Financial markets with uncertain volatility

H. Mete Soner, ETH Zürich
joint with
Nizar Touzi, Ecole Polytechnique, Paris
Jianfeng Zhang, USC

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Outline

1. Introduction
   - Overview
   - Mathematical Formulation
   - Main Result
2. G-expectation
   - "Markov" random variables
   - Closure
3. Proof
   - Construction on $\mathcal{L}_{ip}$
   - Spaces
   - Estimates

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Foreword

In very general terms, we are interested in stochastic analysis simultaneously for a large class of probability measures that are singular to each other. There are several possible applications Today I chose to describe the results in an uncertain volatility for specificity.

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Cheredito, Soner, Touzi, Victoir (CPAM, 2002) on second order backward stochastic differential equations (2BSDE). This connects the BSDEs to fully nonlinear parabolic equations.

Denis & Martini (Annals of Applied Probability, 2006) on quasi-sure stochastic analysis using capacity theory.

Peng (2007) on G-expectations which is the generalization of BSDEs using the viscosity theory.

Uncertain volatility

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Here we consider the problem of hedging in this market. As well known this question is essentially equivalent to the martingale representation theorem.

Motivations and Connections

This problem is a specific problem in a general program.

1. Probabilistic representation of fully nonlinear, scalar, parabolic equations.
2. To generalize the theory of backward stochastic differential equations (BSDE). Indeed, G-expectation of Peng is a direct generalization of g-expectation for BSDEs.
3. Duality theory for stochastic optimal control.
4. Include model uncertainty into convex risk measures.

Model

Let $\Omega := C([0,1] : \boldsymbol{R}^d)$ and $B_t(\omega) = \omega_t$ be the canonical map. In the classical theory, we consider only one measure which is the Wiener measure. However, in our model we consider all measures $\mathbb{P}$ so that under this measure the canonical map is a martingale and the quadratic variation $\langle B \rangle_t$ satisfies

$$\underline{a} \leq \frac{d}{dt}\langle B \rangle_t \leq \overline{a},$$

where $0 \leq \underline{a} \leq \overline{a}$ are given volatility bounds. $\mathcal{P}$ is the set of all such measures.

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We use the filtration generated by B without completing.

Problem of super-replication

Assume a bounded random variable $\xi \in \mathcal{F}_1$ is given. The minimal super-replication cost is given by

$$v(\xi) := \inf \left\{ x | \exists\ H \text{ s.t., } x + \int_0^1 H_s dB_s \geq \xi,\ \mathbb{P} - a.s.,\ \forall\ \mathbb{P} \in \mathcal{P} \right\}.$$

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Here the difficulty emanates from the fact that $\mathcal{P}$ contains uncountably many measures that are singular to each other. Moreover, we want to describe the hedge for all measures and not only for the "optimal" one.

Description of the value

For any $\mathbb{P} \in \mathcal{P}$ and $x > v(\xi)$, we have $x + \int_0^1 H_s dB_s \geq \xi$ for some Z. Take expected value $x \geq \mathbb{E}^{\mathbb{P}}[\xi]$, forall $\mathbb{P} \in \mathcal{P}$, and $x > v(\xi)$. Hence,

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By analogy with the results of El Karoui & Quenez (1993) for super-replication in an incomplete market, it is proved by Denis & Martini and Denis, Hu & Peng (2009) that they are equal :

$$v(\xi) = \mathbb{E}^{G}[\xi] := \sup_{\mathcal{P}} \ \mathbb{E}^{\mathbb{P}}[\xi],$$

where $\mathbb{E}^{G}$ is the G-expectation of Peng.

