Calibration of Stock Betas from Skews of Implied Volatilities

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Capital Asset Pricing Model

The original **discrete time CAPM model** defined the log price return on **individual asset** R_a as a linear function of the risk free interest rate R_f, the log return of the **market** R_M, and a Gaussian error term:

$$R_a - R_f \quad = \quad \beta_{\mathbf{a}}(R_M - R_f) + \epsilon_a$$

The beta coefficient $\beta_{\mathbf{a}}$ was originally estimated **using historical** returns on the asset and market index, by a simple linear regression of asset returns on market returns.

Fundamental flaw: it is inherently **backward looking**, and used in **forward looking portfolio** construction.

Previous Attempt to Forward Looking Betas

Christoffersen, Jacobs, and Vainberg (2008, McGill University, Canada) have attempted to extract the beta parameter from option prices on the underlying market and asset processes:

$$\beta_a = \left(\frac{SKEW_a}{SKEW_M}\right)^{\frac{1}{3}} \left(\frac{VAR_a}{VAR_M}\right)^{\frac{1}{2}},$$

where VAR_a (resp. VAR_M), and $SKEW_a$ (resp. $SKEW_M$) are the **variance**, and the **risk-neutral skewness** of returns of the asset (resp. of the market).

Then, they use results from Carr and Madan (2001) which relate these moments to options prices (**Quad** and **Cubic**), the **Call-transform**: $\mathbb{E}^\star\{h(S_T)\} = e^{rT}\int_0^\infty h''(K)C_{BS}(T,K)dK$

The advantage of this approach is that option prices are inherently **forward looking** on the underlying price processes.

Continuous Time CAPM

The market price M_t and an asset price X_t evolve as follows:

$$\begin{aligned}
\frac{dM_t}{M_t} &= \mu dt + \sigma_m dW_t^{(1)}, \\
\frac{dX_t}{X_t} &= \beta \frac{dM_t}{M_t} + \sigma dW_t^{(2)},
\end{aligned}$$

for constant positive volatilities σ_m and σ. In this model we assume independence between the Brownian motions driving the market and asset price processes: $d\langle W^{(1)}, W^{(2)}\rangle_t = 0$, so that

$$\begin{aligned}
\frac{Cov\left(\frac{dX_t}{X_t}, \frac{dM_t}{M_t}\right)}{Var\,\frac{dM_t}{M_t}} &= \frac{Cov\left(\beta\frac{dM_t}{M_t} + \sigma dW_t^{(2)}, \frac{dM_t}{M_t}\right)}{Var\,\frac{dM_t}{M_t}} \\
&= \frac{Cov\left(\beta\frac{dM_t}{M_t}, \frac{dM_t}{M_t}\right)}{Var\,\frac{dM_t}{M_t}} = \beta\,.
\end{aligned}$$

Beta Estimation with Constant Volatility CAPM

Observe that the evolution of X_t is given by

$$\frac{dX_t}{X_t} = \beta\mu dt + \beta\sigma_m dW_t^{(1)} + \sigma dW_t^{(2)} \,,$$

that is a **geometric Brownian motion** with **volatility**

$$\sqrt{\beta^2\sigma_m^2 + \sigma^2}$$

Even if this quantity is known, along with the volatility σ_m of the market process, one cannot disentangle β and σ.
Then, one has to rely on historical returns data.

This drawback, along with the fact that **constant volatility does not generate skews**, motivates us to introduce **stochastic volatility** in the model.

Stochastic Volatility in Continuous Time CAPM

We introduce a stochastic volatility component to the market price process, that is we replace σ_m by a stochastic process $\sigma_t = f(Y_t)$:

$$\begin{aligned}
\frac{dM_t}{M_t} &= \mu dt + f(Y_t) dW_t^{(1)}, \\
\frac{dX_t}{X_t} &= \beta \frac{dM_t}{M_t} + \sigma dW_t^{(2)}, \\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dZ_t\,.
\end{aligned}$$

In this model, the volatility process is driven by a mean-reverting OU process Y_t with a **large mean-reversion rate $1/\varepsilon$** and the invariant (long-run) distribution $\mathcal{N}(m, \nu^2)$. This model also implies stochastic volatility in the asset price through its dependence on the market return. It allows **leverage**: $d\langle W^{(1)}, Z\rangle_t = \rho\, dt$. However, we continue to assume independence between $W_t^{(2)}$ and the other two Brownian motions $W_t^{(1)}$ and Z_t in order to preserve the interpretation of β.

Pricing Risk-Neutral Measure

The market (or index) and the asset being both **tradable**, their discounted prices need to be **martingales** under a **pricing risk-neutral measure**. Setting

$$Z_t = \rho\, dW_t^{(1)} + \sqrt{1-\rho^2}\, dW_t^{(3)},$$

with $(W_t^{(1)}, W_t^{(2)}, W_t^{(3)})$ being three independent BMs, we write:

$$\begin{aligned}
\frac{dM_t}{M_t} &= r dt + f(Y_t)\left(dW_t^{(1)} + \frac{\mu - r}{f(Y_t)} dt\right), \\
\frac{dX_t}{X_t} &= r dt + \beta f(Y_t)\left(dW_t^{(1)} + \frac{\mu - r}{f(Y_t)} dt\right) + \sigma\left(dW_t^{(2)} + \frac{(\beta-1)r}{\sigma} dt\right), \\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\left[\rho\,\frac{\mu - r}{f(Y_t)} + \sqrt{1-\rho^2}\,\gamma(Y_t)\right] dt \\
&\quad + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\left[\rho\left(dW_t^{(1)} + \frac{\mu - r}{f(Y_t)} dt\right) + \sqrt{1-\rho^2}\left(dW_t^{(3)} + \gamma(Y_t) dt\right)\right].
\end{aligned}$$

Market price of risk and risk-neutral measure

$\gamma(Y_t)$ is a **market price of volatility risk**, and we defined the **combined market price of risk**:

$$\Lambda(Y_t) = \rho\,\frac{\mu - r}{f(Y_t)} + \sqrt{1-\rho^2}\,\gamma(Y_t).$$

Setting

$$\begin{aligned} dW_t^{(1)*} &= dW_t^{(1)} + \frac{\mu - r}{f(Y_t)}dt\,, \\ dW_t^{(2)*} &= dW_t^{(2)} + \frac{(\beta-1)r}{\sigma}dt\,, \\ dW_t^{(3)*} &= dW_t^{(3)} + \gamma(Y_t)dt\,, \end{aligned}$$

by **Girsanov theorem**, there is an equivalent probability $I\!P^{\star(\gamma)}$ such that $(W_t^{(1)*}, W_t^{(2)*}, W_t^{(3)*})$ are **independent BMs** under $I\!P^{\star(\gamma)}$, called the **pricing equivalent martingale measure** and determined by the market price of volatility risk γ.

