Investment performance measurement under time-monotone criteria

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Performance measurement of investment strategies

Market environment

Riskless and risky securities

- $(\Omega, \mathcal{F}, \mathbb{P})$; $W = (W^1, \ldots, W^d)$ standard Brownian Motion

- Traded securities

$$1 \le i \le k \qquad \begin{cases} dS_t^i = S_t^i\Big(\mu_t^i dt + \sigma_t^i \cdot dW_t\Big)\,, & S_0^i > 0 \\ dB_t = r_t B_t dt\,, & B_0 = 1 \end{cases}$$

$\mu_t, r_t \in \mathbb{R}$, $\sigma_t^i \in \mathbb{R}^d$ bounded and $\mathcal{F}_t$-predictable stochastic processes

- Postulate existence of an $\mathcal{F}_t$-predictable stochastic process $\lambda_t \in \mathbb{R}^d$ satisfying

$$\mu_t - r_t\,\mathbb{1} = \sigma_t^T \lambda_t$$

- No assumptions on market completeness

Market environment

- Self-financing investment strategies π_t^0, $\pi_t = (\pi_t^1, \ldots, \pi_t^i, \ldots, \pi_t^k)$
- Present value of this allocation

$$X_t = \sum_{i=0}^{k} \pi_t^i$$

$$dX_t = \sum_{i=1}^{k} \pi_t^i \sigma_t^i \cdot (\lambda_t \, dt + dW_t)$$

$$= \sigma_t \pi_t \cdot (\lambda_t \, dt + dW_t)$$

Investment performance measurement

Investment performance process

$U(x,t)$ is an $\mathcal{F}_t$-predictable process, $\qquad t \geq 0$

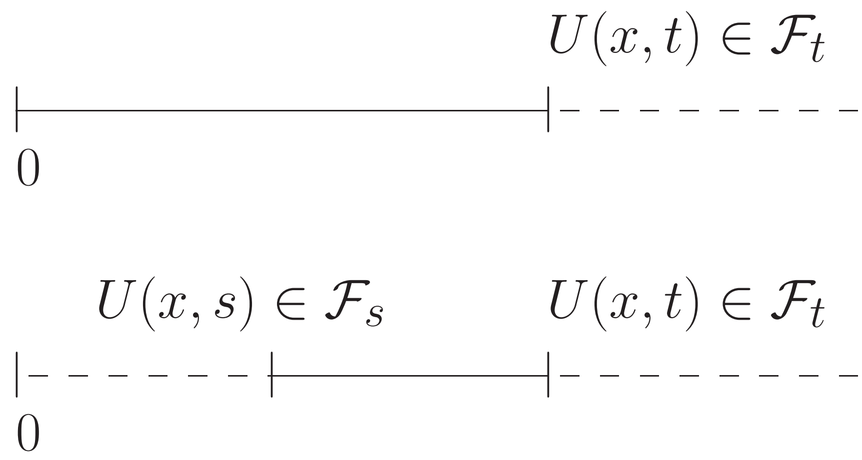
- The mapping $x \rightarrow U(x,t)$ is increasing and concave
- For each self-financing strategy, represented by π, the associated (discounted) wealth X_t^π satisfies

$$E_{\mathbb{P}}(U(X_t^\pi, t) \mid \mathcal{F}_s) \leq U(X_s^\pi, s), \qquad 0 \leq s \leq t$$

- There exists a self-financing strategy, represented by π^*, for which the associated (discounted) wealth $X_t^{\pi^*}$ satisfies

$$E_{\mathbb{P}}(U(X_t^{\pi^*}, t) \mid \mathcal{F}_s) = U(X_s^{\pi^*}, s), \qquad 0 \leq s \leq t$$

Optimality across times



$$U(x,s) = \sup_{\mathcal{A}} E(U(X_t^{\pi}, t) | \mathcal{F}_s,\ X_s = x)$$

- What is the meaning of this process?
- Does such a process aways exist?
- Is it unique?
- Axiomatic construction?

Forward performance process

A datum $u_0(x)$ is assigned at the beginning of

the trading horizon, $t = 0$

$$U(x, 0) = u_0(x)$$

Forward in time criteria

$$E_{\mathbb{P}}(U(X_t^{\pi}, t) | \mathcal{F}_s) \leq U(X_s^{\pi}, s), \qquad 0 \leq s \leq t$$

$$E_{\mathbb{P}}(U(X_t^{\pi^*}, t) | \mathcal{F}_s) = U(X_s^{\pi^*}, s), \qquad 0 \leq s \leq t$$

Many difficulties due to "inverse in time"

nature of the problem

The forward performance SPDE

The forward performance SPDE

Let $U(x,t)$ be an $\mathcal{F}_t-$predictable process such that the mapping $x \rightarrow U(x,t)$ is increasing and concave. Let also $U = U(x,t)$ be the solution of the stochastic partial differential equation

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}(U\lambda + a)\right|^{2}}{\mathcal{A}^{2}U}dt + a \cdot dW$$

where $a = a(x,t)$ is an $\mathcal{F}_t-$predictable process, while $\mathcal{A} = \frac{\partial}{\partial x}$.