Martingale Representation

Theorem

There are (Y_t, Z_t, K_t) *satisfying for all* $\mathbb{P} \in \mathcal{P}$

$$\begin{aligned} Y_t := \mathbb{E}_t^G[\xi] &= \xi - \int_t^1 H_s dB_s + K_1 - K_t, \qquad \mathbb{P} - a.s. \\ &= \mathbb{E}^G[\xi] + \int_0^t H_s dB_s - K_t, \qquad \mathbb{P} - a.s., \end{aligned}$$

where $\mathbb{E}_t^G$ *is the* G*-conditional expectation of Peng. Moreover,* K_t *is non-decreasing with* $K_0 = 0$ *and it is minimal in the sense that it is a* G*-martingale :*

$$K_t = \mathbb{E}_t^G[K_s], \qquad 0 \le t \le s \le 1.$$

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PDE

Peng used the following partial differential equation

$$-\partial_t u - G(D^2 u) = 0 \quad \text{on } [0,1),$$

where for given symmetric matrices $0 \le \underline{a} \le \overline{a}$, the nonlinearity G is defined by,

$$G(\gamma) := \frac{1}{2} \sup\{\ [\gamma : a] \mid \underline{a} \le a \le \overline{a}\}.$$

This is the dynamic programming equation for the optimal control problem in which the diffusion coefficient a_t can be chosen in the interval $\underline{a} \le a_t \le \overline{a}$.

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This equation was obtained by Avellaneda & Panas (1995) and named it Black-Scholes-Barenblatt equation.

$\xi = \varphi(B_1)$

Assume that φ is bounded and Lipschitz continuous on $\mathbb{R}^d$. Let u be the unique bounded viscosity solution of the following parabolic equation,

$$-u_t - G(D^2 u) = 0 \quad \text{on } [0,1), \quad \text{and } u(1,x) = \varphi(x).$$

Then, Peng defines the conditional G-expectation of the random variable $\varphi(B_1)$ at t by

$$\mathbb{E}_t^G\left[\varphi(B_1)\right] := u\left(t, B_t\right).$$

In particular, the G-expectation of $\varphi(B_1)$ is given by

$$\mathbb{E}^G[\varphi(B_1)] := \mathbb{E}_0^G[\varphi(B_1)] = u(0,0).$$

$\xi = \varphi(B_{t_1}, \ldots, B_1)$

We essentially proceed the same way. Indeed, in the interval (t_{i-1}, t_i), let

$$\mathbb{E}_t^G[\xi] = \mathbb{E}_t^G[\varphi(B_{t_1}, \ldots, B_{t_n})] := v_i(t, B_{t_1}, \ldots, B_{t_{i-1}}, B_t),$$

where $\{v_i\}_{i=1,\ldots,n-1}$ is the unique, bounded, Lipschitz viscosity solution of the following equation,

$$\begin{aligned} -\partial_t v_i - G\left(D^2 v_i\right) &= 0, \qquad t_{i-1} \le t < t_i \qquad \text{and} \\ v_i\left(t_i, x_1, \ldots, x_{i-1}, x\right) &= v_{i+1}\left(t_i, x_1, \ldots, x_{i-1}, x, x\right), \end{aligned}$$

and v_n solves the above equation with final data $v_n(1, x_1, \ldots, x_{n-1}, x) = \varphi(x_1, \ldots, x_{n-1}, x)$.

The space $\mathcal{L}^p_G$

Let $\mathcal{L}_{ip}$ be the space of all random variables $\xi = \varphi(B_{t_1}, \ldots, B_1)$ with a bounded and Lipschitz φ. Then on $\mathcal{L}_{ip}$ we have just defined $\mathbb{E}^G_t$. Therefore, on $\mathcal{L}_{ip}$ we can also define a semi-norm

$$\|\xi\|^p_{\mathcal{L}^p_G} := \mathbb{E}^G[\ |\xi|^p\ ].$$

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$$\|\xi\|^p_{\mathcal{L}^p_G} := \mathbb{E}^G[\ |\xi|^p\ ].$$

We then define the integrability class $\mathcal{L}^p_G$ as the closure of $\mathcal{L}_{ip}$ under this norm.

It is clear that G-expectations can be defined on $\mathcal{L}^p_G$ by a closure argument.

