Interest Rate Models: 
from Parametric Statistics to 
Infinite Dimensional Stochastic Analysis

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IPAM / Financial Math
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Motivation (*where it all started*)

- September 99, Cont’s talk in Ascona
- Winter 99, Cont visits Princeton
- Graduate Seminar
- Filipovic’s PhD
Tutorial Prerequisites / Objectives

• Prerequisites
  1. Mathematical: Some Probability, including Wiener Process and Stochastic Calculus
  2. Professional Experience: NONE

• Objectives
  1. Read Professional & Mathematical Publications
  2. Dive into Mathematical Research Problems
Tutorial Contents

• The Term Structure of Interest Rates
  1. The Term Structure of Interest Rates: A Crash Course
  2. Statistical Estimation of the Term Structure
  3. First Term Structure Models

• Infinite Dimensional Stochastic Analysis
  1. Infinite Dimensional Integration Theory
  2. Infinite Dimensional Stochastic Integration
  3. Infinite Dimensional Ornstein Uhlenbeck Processes

• More Stochastic Models for the Term Structure
  1. Infinite Dimensional HJM Models
  2. Problems
Chapter 1

The Term Structure of Interest Rates:
A Crash Course
The Time Value of Money

One dollar is worth more now than later

Simplest possible fixed income instrument.

- Cash flow: ONE SINGLE PAYMENT (principal or nominal $X$)
- At a given date in the future (maturity date) say $n$ years from now

- Present value:

$$P(X, n) = \frac{1}{(1 + r)^n}X$$
Discount Bond

\[ P(X, n) = \frac{1}{(1 + r)^n} X \]

Present value of a nominal amount \( X \) due in \( n \) years time.

- **discount bond** or **zero coupon bond**
  - only cash exchange at the end of the life of the instrument
- \( r \) is called the (yearly) **discount rate** or **spot interest rate**
Treasury Bills

Securities issues by the US government with a time to maturity of one year or less.

**NO coupon payments**

**Example**

- Investor buys a $100,000 13-week T-bill at a 6% yield
- Investor pays $98,500 at the inception of the contract
- Investor receives the nominal value $100,000 at maturity 13 weeks later
Computations

$13 = \frac{52}{4}$ weeks = one quarter, and 6% is an annual rate, so discount is $100,000 \times \frac{.06}{4} = 1,500$

Terminology

Rates, yields, spreads, . . . are usually quoted in basis points. There are 100 basis points in one percentage point.
Auctions

Maturities of Treasury Bill issues:

- **13 weeks**: *three-month bills* – Auctioned off every Monday
- **26 weeks**: *six-month bills* – Auctioned off every Monday
- **52 weeks**: *one-year bills* – Auctioned off every Month

Accurate only at inception!
Treasury Sets Offering
Of About $22 Billion
In Short-Term Bills

Dow Jones Newswires

WASHINGTON—The Treasury Department plans to raise $4.98 billion in new cash with the sale on Tuesday of about $22 billion in short-term bills to redeem $17.02 billion in maturing bills.

The offering will be divided between $12 billion 13-week and $10 billion 26-week bills maturing on March 29, 2001, and June 28, 2001, respectively. The Cusip number for the three-month bills is 91279FZ9. The Cusip number for the six-month bills is 91279GN5.

Noncompetitive tenders for the bills, available in minimum $1,000 denominations, must be received by noon EST Tuesday at the Treasury or at Federal Reserve banks or branches. Competitive tenders for the bills must be received by 1 p.m., EST.

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Excerpts from WSJ December 22nd, 2000
Local (Tucson) Paper

Excerpts from a local Tucson paper on December 31st, 2000

- The Treasury will auction 13-week and 26-week Treasury bills Tuesday. Last week's 13-week discount rate was 5.7 percent. The 26-week discount rate was 5.5 percent.
- The next auction of the 52-week bill is not yet scheduled. The last auction was Nov. 28. The discount rate was 5.71 percent.
- The next auction of the two-year note is scheduled for Jan. 24. The last auction was Wednesday. The coupon rate was 5.725 percent with a yield of 5.13 percent.
- The next auction of the five-year note is not yet scheduled. The last auction of the five-year note was Nov. 7. The coupon rate was 5.75 percent with a yield of 5.87 percent.
- The next auction of the 30-year bond is not yet scheduled. The last auction of the 30-year bond was Aug. 16. The coupon rate was 6.25 percent with a yield of 5.997 percent.

The Department of the Treasury Bureau of the Public Debt home page address is: www.publicdebt.treas.gov

For a packet of information and terms to invest in the various instruments call (800) 728-1988 from 9 a.m. to 5:15 p.m. PST Monday through Friday. More terms also can be obtained by writing the Bureau of the Public Debt Department A, Washington, DC 20225, 1000.
The Discount Factor

The present value of any future cashflow can be computed by multiplying its nominal value by the appropriate value of the discount factor

\[ d_{t,m} = \delta(t, T) = \frac{1}{(1 + r_{t,m})^m} \]

- current time \( t \)
- time to maturity \( m \)
- maturity date \( T = t + m \)
- \( r_{t,m} \) yearly interest rate in force at time \( t \) for this time to maturity.
It assumed (implicitly) that the time to maturity $T - t$ is a whole number $m$ of years.

$$\log(1 + r_{t,m}) = -\frac{1}{m} \log d_{t,m}$$

**using** $\log(1 + x) \sim x$ **when** $x$ **is small**

$$r_{t,m} \sim -\frac{1}{m} \log d_{t,m}$$

equality if we use continuous compounding.

