	01/09	101-03/04	+ 2	3.712	1060
4.250	01/10	104-07/08	+ 3	3.694	1033
3.625	04/28	98-19/20	+ 3	3.705	1075
3.875	04/29	103-01/02	+ 3	3.699	1057

*Yld. to maturity on accrued principal.

Source: Wall Street Journal, December 22, 2000

- Corporate Bonds

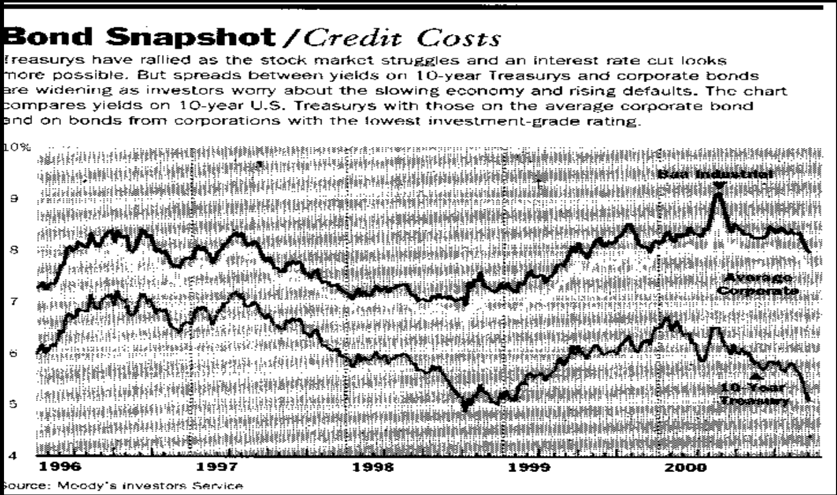
  – Corporation raising funds

  – Issues are rated by S&P and/or Moody's (ratings updated periodically)

  – Ratings quantify the credit risk associated with the bonds.

    * Poor ratings = non-investment grade bonds or junk bonds. Spread over Treasury is usually relatively high (high yield)

* Good ratings = investment grade bonds



Comparison of several yield curves. Source: Wall Street Journal December 22, 2000.

– Complex indentures: pricing can be a challenge
  * Callable bonds (some Treasury issues do have this feature)
  * Convertible bonds.

- Asset Backed Securities
  – Mortgage loans are bundled and packaged (i.e. securitized) as bond issues

backed by the interest income of the mortgages.

– Prepayments and default risks are the main factors in pricing of these securities.

– Securitization of many other risky future incomes

  * catastrophic risk (natural disasters such as earthquakes and hurricanes)
  * intellectual property (Bowie bond issued by the rock star borrowing on the future cash flow expected from its rights and record sales)