Then $U(x,t)$ is a forward performance process.

The volatility of the investment performance process

This is the novel element in the new approach of forward investment performance measurement

- The volatility models how the current shape of the performance process is going to diffuse in the next trading period
- The volatility is up to the investor to choose, in contrast to the classical approach in which it is uniquely determined via the backward construction of the value function process
- When the volatility is not state-dependent, we are essentially in the zero volatility case
- The volatility process results in non-myopic portfolios
- The volatility's dependence on the risk premium is intriguing
- The process a may depend on $t, x, U,$ its spatial derivatives etc.

Specifying the appropriate class of volatility processes is very challenging but extremely didactic!

Optimal portfolios and wealth

At the optimum

- The optimal portfolio vector π^* is given in the feedback form

$$\pi_t^* = \pi^*\left(X_t^*, t\right) = -\sigma^+ \frac{\mathcal{A}\left(U\lambda + a\right)}{\mathcal{A}^2 U}\left(X_t^*, t\right)$$

- The optimal wealth process X^* solves

$$dX_t^* = -\sigma\sigma^+ \frac{\mathcal{A}\left(U\lambda + a\right)}{\mathcal{A}^2 U}\left(X_t^*, t\right)\left(\lambda dt + dW_t\right)$$

Solutions to the forward performance SPDE

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U\lambda + a\right)\right|^{2}}{\mathcal{A}^{2}U}\,dt + a\cdot dW$$

Local differential coefficients

$$a\left(x,t\right) = F\left(x,t,U\left(x,t\right),U_{x}\left(x,t\right)\right)$$

Difficulties

- The equation is fully nonlinear
- The diffusion coefficient depends, in general, on U_x and U_{xx}
- The equation is not (degenerate) elliptic

The zero volatility case: $a(x,t) \equiv 0$

Space-time monotone performance process

The forward performance SPDE simplifies to

$$dU = \frac{1}{2}\frac{\left|\sigma\sigma^{+}\mathcal{A}\left(U\lambda\right)\right|^{2}}{\mathcal{A}^{2}U}dt$$

The process

$$U\left(x,t\right) = u\left(x,A_{t}\right) \quad \text{with} \quad A_{t} = \int_{0}^{t}\left|\sigma_{s}\sigma_{s}^{+}\lambda_{s}\right|^{2}ds$$

with $u : \mathbb{R} \times [0,+\infty) \rightarrow \mathbb{R}$, increasing and concave with respect to x, and solving

$$u_{t}u_{xx} = \frac{1}{2}u_{x}^{2}$$

is a solution.

MZ (2006)

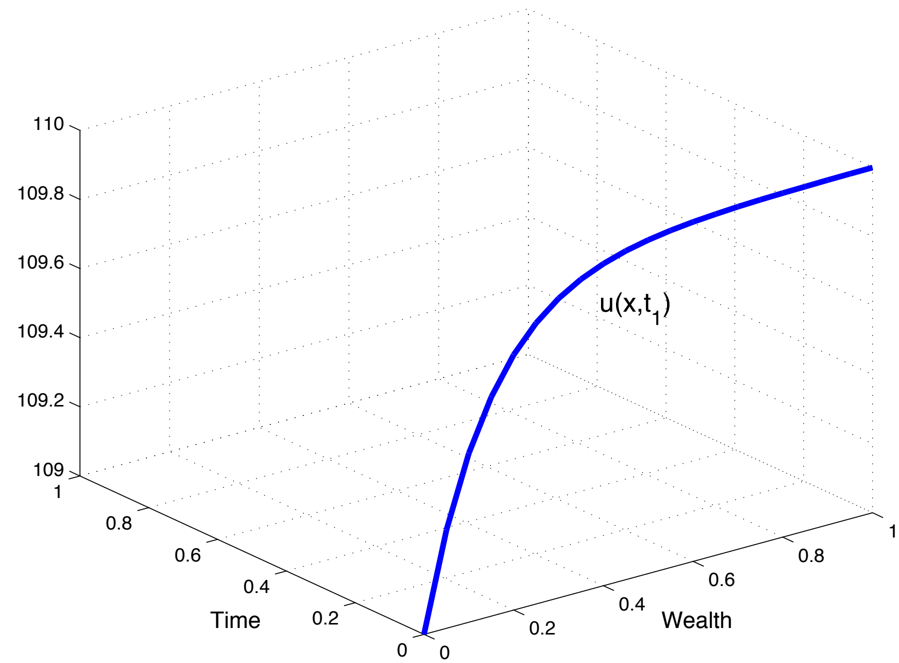
Berrier, Rogers and Tehranchi (2009)