Natural generalization to continuous time models with continuous compounding of the interest.

$$d(t, T) = e^{-(T-t)r(t,T)}.$$
Coupon Bearing Bonds

*Regular stream of future cash flows*

- Payment Amounts $C_1, C_2, \cdots, C_m$,
- at times $T_1, T_2, \cdots, T_m$,
- Terminal payment $X$ at the maturity date $T_m$.

$X$ is called *nominal value* or *face value* or *principal value*

Bond price at time $t$:

$$B(t) = \sum_{t \leq T_j} C_j d(t, T_j) + X d(t, T_m)$$
More on Coupon Bonds

- Coupon payments $C_j$'s are made at regular time intervals.
- Quoted as (annual) percentage $c$ of the face value $X$ of the bond, i.e. $C_j = cX$
- Frequency of six months for Treasuries
- Possibly another periodicity

Notation $n_y$ for the number of coupon payments per year

$r_1, r_2, \cdots, r_m$ the interest rates for the $m$ periods ending with the coupon payments $T_1, T_2, \cdots, T_m$

$$B(t) = \frac{C_1}{1 + r_1/n_y} + \frac{C_2}{(1 + r_2/n_y)^2} + \cdots + \frac{C_m}{(1 + r_m/n_y)^m} + \frac{X}{(1 + r_m/n_y)^m}$$

$$= \frac{cX}{n_y(1 + r_1/n_y)} + \frac{cX}{n_y(1 + r_2/n_y)^2} + \cdots + \frac{cX}{n_y(1 + r_m/n_y)^m} + \frac{X}{(1 + r_m/n_y)^m}$$

bond price equations
Treasury Notes

• Treasury securities with time to maturity ranging from 1 to 10 years at the time of sale
• Unlike bills, they have coupons every six months
• Auctioned on a regular cycle. The Fed acts as agent for the Treasury, awarding competitive bids in decreasing order of price, highest prices first
• Smallest denomination $5,000 for notes with two to three years to maturity at the time of issue
• Smallest denomination $1,000 for notes with four or more years to maturity at the time of issue
Treasury Bonds

- Treasury securities with more than 10 years to maturity at the time of sale
- Sold at auctions
- Bare coupons

Few differences between Treasury notes and bonds
Wall Street Journal Treasury notes and bond quotes on December 22, 2000

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<th>ASKED</th>
<th>CHG.</th>
<th>YLD.</th>
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<td>+ 5</td>
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Remarks

• "n" is used when the instrument is a T-Note
• None of the decimal parts happen to be greater than $\frac{31}{32}$ (compare with T-Bills)
• Prices of Treasury notes and bonds are quoted in percentage points and $\frac{32}{32}$nds of a percentage point

99.28 is actually $99 + \frac{28}{32} = 99.875$

which represents

$998,750$

per million of dollars of nominal amount
STRIPS

A coupon bearing bond is a composite instrument comprising a zero coupon bond with maturity $T_m$ and a set of zero coupon bonds whose maturity dates $T_j$ and face values $cX/n_y$

The principal and the interest components (of eligible issues) can be traded separately under the Treasury STRIPS program

Separate Trading of Registered Interest and Principal Securities
### U.S. TREASURY STRIPS

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<th>MATURITY</th>
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Wall Street Journal Treasury STRIPS quotes on December 22, 2000
Accrued Interests

- Formulae implicitly assumed that $t$ was the time of a coupon payment
- Bond price jumps by $cX/n_y$ at the times $T_j$ of the coupon payments
- Smooth out the discontinuities by including *accrued interest* earned by the bond holder since the time of the last coupon payment
- If last coupon payment (before the present time $t$) was on $T_n$, then the accrued interest:

$$AI(T_n, t) = \frac{t - T_n}{T_{n+1} - T_n} \frac{cX}{n_y},$$
Clean Prices

transaction price = clean price + the accrued interest.

\[ CP(t, T_m) = P_{X,C}(t, T_m) - AI(t, T_n) \]

if \( T_n \leq t < T_{n+1} \)
The Spot (Zero Coupon) Yield Curve

\[ d_{t,m} = \frac{1}{(1 + r_{t,m})^m} \quad \text{or} \quad r_{t,m} = \left( \frac{1}{d_{t,m}^{1/m}} - 1 \right) \]

- \( r_{t,m} \) is called the zero coupon yield as the yield to maturity on a zero coupon bond.

- The sequence of spot rates \( \left\{ r_{t,j}; \ j = 1, \ldots, m \right\} \) where \( m \) is a distant maturity is called the term structure of (spot) interest rate or the zero coupon yield curve at time \( t \).