Dual formula

As mentioned earlier, Denis, Martini & Peng proved that

$$\|\xi\|^p_{\mathcal{L}^p_G} := \mathbb{E}^G[\ |\xi|^p\ ] = \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}[|\xi|^p].$$

Moreover, for any $\xi \in \mathcal{L}^1_G$ and $t \in [0,1]$,

$$\mathbb{E}^G_t[\xi] = \operatorname*{esssup}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\ \xi \mid \mathcal{F}_t], \qquad \mathbb{P}-a.s.,\ \forall\ \mathbb{P}\in\mathcal{P},$$

where

$$\mathcal{P}(t,\mathbb{P}) := \left\{\mathbb{P}'\in\mathcal{P} :\ \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t\right\}.$$

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$\xi = \varphi(B_1)$

Assume that there is a smooth solution u of

$$-u_t - G(D^2 u) = 0 \;\text{ on } [0,1), \;\text{ and } u(1,x) = \varphi(x).$$

Then, we define on all of Ω

$$\begin{aligned} Y_t &:= \mathbb{E}_t^G[\xi] = u(t, B_t) \\ H_t &:= \nabla u(t, B_t) \\ K_t &:= \int_0^t \left(G(D^2 u(s, B_s)) - \frac{1}{2}\left[\hat{a}_s : D^2 u(s, B_s)\right] \right) ds, \end{aligned}$$

where $\hat{a}_t := d\langle B\rangle_t / dt$ is the universal quadratic variation process.

$\xi = \varphi(B_1)$ continued

By a direct application of the Ito's formula and the PDE, we conclude that for every $\mathbb{P} \in \mathcal{P}$,

$$dY_t = H_t dB_t - dK_t, \qquad \mathbb{P} - a.s..$$

Hence the martingale representation is proved once we show the properties of the process K. Then the definition of G yields G

$$\begin{aligned} dK_t &= G(D^2 u(s, B_s)) - \frac{1}{2} \left[\hat{a}_s : D^2 u(s, B_s) \right] \\ &= \frac{1}{2} \left[\sup_{\underline{a} \le a \le \overline{a}} (a : D^2 u(s, B_s)) - (\hat{a}_s : D^2 u(s, B_s)) \right]. \end{aligned}$$

Since for $\mathbb{P} \in \mathcal{P}$, $\underline{a} \le \hat{a}_s \le \overline{a}$, $dK_t \ge 0$.

In general

For $\xi = \varphi(B_{t_1}, \ldots, B_1)$, we use the same construction.

To simplify, I assumed that the PDE has a smooth solution. This may not be the case and a routine approximation procedure is needed.

Norms

For $p \geq 1$, and a non-negative $\xi \in \mathcal{L}_{ip}$,

$$\begin{aligned} \|\xi\|^p_{\mathbb{L}^p_{\mathcal{P}}} &:= \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{esssup}_{t\in[0,1]} \left(M^{\mathbb{P}}_t(\xi)\right)^p\right], \\ M^{\mathbb{P}}_t(\xi) &:= \operatorname*{esssup}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}\left[\xi|\mathcal{F}_t\right]. \end{aligned}$$

Let $\mathbb{L}^p$ be the closure of $\mathcal{L}_{ip}$ under this norm.

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\end{aligned}$$

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We prove the martingale representation for $\xi \in \mathbb{L}^2$.

Norms

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$$\|\xi\|^p_{\mathbb{L}^p_{\mathcal{P}}} := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{esssup}_{t\in[0,1]} \left(M_t^{\mathbb{P}}(\xi)\right)^p\right],$$

$$M_t^{\mathbb{P}}(\xi) := \operatorname*{esssup}_{\mathbb{P}'\in\mathcal{P}(t,\mathbb{P})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{F}_t].$$

Let $\mathbb{L}^p$ be the closure of $\mathcal{L}_{ip}$ under this norm.

We prove the martingale representation for $\xi \in \mathbb{L}^2$.