Performance measurement

time t_1, information $\mathcal{F}_{t_1}$

risk premium

$$A_{t_1} = \int_0^{t_1} |\lambda|^2 \, ds \quad \longrightarrow$$



$$A_{t_1} \quad \dashrightarrow \quad + \quad \dashleftarrow \quad u(x, t_1)$$

$$\Downarrow$$

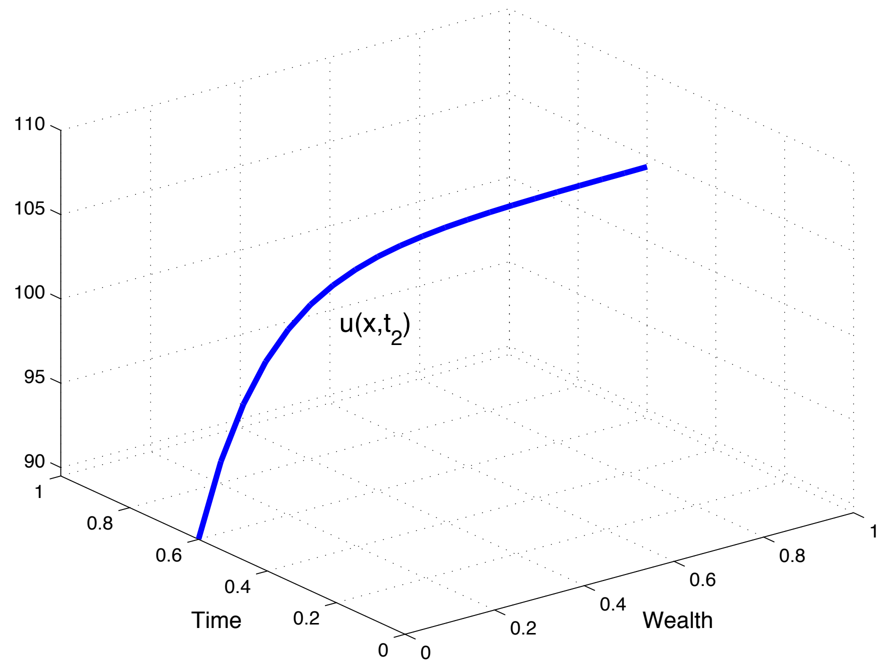
$$\boxed{U(x, t_1) = u(x, A_{t_1}) \in \mathcal{F}_{t_1}}$$

Performance measurement

time t_2, information $\mathcal{F}_{t_2}$

risk premium

$$A_{t_2} = \int_0^{t_2} |\lambda|^2 \, ds \quad \longrightarrow$$



$$A_{t_2} \dashrightarrow + \dashleftarrow u(x,t_2)$$

$$\Downarrow$$

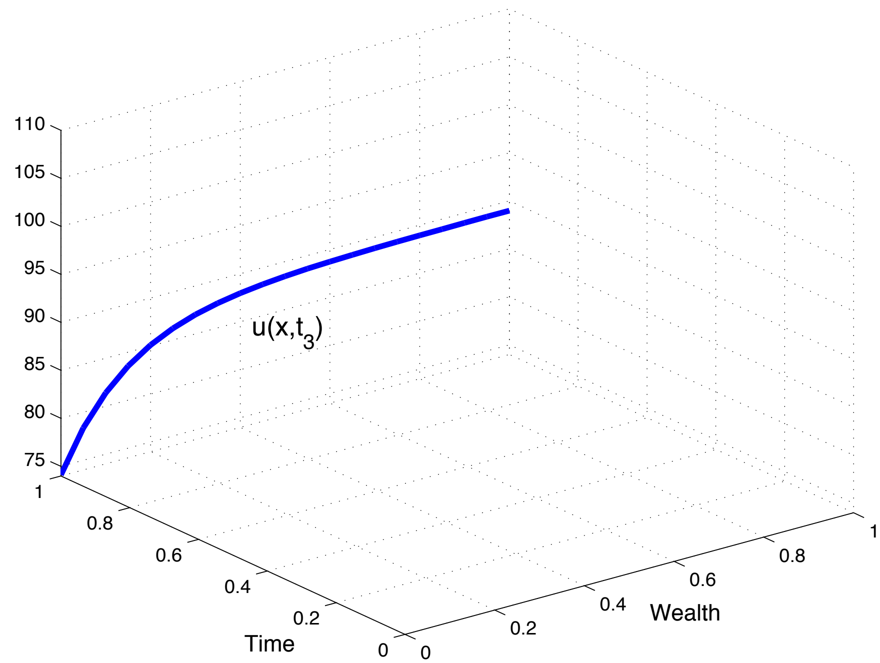
$$\boxed{U(x,t_2) = u(x, A_{t_2}) \in \mathcal{F}_{t_2}}$$