Dynamics under the risk-neutral measure

Under $I\!P^{\star(\gamma)}$, the model becomes:

$$
\begin{aligned}
\frac{dM_t}{M_t} &= r dt + f(Y_t) dW_t^{(1)*}, \\
\frac{dX_t}{X_t} &= r dt + \beta f(Y_t) dW_t^{(1)*} + \sigma dW_t^{(2)*}, \\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \Lambda(Y_t) dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dZ_t^*, \\
Z_t^* &= \rho W_t^{(1)*} + \sqrt{1-\rho^2}\, W_t^{(3)*}.
\end{aligned}
$$

We take the point of view that by pricing options on the index M and on the particular asset X, the market is **"completing itself"** and indirectly choosing the market price of volatility risk γ.

Market Option Prices

Let $P^{M,\epsilon}$ denote the price of a **European option written on the market index M**, with maturity T and payoff h, evaluated at time $t < T$ with current value $M_t = \xi$. Then, we have

$$P^{M,\epsilon} = I\!E^{*(\gamma)} \left\{ e^{-r(T-t)} h(M_T) \mid \mathcal{F}_t \right\} = P^{M,\epsilon}(t, M_t, Y_t)\,,$$

By the Feynman-Kac formula, the function $P^{M,\epsilon}(t,\xi,y)$ satisfies the **partial differential equation**:

$$\begin{aligned} \mathcal{L}^\epsilon P^{M,\epsilon} &= 0, \\ P^{M,\epsilon}(T,\xi,y) &= h(\xi), \end{aligned}$$

where

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2$$

Operator Notation

$$
\begin{aligned}
\mathcal{L}_0 &= \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \equiv \mathcal{L}_{OU} \\
\mathcal{L}_1 &= \rho\nu\sqrt{2} f(y)\xi \frac{\partial^2}{\partial \xi \partial y} - \nu\sqrt{2}\Lambda(y)\frac{\partial}{\partial y} \\
\mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2} f(y)^2 \xi^2 \frac{\partial^2}{\partial \xi^2} + r(\xi\frac{\partial}{\partial \xi} - \cdot) \equiv \mathcal{L}_{BS}(f(y))
\end{aligned}
$$

Here $\mathcal{L}_{BS}(\sigma)$ denotes the **Black-Scholes operator** with volatility parameter σ.

The next step is to **expand** $P^{M,\epsilon}$ in powers of $\sqrt{\epsilon}$

$$P^{M,\epsilon} = P_0^M + \sqrt{\epsilon}P_1^M + \epsilon P_2^M + \epsilon^{3/2}P_3^M + \cdots$$

Expansion of the solution

Expanding

$$(\frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2)(P_0^M + \sqrt{\epsilon}P_1^M + \epsilon P_2^M + \epsilon^{3/2}P_3^M + \cdots) = 0,$$

one cancel the terms in $1/\varepsilon$ and $1/\sqrt{\varepsilon}$ by choosing **P_0^M and P_1^M independent of y** (observe that $\mathcal{L}_1$ takes derivatives with respect y). The terms of order ε^0 lead to

$$\mathcal{L}_0 P_2^M + \mathcal{L}_2 P_0^M = 0,$$

which is a **Poisson equation** associated with $\mathcal{L}_0$. The **centering condition** for this equation is

$$\langle \mathcal{L}_2 P_0^M \rangle = \langle \mathcal{L}_2 \rangle P_0^M = 0,$$

where $\langle\cdot\rangle$ denotes the **averaging with respect to the invariant distribution of Y_t** with infinitesimal generator $\mathcal{L}_0$.

Leading order term

Noting that

$$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2}\bar{\sigma}^2 \xi^2 \frac{\partial^2}{\partial \xi^2} + r(\xi \frac{\partial}{\partial \xi} - \cdot) = \mathcal{L}_{BS}(\bar{\sigma}),$$

with $\bar{\sigma}^2 = \langle f^2 \rangle$, and imposing the terminal condition $P_0^M(T, \xi) = h(\xi)$, we deduce that P_0^M **is the Black-Scholes price** of the option computed with the **constant effective volatility** $\bar{\sigma}$.

We also have

$$P_2^M = -\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_0^M.$$

so that the terms of order $\sqrt{\varepsilon}$ lead to

$$\mathcal{L}_0 P_3^M + \mathcal{L}_1 P_2^M + \mathcal{L}_2 P_1^M = 0,$$

which is again a **Poisson equation** in P_3^M which requires the **solvability condition** $\langle \mathcal{L}_1 P_2^M + \mathcal{L}_2 P_1^M \rangle = 0$.

Equation for the first correction

$$\langle \mathcal{L}_2 \rangle P_1^M + \langle \mathcal{L}_1 P_2^M \rangle = \langle \mathcal{L}_2 \rangle P_1^M - \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0^M = 0.$$

Therefore P_1^M is the solution to the Black-Scholes equation with constant volatility $\bar{\sigma}$, with a zero terminal condition, and a **source term** given by $\langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0^M$. In order to compute this source term we introduce a solution $\phi(y)$ of the Poisson equation

$$\mathcal{L}_0 \phi(y) = f(y)^2 - \langle f^2 \rangle, \qquad \text{so that}$$

$$\begin{aligned}
\langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle P_0^M &= \left\langle \mathcal{L}_1 \mathcal{L}_0^{-1} \left(\frac{1}{2} (f(y)^2 - \langle f^2 \rangle) \xi^2 \frac{\partial^2}{\partial \xi^2} \right) \right\rangle P_0^M \\
= \left\langle \mathcal{L}_1 \left(\frac{1}{2} \phi(y) \xi^2 \frac{\partial^2}{\partial \xi^2} \right) \right\rangle P_0^M \quad &= \quad \frac{1}{2} \langle \mathcal{L}_1 \phi \rangle \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \\
= \frac{\rho \nu}{\sqrt{2}} \langle \phi' f \rangle \xi \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) &- \frac{\nu}{\sqrt{2}} \langle \phi' \Lambda \rangle \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2}
\end{aligned}$$

First correction and market parameters

The **first correction term** $\sqrt{\varepsilon}\, P_1^M$ solves the following problem:

$$\langle \mathcal{L}_2 \rangle (\sqrt{\varepsilon}\, P_1^M) + V_2^{M,\varepsilon} \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} + V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) = 0,$$

$$(\sqrt{\varepsilon}\, P_1^M)(T, \xi) = 0.$$

with

$$V_2^{M,\epsilon} = \frac{\sqrt{\epsilon}\nu}{\sqrt{2}} \langle \phi' \Lambda \rangle \qquad \text{and} \qquad V_3^{M,\epsilon} = -\frac{\sqrt{\epsilon}\rho\nu}{\sqrt{2}} \langle \phi' f \rangle.$$

In fact, the solution is **given explicitly** by

$$\sqrt{\varepsilon}\, P_1^M = (T - t) \left(V_2^{M,\varepsilon} \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} + V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) \right).$$

One can then deduce the **price approximation**

$$P^{M,\varepsilon} = P_0^M + (T - t) \left(V_2^{M,\varepsilon} \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} + V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left(\xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) \right) + \mathcal{O}(\varepsilon).$$

Parameter Reduction

One of the inherent advantages of this approximation is parameter reduction. While the full stochastic volatility model requires the four parameters (ϵ, ν, ρ, m) and the two functions f and γ, our approximated option price requires only the **three group parameters**:

- The **effective historical volatility** $\bar{\sigma}$
- The **volatility level correction** $V_2^{M,\epsilon}$ due to the market price of volatility risk
- The **skew parameter** $V_3^{M,\epsilon}$ proportional to ρ

We can further reduce to only two parameters by noting that $V_2^{M,\epsilon}$ is associated with a second order derivative with respect to the current market price ξ. As such, it can be considered as a *volatility level correction* and absorbed into the volatility of the leading order Black-Scholes price.