- It is usually plotted against the time to maturity \( x_j = T_j - t \) in years.
Wall Street Journal Yield Curves

Treasury Yield Curve
Yields as of 4:30 p.m. Eastern time

Source: Reuters


The Par Yield Curve

- A coupon paying bond is said to be priced at par if its current market price equals its face (or par) value.
- If a bond price is less than its face value, we say the bond trades at a discount if its yield is higher than the coupon rate.
- If its price is higher than its face value, it is said to trade at a premium if the yield is lower than the coupon rate.

Higher yields correspond to lower prices.

- The par yield is defined as the yield of a bond priced at par. It is the value of $y$ for which we have the following equality:

$$P = \sum_{j=1}^{m} \frac{yP}{n_y(1 + y/n_y)^j} + \frac{P}{(1 + y/n_y)^m}.$$
The Par Yield Curve

The Par yield curve is the curve

$$\{y(T_j); T_0 < T_1 < \cdots < T_m\}$$

which describes the coupons $y(T_j)$ required on a (hypothetical) coupon paying bond with maturity $T_j$ for that bond to trade at par.
The Forward Rate Curve

Notation: $f_{t,m}$ rate applicable at time $t$ for the period from the end of the $(m - 1)$-th period to the end of the $m$-th period.

$$\frac{1}{d_{t,1}} = 1 + r_{t,1} = 1 + f_{t,1}$$
$$\frac{1}{d_{t,2}} = (1 + r_{t,2})^2 = (1 + f_{t,1})(1 + f_{t,2})$$
$$\ldots = \ldots \ldots$$

$$\frac{1}{d_{t,j-1}} = (1 + r_{t,j-1})^{j-1} = (1 + f_{t,1})(1 + f_{t,2}) \cdots (1 + f_{t,j-1})$$
$$\frac{1}{d_{t,j}} = (1 + r_{t,j})^j = (1 + f_{t,1})(1 + f_{t,2}) \cdots (1 + f_{t,j-1})(1 + f_{t,j})$$
The Forward Rate Curve (cont)

Computing the ratio of the last two equations gives:

\[
\frac{d_{t,j-1}}{d_{t,j}} = 1 + f_{t,j}
\]

or equivalently:

\[
f_{t,j} = \frac{d_{t,j-1} - d_{t,j}}{d_{t,j}} = -\frac{\Delta d_{t,j}}{d_{t,j}}
\]

Rates \( f_{t,1}, f_{t,2}, \ldots, f_{t,j} \) implied by the discount factors \( d_{t,1}, d_{t,2}, \ldots, d_{t,j} \) are called the **implied forward interest rates**
Continuous Time Analogs

Discount factor (for continuous compounding)

\[ x \leftrightarrow d(t, x) \]

where \( x = T - t \) time to maturity

Forward rate

\[ f(t, x) = -\frac{d'(t, x)}{d(t, x)} \]

Equivalently

\[ d(t, x) = e^{-\int_0^x f(t, s) ds} \]

Spot rate/yield

\[ r(t, x) = -\frac{1}{x} \int_0^x f(t, s) ds. \]

Equivalently

\[ f(t, x) = r(t, x) + x r'(t, x). \]
Extensions

- **Tax Issues** Complex (ignored here)
- **Municipal Bonds**: debt securities issued by states, cities, townships, counties, US Territories and their agencies
  - Complex Tax Status
  - Credit Risk with High profile defaults: NY City, Orange County
  - Ratings (S&P and Moody’s)
  - Insurance Contracts which pay interest and principal in case of default of the issuer.
  - Quoted as spread over Treasury (in basis points)

- **Index Linked Bonds** to guarantee real returns and protect the cash flows from inflation.
  - Four types of indexing
    1. *indexed principal bonds* for which both coupons and principal are adjusted for inflation
    2. *indexed coupon bonds* for which only the coupons are adjusted for inflation
    3. *zero coupon bonds* which pay no coupon but for which the principal is adjusted for inflation
4. *indexed annuity bonds* which pay inflation adjusted coupons and no principal on redemption
   - Most common index used is the Consumer Price Index (CPI)
   - More popular in Europe than in the US.

![INFLATION-INDEXED TREASURY SECURITIES](image)


- **Corporate Bonds**
  - Corporation raising funds
  - Issues are rated by S&P and/or Moody’s (ratings updated periodically)
  - Ratings quantify the credit risk associated with the bonds.

  * Poor ratings = *non-investment grade* bonds or *junk* bonds. Spread over Treasury is usually relatively high (*high yield*)
Good ratings = investment grade bonds


- Complex indentures: pricing can be a challenge
  * Callable bonds (some Treasury issues do have this feature)
  * Convertible bonds.

- Asset Backed Securities
  - Mortgage loans are bundled and packaged (i.e. securitized) as bond issues
backed by the interest income of the mortgages.

– Prepayments and default risks are the main factors in pricing of these securities.

– Securitization of many other risky future incomes
  * catastrophic risk (natural disasters such as earthquakes and hurricanes)
  * intellectual property (Bowie bond issued by the rock star borrowing on the future cash flow expected from its rights and record sales)