Performance measurement

time t_3, information $\mathcal{F}_{t_3}$

risk premium

$$A_{t_3} = \int_0^{t_3} |\lambda|^2 \, ds \quad \longrightarrow$$



$$A_{t_3} \quad \dashrightarrow \quad + \quad \dashleftarrow \quad u(x, t_3)$$

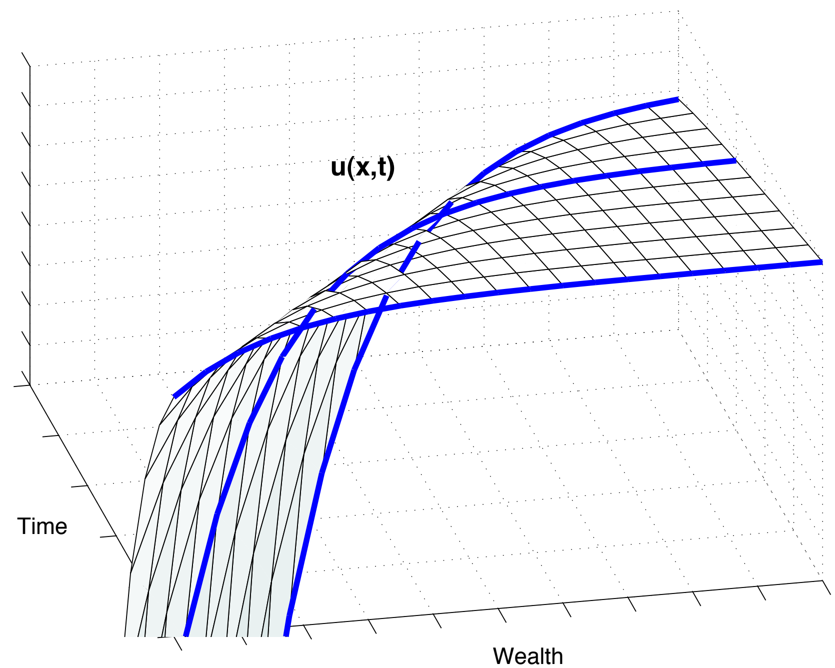
$$\Downarrow$$

$$\boxed{U(x, t_3) = u(x, A_{t_3}) \in \mathcal{F}_{t_3}}$$

Forward performance measurement

time t, information $\mathcal{F}_t$

market



$$MI(t) \quad \dashrightarrow \quad + \quad \dashleftarrow \quad u(x,t)$$

$$\Downarrow$$

$$\boxed{U(x,t) = u(x, A_t) \in \mathcal{F}_t}$$

Properties of the performance process

$$U(x,t) = u(x, A_t)$$

- $U(x,t)$ is decreasing in time if $\lambda \neq 0$

- the deterministic risk preferences $u(x,t)$ are compiled with the stochastic market input $A_t = \int_0^t |\lambda|^2 \, ds$

- the evolution of preferences is "deterministic"

- if $\lambda = 0, \quad U(x,t) = U(x,0)$

- if λ large, the investor is heavily penalized if he does not invest

Optimal allocations

Optimal allocations

- Let X_t^* be the optimal wealth, and A_t the time-rescaling processes

$$dX_t^* = \sigma_t \pi_t^* \cdot (\lambda_t dt + dW_t)$$
$$dA_t = |\lambda_t|^2 dt$$

- Define

$$R_t^* \triangleq r(X_t^*, A_t) \qquad r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

Optimal portfolios

$$\pi_t^* = \sigma_t^+ \lambda_t R_t^*$$

The optimal portfolio is always myopic

A system of SDEs at the optimum

$$\begin{cases} dX_t^* = r(X_t^*, A_t)\lambda_t \cdot (\lambda_t\, dt + dW_t) \\ dR_t^* = r_x(X_t^*, A_t) dX_t^* \end{cases}$$

$$\pi_t^* = \sigma_t^+ \lambda_t R_t^*$$

The optimal wealth and portfolios are explicitly constructed

if the function $r(x,t)$ is known.