Adjusted effective volatility

We introduce the adjusted effective volatility $\sigma^{M*} = \sqrt{\bar{\sigma}^2 + 2V_2^{M,\varepsilon}}$, and we denote by P^{M*} the corresponding Black-Scholes option price.

Next, we define the first order correction $\sqrt{\varepsilon}P_1^{M*}$ solution to

$$\mathcal{L}_{BS}(\sigma^{M*})(\sqrt{\varepsilon}\,P_1^{M*}) + V_3^{M,\varepsilon}\xi\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial^2 P_0^{M*}}{\partial\xi^2}\right) = 0,$$

$$(\sqrt{\varepsilon}\,P_1^{M*})(T,\xi) = 0.$$

It is indeed given explicitly by

$$\sqrt{\varepsilon}\,P_1^{M*} = (T-t)V_3^{M,\varepsilon}\xi\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial^2 P_0^{M*}}{\partial\xi^2}\right),$$

and one can show that **the order of accuracy is preserved**:

$$P^{M,\varepsilon} = P_0^{M*} + (T-t)V_3^{M,\varepsilon}\xi\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial^2 P_0^{M*}}{\partial\xi^2}\right) + \mathcal{O}(\varepsilon)$$

Proof of order of accuracy

Observe that $\mathcal{L}_{BS}(\sigma^{M*}) = \mathcal{L}_{BS}(\bar{\sigma}) + \frac{1}{2}\,(2V_2^{M,\varepsilon})\xi^2 \frac{\partial^2}{\partial \xi^2},$
and therefore

$$
\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma})(P_0^M - P_0^{M*}) &= V_2^{M,\varepsilon}\xi^2 \frac{\partial^2 P_0^{M*}}{\partial \xi^2}\,, \\
(P_0^M - P_0^{M*})(T,\xi) &= 0\,.
\end{aligned}
$$

Since the source term is $\mathcal{O}(\sqrt{\varepsilon})$ because of the $V_2^{M,\varepsilon}$ factor, the difference $P_0^M - P_0^{M*}$ is also $\mathcal{O}(\sqrt{\varepsilon})$. Next we write

$$
\begin{aligned}
|P^{M,\varepsilon} - (P_0^{M*} + \sqrt{\varepsilon}\,P_1^{M*})| \quad &\leq \quad |P^{M,\varepsilon} - (P_0^M + \sqrt{\varepsilon}\,P_1^M)| \\
&\quad + |(P_0^M + \sqrt{\varepsilon}\,P_1^M) - (P_0^{M*} + \sqrt{\varepsilon}\,P_1^{M*})|\,,
\end{aligned}
$$

which, combined with the previous accuracy result, shows that the only quantity left to be controlled is the **residual**

$$
R \equiv (P_0^M + \sqrt{\varepsilon}\,P_1^M) - (P_0^{M*} + \sqrt{\varepsilon}\,P_1^{M*})\,.
$$

Proof of order of accuracy (continued)

From the equations satisfied by $P_0^M, \sqrt{\varepsilon}\, P_1^M, P_0^{M*}, \sqrt{\varepsilon}\, P_1^{M*}$, it follows that

$$\mathcal{L}_{BS}(\bar{\sigma})(P_0^M + \sqrt{\varepsilon}\, P_1^M) + V_2^{M,\varepsilon}\xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} + V_3^{M,\varepsilon}\xi \frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2}\right) = 0$$

$$\mathcal{L}_{BS}(\sigma^{M*})(P_0^{M*} + \sqrt{\varepsilon}\, P_1^{M*}) + V_3^{M,\varepsilon}\xi \frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial^2 P_0^{M*}}{\partial \xi^2}\right) = 0.$$

Denoting by

$$\begin{aligned}
\mathcal{H}^{\varepsilon} &= V_2^{M,\varepsilon}\xi^2 \frac{\partial^2}{\partial \xi^2} + V_3^{M,\varepsilon}\xi \frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial^2}{\partial \xi^2}\right), \\
\mathcal{H}^{\varepsilon *} &= V_3^{M,\varepsilon}\xi \frac{\partial}{\partial \xi}\left(\xi^2 \frac{\partial^2}{\partial \xi^2}\right),
\end{aligned}$$

one deduces that **the residual R satisfies the equation**:

$$
\begin{aligned}
\mathcal{L}_{BS}(\bar{\sigma})(R) &= -\mathcal{H}^{\varepsilon} P_0^M - \left(\mathcal{L}_{BS}(\sigma^{M*}) - V_2^{M,\varepsilon}\xi^2 \frac{\partial^2}{\partial \xi^2}\right)(P_0^{M*} + \sqrt{\varepsilon}\, P_1^{M*}) \\
&= -\mathcal{H}^{\varepsilon} P_0^M + \mathcal{H}^{\varepsilon *} P_0^{M*} + V_2^{M,\varepsilon}\xi^2 \frac{\partial^2}{\partial \xi^2}(P_0^{M*} + \sqrt{\varepsilon}\, P_1^{M*}) \\
&= \mathcal{H}^{\varepsilon *}(P_0^{M*} - P_0^M) + V_2^{M,\varepsilon}\xi^2 \frac{\partial^2}{\partial \xi^2}(P_0^{M*} - P_0^M + \sqrt{\varepsilon}\, P_1^{M*}) \\
&= \mathcal{O}(\varepsilon)\,,
\end{aligned}
$$

where we have used in the last equality that $\mathcal{H}^{\varepsilon *} = \mathcal{O}(\sqrt{\varepsilon})$, $V_2^{M,\varepsilon} = \mathcal{O}(\sqrt{\varepsilon})$, $P_0^{M*} - P_0^M = \mathcal{O}(\sqrt{\varepsilon})$, and $\sqrt{\varepsilon}\, P_1^{M*} = \mathcal{O}(\sqrt{\varepsilon})$. Since R **vanishes at the terminal time** T, we deduce $\mathbf{R} = \mathcal{O}(\varepsilon)$ which concludes the proof.