For every $p > 2$, $\mathcal{L}^p_G \subset \mathbb{L}^2$. Moreover, when $\mathcal{P}$ is a singleton this norm is equivalent to the usual L^2 norm due to the BDG inequality.

Norms, continued

For an adapted integrand H and a stochastic process Y, we set

$$\|H\|^p_{\mathbb{H}^p} := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\left(\int_0^1 (d\langle B\rangle_t H_t \cdot H_t)\right)^{\frac{p}{2}}\right],$$

$$\|Y\|^p_{\mathbb{S}^p} := \sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\sup_{0\le t\le 1} |Y_t|^p\right].$$

If $Y_t = \mathbb{E}^G_t[|\xi|]$ for some $\xi \in \mathcal{L}^1_G$, then $\|Y\|^p_{\mathbb{S}^p} = \|\xi\|^p_{\mathbb{L}^p}$. This identity also motivates the definition of the norm $\mathbb{L}^p$.

Spaces

- $\mathbb{H}^p$ is the set of all integrands with a finite $\|\cdot\|_{\mathbb{H}^p}$-norm,
- $\bar{\mathcal{H}}_G^2$ is the closure of $\mathcal{H}_G^2$ under the norm $\|\cdot\|_{\mathbb{H}^2}$,
- $\mathbb{S}^p$ is the set of all *continuous* processes with finite $\|\cdot\|_{\mathbb{S}^p}$-norm,
- $\mathbb{I}^p$ is the set of all *continuous and non-decreasing* processes with finite $\|\cdot\|_{\mathbb{S}^p}$-norm and $X_0 = 0$.

Exact statement

Now I can be precise about the spaces in the theorem. Recall

$$dY_t = H_t dB_t - dK_t, \qquad Y_1 = \xi.$$

- $\xi \in \mathbb{L}^2$ (This is an assumption).
- $H \in \bar{\mathcal{H}}_G^2$ and the stochastic integral is define quasi-surely.
- $Y \in \mathbb{S}^p$.
- $K \in \mathbb{I}^p$ and $-K$ is a G-martingale.

A priori estimate

$$dY_t = H_t dB_t - dK_t, \qquad Y_1 = \xi.$$

Following the techniques in El Karoui, Kapoudjian, Pardoux, Peng, Quenez (1997), we get the following estimate

$$\|H\|_{\mathbb{H}^2} + \|K\|_{\mathbb{S}^2} \leq C\|Y\|_{\mathbb{S}^2} \leq C\|\xi\|_{\mathbb{L}^2}.$$

Difference estimate

$$dY^i_t = H^i_t dB_t - dK^i_t, \qquad Y^i_1 = \xi^i, \quad i = 1, 2.$$

Set $\delta Y := Y^1 - Y^2$, $\delta H := H^1 - H^2$, $\delta K := K^1 - K^2$. The following estimate is used repeatedly in the proofs.

$$\|\delta H\|^2_{\mathbb{H}^2} + \|\delta K\|^2_{\mathbb{S}^2} \le C\left[\|\delta Y\|^2_{\mathbb{S}^2} + \|\delta Y\|_{\mathbb{S}^2}\left(\|K^1\|_{\mathbb{S}^2} + \|K^2\|_{\mathbb{S}^2}\right)\right].$$

Moreover

$$\|\delta Y\|_{\mathbb{S}^2} \le \|\delta\xi\|_{\mathbb{L}^2}.$$

Proof of Theorem

For $\xi \in \mathbb{L}^2$, there is a sequence $\xi_n \in \mathcal{L}_{ip}$ converging to ξ in the $\mathbb{L}^2$-norm.

For $\xi_n \in \mathcal{L}_{ip}$ we have the representation

$$dY_t^n = H_t^n dB_t - dK_t^n, \qquad Y_1^n = \xi_n.$$

The previous estimates imply that all the above processes converge in the appropriate spaces. We then check that all properties also remain under this convergence.