Should we model $r(x,t)$ instead of $u(x,t)$?

Concave utility inputs and increasing harmonic functions

Concave utility inputs and increasing harmonic functions

There is a one-to-one correspondence between strictly concave solutions $u(x,t)$ to

$$u_t = \frac{1}{2}\,\frac{u_x^2}{u_{xx}}$$

and strictly increasing solutions to

$$h_t + \frac{1}{2}h_{xx} = 0$$

via the Cole-Hopf transformation

$$u(h(x,t),t) = e^{-x+\frac{t}{2}}$$

Concave utility inputs and increasing harmonic functions

$$x \longrightarrow h(x,t) \longrightarrow u(h(x,t),t)$$

- The harmonic function $h(x,t)$ is defined on $\mathbb{R} \times [0 + \infty)$ and represents the investor's wealth
- The range of $h(x,t)$ reflects the wealth state constraints (e.g., wealth bounded from below)
- The harmonic function $h(x,t)$ is increasing in x

Concave utility inputs and increasing harmonic functions

- If $h(x,t)$ harmonic then $h_x(x,t)$ is also harmonic

$$(h_x)_t + \frac{1}{2}(h_{xx})_x = 0$$

- Because $h_x(x,t)$ is positive harmonic it can be represented via Widder's theorem as

$$h_x(x,t) = \int_{\mathbb{R}} e^{xy - \frac{1}{2}y^2 t} \nu(dy)$$

- The wealth function $h(x,t)$ is then constructed from $h_x(x,t)$

- Boundary and asymptotic behavior of $h_x(x,t)$ not obvious

Concave utility inputs and increasing harmonic functions

The measure ν becomes the defining element

- Its support plays a key role in the form of the range of $h(x,t)$ and, as a result, in the form of the domain and range of $u(x,t)$ as well as in its asymptotic behavior (Inada conditions)
- It defines the class of initial conditions
- Can it be inferred from the investor's desired investment targets?

Range of $h(x,t) = (-\infty, +\infty)$

Assumption on ν: $\displaystyle\int_{-\infty}^{+\infty} e^{yx}\nu(dy) < +\infty \ , \quad x \in \mathbb{R}$

- $\nu(\{0\}) > 0$
- $\nu(\{0\}) = 0 \ , \quad \nu\Big((-\infty, 0)\Big) = 0 \ , \quad \nu\Big((0, +\infty)\Big) > 0$ and $\displaystyle\int_{0^+}^{+\infty} \frac{\nu(dy)}{y} = +\infty$
- $\nu(\{0\}) = 0 \ , \quad \nu\Big((0, +\infty)\Big) = 0 \ , \quad \nu\Big((-\infty, 0)\Big) > 0$ and $\displaystyle\int_{-\infty}^{0^-} \frac{\nu(dy)}{y} = -\infty$

Concave utility inputs and increasing harmonic functions

- Increasing harmonic function $h : \mathbb{R} \times [0, +\infty) \longrightarrow \mathbb{R}$ is represented as

$$h(x,t) = \int_{\mathbb{R}} \frac{e^{yx - \frac{1}{2}y^2 t} - 1}{y} \nu(dy)$$

- The associated utility input $u : \mathbb{R} \times [0, +\infty) \longrightarrow \mathbb{R}$ is then given by the concave function

$$u(x,t) = -\frac{1}{2} \int_0^t e^{-h^{(-1)}(x,s) + \frac{s}{2}} h_x\left(h^{(-1)}(x,s), s\right) ds + \int_0^x e^{-h^{(-1)}(z,0)} dz$$

Measure ν has compact support

$\nu(dy) = \delta_0$, where δ_0 is a Dirac measure at 0

Then,

$$h\left(x,t\right) = \int_{\mathbb{R}} \frac{e^{yx-\frac{1}{2}y^2t}-1}{y}\delta_0 = x$$

and

$$u\left(x,t\right) = -\frac{1}{2}\int_0^t e^{-x+\frac{s}{2}}ds + \int_0^x e^{-z}dz = 1 - e^{-x+\frac{t}{2}}$$

Measure ν has compact support

$$\nu(dy) = \frac{b}{2}\left(\delta_a + \delta_{-a}\right), \quad a, b > 0$$

$\delta_{\pm a}$ is a Dirac measure at $\pm a$

Then,

$$h(x,t) = \frac{b}{a} e^{-\frac{1}{2}a^2 t} \sinh(ax)$$

If, $a = 1$, then

$$u(x,t) = \frac{1}{2}\left(\ln\left(x + \sqrt{x^2 + b^2 e^{-t}}\right) - \frac{e^t}{b^2} x\left(x - \sqrt{x^2 + b^2 e^{-t}}\right) - \frac{t}{2}\right)$$