The new approximation has now **only two parameters** to be calibrated σ^{M*} and $V_3^{M,\epsilon}$, while preserving the accuracy of approximation.

This parameter reduction is essential in the forward-looking calibration procedure presented next.

Asset Option Approximation

Let $P^{a,\epsilon}$ denote the price of a **European option written on the asset X**, with maturity T and payoff h, evaluated at time $t < T$ with current value $X_t = x$. Then, we have

$$P^{a,\epsilon} = \mathbb{E}^{*(\gamma)} \left\{ e^{-r(T-t)} h(X_T) \mid \mathcal{F}_t \right\} = P^{a,\epsilon}(t, X_t, Y_t) \, .$$

By the Feynman-Kac formula, the function $P^{a,\epsilon}(t,x,y)$ satisfies the **partial differential equation**:

$$\begin{array}{rcl} \mathcal{L}^{a,\epsilon} P^{a,\epsilon} & = & 0, \\ P^{a,\epsilon}(T,x,y) & = & h(x), \end{array}$$

where

$$\mathcal{L}^{a,\epsilon} = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1^a + \mathcal{L}_2^a,$$

with

$$\begin{aligned}
\mathcal{L}_0 &= \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \\
\mathcal{L}_1^a &= \rho\nu\sqrt{2}\beta f(y) x \frac{\partial^2}{\partial x \partial y} - \nu\sqrt{2}\Lambda(y)\frac{\partial}{\partial y}, \\
\mathcal{L}_2^a &= \frac{\partial}{\partial t} + \frac{1}{2}\left(\beta^2 f(y)^2 + \sigma^2\right) x^2 \frac{\partial^2}{\partial x^2} + r(x\frac{\partial}{\partial x} - \cdot) \equiv \mathcal{L}_{BS}(\sqrt{\beta^2 f(y)^2 + \sigma^2}).
\end{aligned}$$

Observe that the only differences with options on the market index is the **factor β in $\mathcal{L}_1^a$**, and the **modified square volatility $\beta^2 f(y)^2 + \sigma^2$ in $\mathcal{L}_2^a$**. it is easy to see that the only modifications in the approximation are:

1. $\bar{\sigma}^2$ is replaced by $\bar{\sigma}_a^2 = \beta^2\bar{\sigma}^2 + \sigma^2$

2. $V_2^{M,\varepsilon}$ is replaced by $V_2^{a,\varepsilon} = \beta^2 V_2^{M,\varepsilon} \Longrightarrow \quad V_2^{a,\epsilon} = \frac{\beta^2\sqrt{\epsilon}\nu}{\sqrt{2}}\langle\phi'\Lambda\rangle$

3. $V_3^{M,\varepsilon}$ is replaced by $V_3^{a,\varepsilon} = \beta^3 V_3^{M,\varepsilon} \Longrightarrow V_3^{a,\epsilon} = -\frac{\beta^3\sqrt{\epsilon}\rho\nu}{\sqrt{2}}\langle\phi' f\rangle$

4. σ^{M*} is replaced by $\sigma^{a*} = \sqrt{\beta^2\bar{\sigma}^2 + \sigma^2 + 2V_2^{a,\varepsilon}}$

5. The option price approximation becomes

$$P^{a,\varepsilon} = P_0^{a*} + (T-t)V_3^{a,\varepsilon} x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2 P_0^{a*}}{\partial x^2}\right) + \mathcal{O}(\varepsilon),$$

   where P_0^{a*} is the Black-Scholes price with volatility σ^{a*}

6. **Only the parameters $V_3^{a,\varepsilon}$ and σ^{a*} need to be calibrated**

Beta Estimation

From the expressions for $V_3^{M,\varepsilon}$ and $V_3^{a,\varepsilon}$, one deduces that

$$V_3^{a,\varepsilon} = \beta^3 V_3^{M,\varepsilon}.$$

It is then natural to propose the following **estimator for β**:

$$\beta = \left(\frac{V_3^{a,\epsilon}}{V_3^{M,\epsilon}} \right)^{\frac{1}{3}}.$$

Therefore in order to estimate the market beta parameter in a forward looking fashion using the implied skew parameters from option prices we must calibrate our two parameters $V_3^{a,\epsilon}$ and $V_3^{M,\epsilon}$.

Next we show how to do that by using the **implied volatility surfaces** from options data.

Calibration Method

We know that a first order approximation of an option price (on the market or the individual asset) with time to maturity $\tau = T - t$, and in the presence of **fast mean-reverting stochastic volatility**, takes the following form:

$$P^{\epsilon} \sim P^*_{BS} + \tau V_3^{\epsilon} x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P^*_{BS}}{\partial x^2}\right),$$

where P^*_{BS} **is the Black-Scholes price with volatility** σ^*.

The European call option price P^*_{BS} with current price x, time to maturity τ, and strike price K is given by the Black-Scholes formula

$$P^*_{BS} = xN(d_1^*) - Ke^{-r\tau}N(d_2^*),$$

where N is the cumulative standard normal distribution and

$$d^*_{1,2} = \frac{log(x/K) + (r \pm \frac{1}{2}\sigma^{*2})\tau}{\sigma^*\sqrt{\tau}}.$$

Recall the relationship between *Vega* and *Gamma* for **plain vanilla European options**:

$$\frac{\partial P^*_{BS}}{\partial \sigma} = \tau \sigma^* x^2 \frac{\partial^2 P^*_{BS}}{\partial x^2},$$

and rewrite our price approximation as

$$P^\epsilon \sim P^*_{BS} + \frac{V_3^\epsilon}{\sigma^*} x \frac{\partial}{\partial x} \left(\frac{\partial P^*_{BS}}{\partial \sigma} \right).$$

Using the definition of the **implied volatility** $P_{BS}(I) = P^\varepsilon$, and expanding the implied volatility as

$$I = \sigma^* + \sqrt{\epsilon} I_1 + \epsilon I_2 + \cdots ,$$

we obtain:

$$P_{BS}(\sigma^*) + \sqrt{\epsilon} I_1 \frac{\partial P_{BS}(\sigma^*)}{\partial \sigma} + \cdots = P^*_{BS} + \frac{V_3^\epsilon}{\sigma^*} x \frac{\partial}{\partial x} \left(\frac{\partial P^*_{BS}}{\partial \sigma} \right) + \cdots .$$

By definition $P_{BS}(\sigma^*) = P^*_{BS}$, so that

$$\sqrt{\epsilon} I_1 = \frac{V_3^\epsilon}{\sigma^*} \left(\frac{\partial P^*_{BS}}{\partial \sigma} \right)^{-1} x \frac{\partial}{\partial x} \left(\frac{\partial P^*_{BS}}{\partial \sigma} \right).$$