If $a \neq 1$, then

$$u(x,t) = \frac{\left(\sqrt{a}\right)^{1+\frac{1}{\sqrt{a}}}}{a-1}\, e^{\frac{1-\sqrt{a}}{2}t}\, \frac{\frac{\beta}{\sqrt{a}} e^{-at} + (1+\sqrt{a}\,)x\left(\sqrt{a}\,x + \sqrt{ax^2 + \beta e^{-at}}\right)}{\left(\sqrt{a}\,x + \sqrt{ax^2 + \beta e^{-at}}\right)^{1+\frac{1}{\sqrt{a}}}}$$

Measure ν has infinite support

$$\nu(dy) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}y^2}\, dy$$

Then

$$h(x,t) = F\left(\frac{x}{\sqrt{t+1}}\right) \qquad\qquad F(x) = \int_0^x e^{\frac{z^2}{2}}\, dz$$

and

$$u(x,t) = F\left(F^{(-1)}(x) - \sqrt{t+1}\,\right)$$

Range of $h(x,t) = (0,+\infty)$

$$\nu\Big((-\infty,0)\Big) = 0\ , \quad \nu(\{0\}) = 0\ , \quad \nu\Big((0,+\infty)\Big) > 0$$

$$\int_{0^+}^{+\infty} \frac{\nu(dy)}{y}\ \nu(dy) < +\infty$$

Then

$$h(x,t) = \int_{0^+}^{+\infty} \frac{e^{yx-\frac{1}{2}y^2 t}}{y}\ \nu(dy)$$

$$h_t(x,t) < 0 \quad \text{and} \quad h_{xx}(x,t) > 0$$

Range of $h(x,t) = (0,+\infty)$

$$\nu\Big((0,1]\Big) = 0 \quad \text{and} \quad \int_{1^+}^{+\infty} \frac{\nu(dy)}{y-1} < +\infty$$

$$h(x,t) = \int_{1^+}^{+\infty} \frac{e^{yx-\frac{1}{2}y^2 t}}{y}\, \nu(dy)$$

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}}\, h_x(h^{(-1)}(x,s),s)\,ds + \int_0^x e^{-h^{(-1)}(z,0)}\,dz$$

$$\lim_{x\to 0} u(x,t) = 0\ , \quad \lim_{x\to 0} u_x(x,t) = +\infty\ , \quad \lim_{x\to+\infty} u_x(x,t) = 0$$

Example when $\nu\Big((0,1]\Big) = 0$

$$\nu(dy) = \delta_\gamma \ , \qquad \gamma > 1$$

$$h(x,t) = \int_{0^+}^{+\infty} \frac{e^{yx - \frac{1}{2}y^2 t}}{y} \, \nu(dy) = \frac{1}{\gamma} e^{\gamma x - \frac{1}{2}\gamma^2 t}$$

$$u(x,t) = \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{\gamma - 1} \, x^{\frac{\gamma-1}{\gamma}} \, e^{-\frac{\gamma-1}{2}t}$$

Range of $h(x,t) : (0,+\infty)$

$$\nu\Big((0,1]\Big) > 0 \quad \text{or} \quad \nu\Big((0,1]\Big) = 0 \quad \text{and} \quad \int_{1^+}^{+\infty} \frac{\nu(dy)}{y-1} = +\infty$$

$$h(x,t) = \int_{0^+}^{+\infty} \frac{e^{yx-\frac{1}{2}y^2 t}}{y}\, \nu(dy)$$

$$u(x,t) = -\frac{1}{2}\int_0^t e^{-h^{(-1)}(x,s)+\frac{s}{2}}\, h_x(h^{(-1)}(x,s),s)\, ds + \int_{x_0}^{x} e^{-h^{(-1)}} dz\ , \qquad x_0 > 0$$

$$\lim_{x\to 0} u(x,t) = -\infty\ , \quad \lim_{x\to 0} u_x(x,t) = +\infty\ , \quad \lim_{x\to+\infty} u_x(x,t) = 0$$

Examples when $\nu\Big((0,1]\Big) > 0$

- $$\nu(dy) = \delta_\gamma \qquad \gamma = 1$$

$$h(x,t) = \int_{0^+}^{+\infty} \frac{e^{yx - \frac{1}{2}y^2 t}}{y}\, \nu(dy) = e^{x - \frac{1}{2}t}$$

$$u(x,t) = \ln\, \frac{x}{x_0} - \frac{t}{2}$$

- $$\nu(dy) = \delta_\gamma \qquad \gamma \in (0,1)$$

$$u(x,t) = -\frac{\gamma^{\frac{\gamma-1}{\gamma}}}{1-\gamma}\, x^{\frac{\gamma-1}{\gamma}}\, e^{\frac{1-\gamma}{2}t} + \frac{\gamma^{\frac{\gamma-1}{\gamma}}}{1-\gamma}\, x_0^{\frac{\gamma-1}{\gamma}}$$

Optimal processes and increasing harmonic functions

Optimal processes and risk tolerance

$$\begin{cases} dX_t^* = r(X_t^*, A_t)\lambda_t \cdot (\lambda_t \, dt + dW_t) \\ dR_t^* = r_x(X_t, A_t) \, dX_t^* \end{cases}$$

Local risk tolerance function and fast diffusion equation

$$r_t + \frac{1}{2} r^2 r_{xx} = 0$$

$$r(x,t) = -\frac{u_x(x,t)}{u_{xx}(x,t)}$$

Local risk tolerance and increasing harmonic functions

If $h:\mathbb{R}\times[0,+\infty)\longrightarrow\mathbb{R}$ is an increasing harmonic function then

$$r:\mathbb{R}\times[0,+\infty)\longrightarrow\mathbb{R}^{+} \text{ given by}$$

$$r\left(x,t\right)=h_{x}\left(h^{(-1)}\left(x,t\right),t\right)=\int_{\mathbb{R}}e^{yh^{(-1)}(x,t)-\frac{1}{2}y^{2}t}\nu\left(dy\right)$$

is a risk tolerance function solving the FDE

Optimal portfolio and optimal wealth

- Let h be an increasing solution of the backward heat equation

$$h_t + \frac{1}{2} h_{xx} = 0$$

and $h^{(-1)}$ stands for its spatial inverse

- Let the market input processes A and M by defined by

$$A_t = \int_0^t |\lambda_s|^2 \, ds \quad \text{and} \quad M_t = \int_0^t \lambda_s \cdot dW_s$$

- Then the optimal wealth and optimal portfolio processes are given by

$$X_t^{*,x} = h\left(h^{(-1)}(x,0) + A_t + M_t, A_t\right)$$

and

$$\pi_t^* = h_x\left(h^{(-1)}\left(X_t^{*,x}, A_t\right), A_t\right) \sigma_t^+ \lambda_t$$

Complete construction

Utility inputs and harmonic functions

$$u_t = \frac{1}{2}\,\frac{u_x^2}{u_{xx}} \qquad \Longleftrightarrow \qquad h_t + \frac{1}{2}\,h_{xx} = 0$$

Harmonic functions and positive Borel measures

$$h(x,t) \qquad \Longleftrightarrow \qquad \nu(dy)$$

Optimal wealth process

$$X_t^{*,x} = h\left(h^{(-1)}\left(x,0\right) + A_t + M_t, A_t\right) \qquad M = \int_0^t \lambda_s \cdot dW_s, \quad \langle M_t \rangle = A_t$$

Optimal portfolio process

$$\pi_t^{*,x} = h_x\left(h^{(-1)}\left(X_t^{*,x}, A_t\right), A_t\right)\sigma_t^{+}\lambda_t$$

The measure ν emerges as the defining element

$$\nu \quad \Rightarrow \quad h \quad \Rightarrow \quad u$$

How do we choose ν and what does it represent for the investor's risk attitude?

Qualitative analysis of key quantities

Dependence of optimal wealth and investment processes on initial wealth

Differentiating the explicit formulae for X_t^* and π_t^* we deduce

- $$\frac{\partial}{\partial x} X_t^{*,x} = \frac{r(X_t^{*,x}, A_t)}{r(x,0)}$$

- $$\frac{\partial}{\partial x} \pi_t^{*,x} = r_x(X_t^{*,x}, A_t)\, \frac{r(X_t^{*,x}, A_t)}{r(x,0)}\, \sigma_t^{+} \lambda_t$$

- $$\frac{\partial^2}{\partial x^2} X_t^{*,x} = \frac{(r_x(X_t^{*,x}, A_t) - r_x(x,0))}{r(x,0)}\, \frac{\partial}{\partial x}\, X_t^{*,x}$$

Monotonicity of optimal investment strategy on current wealth

$$\pi_t^* = r(X_t^{*,x}, A_t)\sigma_t^+ \lambda_t$$

- If $\nu(dy)$ concentrated in $(0, +\infty)$,

$$r_x(x,t) \geq 0$$

- If $\nu(dy)$ concentrated in $(-\infty, 0)$,

$$r_x(x,t) \leq 0$$

Inferring investor's preferences

Calibration of risk preferences to the market

Given the desired distributional properties of his/her optimal wealth in a specific market environment, what can we say about the investor's risk preferences?