Using the explicit computation of the *Vega*

$$\frac{\partial P^*_{BS}}{\partial \sigma} = \frac{x\sqrt{\tau}\, e^{-d_1^*/2}}{\sqrt{2\pi}},$$

and consequently

$$x \frac{\partial}{\partial x} \left(\frac{\partial P^*_{BS}}{\partial \sigma} \right) = \left(1 - \frac{d_1^*}{\sigma^* \sqrt{\tau}} \right) \frac{\partial P^*_{BS}}{\partial \sigma},$$

we deduce by using the definition of d_1^*:

$$\sqrt{\epsilon} I_1 = \frac{V_3^\epsilon}{\sigma^*} \left(1 - \frac{d_1^*}{\sigma^* \sqrt{\tau}} \right) = \frac{V_3^\epsilon}{2\sigma^*} \left(1 - \frac{2r}{\sigma^{*2}} \right) + \frac{V_3^\epsilon}{\sigma^{*3}} \frac{log(K/x)}{\tau}.$$

Log-Moneyness to Maturity Ratio ($LMMR$)

Define

$$LMMR = \frac{log(K/x)}{\tau},$$

we obtain the *affine LMMR formula*

$$I \sim \sigma^* + \sqrt{\epsilon} I_1 = b^* + a^\epsilon LMMR\,,$$

with the intercept b^* and the slope a^ε to be fitted to the skew of options data, and related to our model parameters σ^* and V_3^ε by:

$$\begin{aligned} b^* &= \sigma^* + \frac{V_3^\epsilon}{2\sigma^*}\left(1 - \frac{2r}{\sigma^{*2}}\right), \\ a^\epsilon &= \frac{V_3^\epsilon}{\sigma^{*3}}. \end{aligned}$$

Calibration Formulas for V_3^ε

We know that b^* and σ^* differ from a quantity of order $\sqrt{\varepsilon}$. Therefore by replacing σ^* by b^* in the relation $V_3^\epsilon = a^\epsilon \sigma^{*3}$, the order of accuracy for V_3^ε is still ε since a^ε is also of order $\sqrt{\varepsilon}$. Consequently we deduce

$$V_3^\epsilon = a^\epsilon \sigma^{*3} \sim \mathbf{a}^\epsilon \mathbf{b}^{*3} \equiv \widehat{\mathbf{V}_3^\varepsilon}.$$

It is indeed also possible to extract σ^* as follows.

$$b^* \quad = \quad \sigma^* + \frac{a^\varepsilon \sigma^{*2}}{2}\left(1 - \frac{2r}{\sigma^{*2}}\right) = \sigma^* - a^\varepsilon \left(r - \frac{\sigma^{*2}}{2}\right).$$

Using again the argument that b^* and σ^* differ from a quantity of order $\sqrt{\varepsilon}$ and a^ε is also of order $\sqrt{\varepsilon}$, by replacing σ^* by b^* in the last term in the relation above, the order of accuracy is still ε, and we conclude that

$$\sigma^* \sim \mathbf{b}^* + \mathbf{a}^\epsilon (\mathbf{r} - \frac{\mathbf{b}^{*2}}{\mathbf{2}}) \equiv \widehat{\sigma^*}.$$

Beta Calibration

Defining the market fitted parameters as $a^{M,\epsilon}$ and b^{M*} and the asset parameters as $a^{a,\epsilon}$ and b^{a*}, we obtain our main formula:

$$\hat{\beta} = \left(\frac{\widehat{\mathbf{V_3^{a,\epsilon}}}}{\widehat{\mathbf{V_3^{M,\epsilon}}}} \right)^{\mathbf{1/3}} = \left(\frac{\mathbf{a^{a,\epsilon}}}{\mathbf{a^{M,\epsilon}}} \right)^{\mathbf{1/3}} \left(\frac{\mathbf{b^{a*}}}{\mathbf{b^{M*}}} \right),$$

where $b^{a*} + a^{a,\epsilon}\, LMMR$ (resp. $b^{M*} + a^{M,\epsilon}\, LMMR$) is the linear fit to the skew of implied volatilities for call options on the individual asset (resp. on the market index).

Observe the similarity with the formula

$$\beta_a = \left(\frac{SKEW_a}{SKEW_M} \right)^{\frac{1}{3}} \left(\frac{VAR_a}{VAR_M} \right)^{\frac{1}{2}},$$

used by Christoffersen, Jacobs, and Vainberg (2008).

LMMR **fit examples**

In the following figure:

Implied volatilities of June 17, 2009 maturity options for the S&P 500 and Amgen, plotted against the option's *Log-Moneyness to Maturity Ratio (LMMR)*.

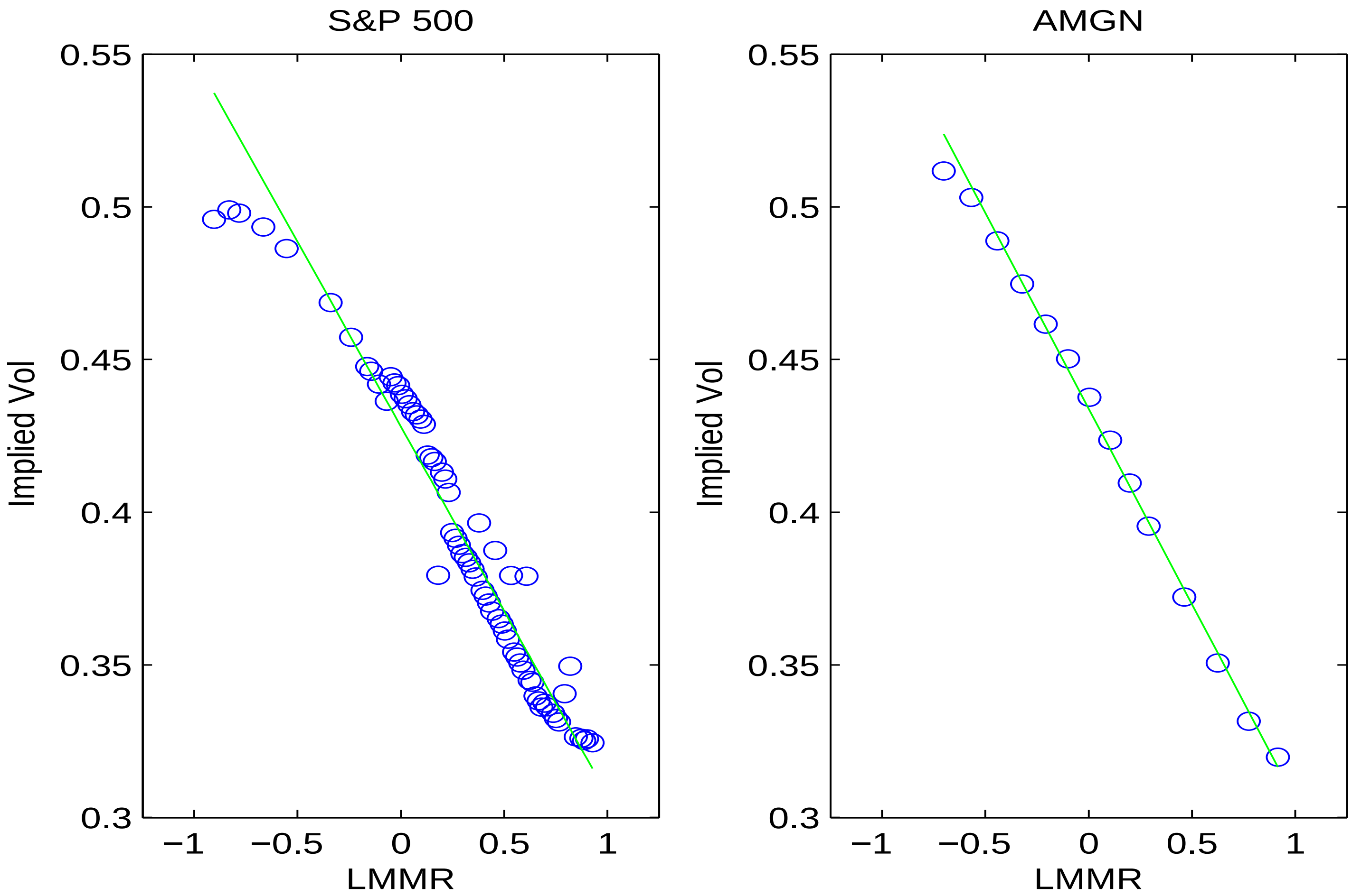
These are for February 18, 2009 option prices. The blue line is the affine fit of implied volatilities on *LMMR* by which the V_3 parameter is fit. The parameters fit for each series are

S&P 500 Fit: $a^{M,\epsilon} = -0.121$ and $b^{M*} = 0.428 \Rightarrow V_3^{M,\epsilon} = -0.0095$

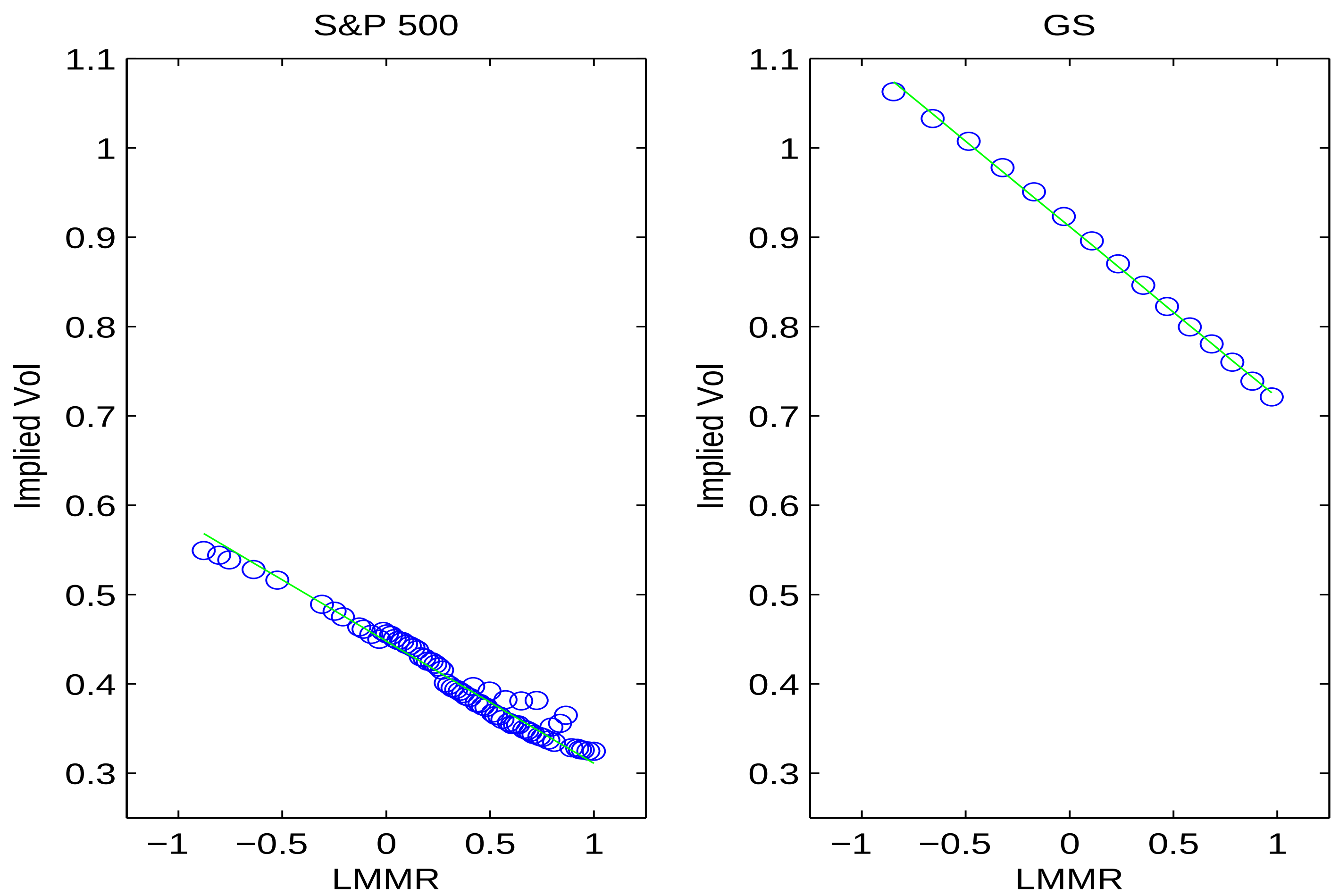
Amgen Fit: $a^{a,\epsilon} = -0.128$ and $b^{a*} = 0.434 \Rightarrow V_3^{a,\epsilon} = -0.010$