Investor's investment targets

- Desired future expected wealth

- Desired distribution

References

Sharpe (2006)

Sharpe-Golstein (2005)

Distributional properties of the optimal wealth process

The case of deterministic market price of risk

Using the explicit representation of $X^{*,x}$ we can compute the distribution, density, quantile and moments of the optimal wealth process.

- $$\mathbb{P}\left(X_t^{*,x} \leq y\right) = N\left(\frac{h^{(-1)}(y, A_t) - h^{(-1)}(x, 0) - A_t}{\sqrt{A_t}}\right)$$

- $$f_{X_t^{*,x}}(y) = n\left(\frac{h^{(-1)}(y, A_t) - h^{(-1)}(x, 0) - A_t}{\sqrt{A_t}}\right)\frac{1}{r(y, A_t)}$$

- $$y_p = h\left(h^{(-1)}(x, 0) + A_t + \sqrt{A_t}N^{(-1)}(p), A_t\right)$$

Properties of the expected optimal wealth

- $$EX_t^{*,x} = h(h^{(-1)}(x,0) + A_t, 0)$$

- $$\frac{\partial}{\partial x} E(X_t^{*,x}) = \frac{r(E(X_t^{*,x}),0)}{r(x,0)} = \frac{r(h(h^{(-1)}(x,0) + A_t, 0), 0)}{r(x,0)}$$

- $$E(r(X_t^{*,x}, A_t)) = r(E(X_t^{*,x}), 0)$$

Target: The mapping $x \to E\left(X_t^{*,x}\right)$ is linear, for all $x > 0$.

Then, there exists a positive constant $\gamma > 0$ such that the investor's forward performance process is given by

$$U(x,t) = \frac{\gamma}{\gamma - 1} x^{\frac{\gamma-1}{\gamma}} e^{-\frac{1}{2}(\gamma-1)A_t}, \quad \text{if} \quad \gamma \neq 1$$

and by

$$U_t(x) = \ln x - \frac{1}{2} A_t, \quad \text{if} \quad \gamma = 1$$

Moreover,

$$E\left(X_t^{*,x}\right) = x e^{\gamma A_t}$$

Calibrating the investor's preferences consists of choosing a time horizon, T, and the level of the mean, mx $(m > 1)$.Then, the corresponding γ must solve $xe^{\gamma A_T} = mx$ and, thus, is given by

$$\gamma = \frac{\ln m}{A_T}$$

The investor can calibrate his expected wealth only for a single time horizon.

Relaxing the linearity assumption

- The linearity of the mapping $x \rightarrow E\left(X_t^{*,x}\right)$ is a very strong assumption. It only allows for calibration of a single parameter, namely, the slope, and only at a single time horizon.

- Therefore, if one intends to calibrate the investor's preferences to more refined information, then one needs to accept a more complicated dependence of $E\left(X_t^{*,x}\right)$ on x.

Target: Fix x_0 and consider calibration to $E\left(X_t^{*,x_0}\right)$, for $t \geq 0$

The investor then chooses an increasing function $m(t)$ (with $m(t) > 1$) to represent $E\left(X_t^{*,x_0}\right)$,

$$E\left(X_t^{*,x_0}\right) = m(t), \quad \text{for } t \geq 0.$$

- What does it say about his preferences?

- Moreover, can he choose an arbitrary increasing function $m(t)$?

Relaxing the linearity assumption

For simplicity, assume $x_0 = 1$ and that ν is a probability measure. Then, $h^{(-1)}(1,0) = 0$ and we deduce that

$$E\left(X_t^{*,1}\right) = h\left(A_t, 0\right) = \int_0^\infty e^{yA_t}\nu\left(dy\right)$$

Clearly, the investor may only specify the function $m\left(t\right)$, $t > 0$, which can be represented, for **some** probability measure ν in the form

$$m\left(t\right) = \int_0^\infty e^{yA_t}\nu\left(dy\right)$$

Conclusions

- Space-time monotone investment performance criteria
- Explicit construction of forward performance process
- Connection with space-time harmonic functions
- Explicit construction of the optimal wealth and optimal portfolio processes
- The "trace" measure as the defining element of the entire construction
- Calibration of the trace to the market
- Inference of dynamic risk preferences