The beta estimate for Amgen on that day is then 1.03

LMMR fits: S&P500 and Amgen, beta estimate is 1.03



LMMR fits: S&P500 and Goldman Sachs, beta estimate is 2.28

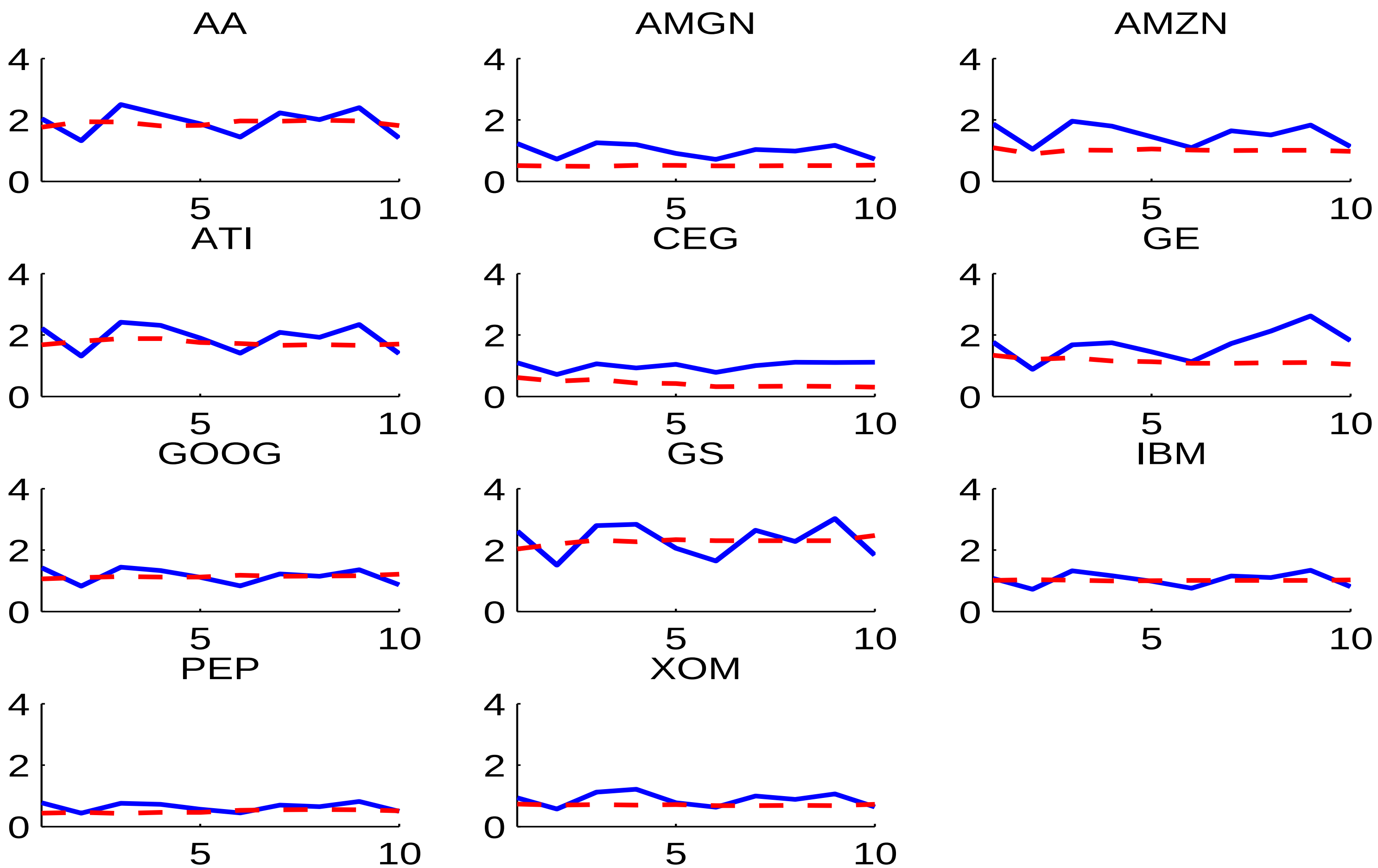


Forward and Backward Looking Betas

In the following figure:

The **solid blue line** is the **forward looking beta** (y-axis) calibrated on June 17, 2009 expiration call options over the course of 10 market days (x-axis) from February 9, 2009 to February 23, 2009.

The **dashed red line** is the corresponding **historical beta** calibrated on a series of historical prices of the same length as the time to maturity of the options.


AA
AMGN
AMZN
ATI
CEG
GE
GOOG
GS
IBM
PEP
XOM

THANKS FOR YOUR ATTENTION ...

unless you want to see a nonlinear case?

Option Pricing Under a Stressed-Beta Model

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Capital Asset Pricing Model (CAPM)

Discrete-time approach

Excess return of asset $R_a - R_f$ is linear function of excess return of market R_M and Gaussian error term:

$$R_a - R_f \quad = \quad \beta(R_M - R_f) + \epsilon$$

Beta coefficient estimated by regressing asset returns on market returns.

Difficulties with CAPM

Some difficulties with this approach, including:

1) Relationship between asset returns, market returns not always linear

2) Estimation of β from history, but future may be quite different

Ultimate goal of this research is to deal with both of these issues

Extending CAPM: Dynamic Beta

Two main approaches:

1) Retain linearity, but beta changes over time; Ferson (1989), Ferson and Harvey (1991), Ferson and Harvey (1993), Ferson and Korajczyk (1995), Jagannathan and Wang (1996)

2) Nonlinear model, by way of state-switching mechanism; Fridman (1994), Akdeniz, L., Salih, A.A., and Caner (2003)

ASC introduces *threshold CAPM model*. Our approach is related.

Estimating Implied Beta

Different approach to estimating β: look to options market

- *Forward-Looking Betas*, 2006
  P Christoffersen, K Jacobs, and G Vainberg
  Discrete-Time Model
- *Calibration of Stock Betas from Skews of Implied Volatilities*, 2009
  J-P Fouque, E Kollman
  Continuous-Time Model, stochastic volatility environment

Example of Time-Dependent Beta

Stock	Industry	Beta (2005-2006)	Beta (2007-2008)
AA	Aluminum	1.75	2.23
GE	Conglomerate	0.30	1.00
JNJ	Pharmaceuticals	-0.30	0.62
JPM	Banking	0.54	0.72
WMT	Retail	0.21	0.29

Larger β means greater sensitivity of stock returns relative to market returns

Regime-Switching Model

We propose a model similar to CAPM, with a key difference:

When market falls below level c, slope increases by δ, where $\delta > 0$

Thus, beta is two-valued

This simple approach keeps the mathematics tractable

Dynamics Under Physical Measure $\mathbb{P}$

M_t value of market at time t
S_t value of asset at time t

$$\frac{dM_t}{M_t} = \mu dt + \sigma_m dW_t \qquad \text{Market Model; const vol, for now}$$

$$\frac{dS_t}{S_t} = \beta(M_t)\frac{dM_t}{M_t} + \sigma dZ_t \qquad \text{Asset Model}$$

$\beta(M_t) = \beta + \delta\,\mathbb{I}_{\{M_t < c\}}$

Brownian motions W_t, Z_t indep: $d\langle W, Z\rangle_t = 0$

Dynamics Under Physical Measure $\mathbb{P}$

Substituting market equation into asset equation:

$$\frac{dS_t}{S_t} = \beta(M_t)\mu dt + \beta(M_t)\sigma_m dW_t + \sigma dZ_t$$

Asset dynamics depend on market level, market volatility σ_m

This is a geometric Brownian motion with volatility $\sqrt{\beta^2(M_t)\sigma_m^2 + \sigma^2}$

Note this is a stochastic volatility model

Dynamics Under Physical Measure $\mathbb{P}$

Process preserves the definition of β:

$$\begin{aligned}
\frac{Cov\left(\frac{dS_t}{S_t}, \frac{dM_t}{M_t}\right)}{Var\left(\frac{dM_t}{M_t}\right)} &= \frac{Cov\left(\beta(M_t)\frac{dM_t}{M_t} + \sigma dZ_t, \frac{dM_t}{M_t}\right)}{Var\left(\frac{dM_t}{M_t}\right)} \\
&= \frac{Cov\left(\beta(M_t)\frac{dM_t}{M_t}, \frac{dM_t}{M_t}\right)}{Var\left(\frac{dM_t}{M_t}\right)} \quad \text{Since BM's indep} \\
&= \beta(M_t)
\end{aligned}$$

Dynamics Under Risk-Neutral Measure $I\!P^{\star}$

Market is complete (M and S both tradeable)

Thus, $\exists$ unique Equivalent Martingale Measure $I\!P^{\star}$ defined as

$$\frac{dI\!P^{\star}}{dI\!P} = exp\left\{-\int_t^T \theta^{(1)} dW_s - \int_t^T \theta^{(2)} dZ_s - \frac{1}{2}\int_t^T \left\{(\theta^{(1)})^2 + (\theta^{(2)})^2\right\} ds\right\}$$

with

$$\begin{aligned} \theta^{(1)} &= \frac{\mu - r}{\sigma_m} \\ \theta^{(2)} &= \frac{r(\beta(M_t) - 1)}{\sigma} \end{aligned}$$

Dynamics Under Risk-Neutral Measure $\mathbb{P}^\star$

$$\begin{aligned}\frac{dM_t}{M_t} &= rdt + \sigma_m dW_t^* \\ \frac{dS_t}{S_t} &= rdt + \beta(M_t)\sigma_m dW_t^* + \sigma dZ_t^*\end{aligned}$$

where

$$\begin{aligned}dW_t^* &= dW_t + \frac{\mu - r}{\sigma_m}dt \\ dZ_t^* &= dZ_t + \frac{r(\beta(M_t) - 1)}{\sigma}dt\end{aligned}$$

By Girsanov's Thm, W_t^*, Z_t^* are indep Brownian motions under $\mathbb{P}^\star$.

Option Pricing

P price of option with expiry T, payoff $h(S_T)$

Option price at time $t < T$ is function of t, M, and S

(M,S) Markovian

Option price discounted expected payoff under risk-neutral measure $\mathbb{P}^*$

$$P(t, M, S) = \mathbb{E}^\star \left\{ e^{-r(T-t)} h(S_T) | M_t = M, S_t = S \right\}$$

State Variables

Define new state variables: $X_t = \log S_t$, $\xi_t = \log M_t$
Initial conditions $X_0 = x$, $\xi_0 = \xi$

Dynamics are:

$$
\begin{aligned}
d\xi_t &= \left(r - \frac{\sigma_m^2}{2}\right) dt + \sigma_m dW_t^* \\
dX_t &= \left(r - \frac{1}{2}(\beta^2(e^{\xi_t})\sigma_m^2 + \sigma^2)\right) dt + \beta(e^{\xi_t})\sigma_m dW_t^* + \sigma dZ_t^*
\end{aligned}
$$

State Variables

WLOG, let $t = 0$

In integral form,

$$\xi_t = \xi + \left(r - \frac{\sigma_m^2}{2}\right) t + \sigma_m W_t^*$$

Next, consider X at expiry (integrate from 0 to T):

$$\begin{aligned} X_T &= x + \left(r - \frac{\sigma^2}{2}\right) T - \frac{\sigma_m^2}{2} \int_0^T \beta^2(e^{\xi_t})dt \\ &+ \sigma_m \int_0^T \beta(e^{\xi_t})dW_t^* + \sigma Z_T^* \end{aligned}$$

Working with X_T

$$M_t < c \quad \Rightarrow \quad e^{\xi_t} < c \quad \Rightarrow \quad \xi_t < \log c$$

$$\beta(M_t) = \beta + \delta\, \mathbb{I}_{\{M_t < c\}} \quad \Rightarrow \quad \beta(e^{\xi_t}) = \beta + \delta\, \mathbb{I}_{\{\xi_t < \log c\}}$$

Using this definition for $\beta(e^{\xi_t})$, X_T becomes

$$\begin{aligned} X_T &= x + \left(r - \frac{\beta^2\sigma_m^2 + \sigma^2}{2}\right)T + \sigma_m \beta W_T^* + \sigma Z_T^* \\ &- (\delta^2 + 2\delta\beta)\frac{\sigma_m^2}{2}\int_0^T \mathbb{I}_{\{\xi_t < \log c\}} dt + \sigma_m \delta \int_0^T \mathbb{I}_{\{\xi_t < \log c\}} dW_t^* \end{aligned}$$

Occupation Time of Brownian Motion

Expression for X_T involves integral $\int_0^T \mathbb{I}_{\{\xi_t < \log c\}} dt$

This is occupation time of Brownian motion with drift

To simplify calculation, apply Girsanov to remove drift from ξ

Occupation Time of Brownian Motion

Consider new probability measure $\widetilde{I\!P}$ defined as

$$\frac{d\widetilde{I\!P}}{dI\!P^{\star}} = exp\left\{-\theta W_T^* - \frac{1}{2}\theta^2 T\right\}$$

$$\theta \quad = \quad \frac{1}{\sigma_m}\left(r - \frac{\sigma_m^2}{2}\right)$$

Under this measure, ξ_t is a martingale with dynamics

$$d\xi_t \quad = \quad \sigma_m d\widetilde{W}_t$$

$$d\widetilde{W}_t \quad = \quad dW_t^* + \frac{1}{\sigma_m}\left(r - \frac{\sigma_m^2}{2}\right) dt$$

Changing Measure: $\mathbb{P}^\star \to \widetilde{\mathbb{P}}$

Since W^* and Z^* indep, Z^* not affected by change of measure

Can replace Z^* with $\widetilde{Z}$

Under $\widetilde{\mathbb{P}}$,

$$\begin{aligned} X_T &= x + A_1 T + \sigma_m \beta \widetilde{W}_T \\ &+ \sigma \widetilde{Z}_T - A_2 \int_0^T \mathbb{I}_{\{\xi_t < \log c\}} dt \\ &+ \sigma_m \delta \int_0^T \mathbb{I}_{\{\xi_t < \log c\}} d\widetilde{W}_t \end{aligned}$$

where constants A_1, A_2 defined as

$$\begin{aligned} A_1 &= r(1-\beta) - \frac{\sigma_m^2(\beta^2 - \beta) + \sigma^2}{2} \\ A_2 &= \delta(\delta + 2\beta - 1)\frac{\sigma_m^2}{2} + \delta r \end{aligned}$$

First Passage Time

Now that ξ_t is driftless, easier to work with occupation time

Run process until first time it hits level $log\, c$

Denote this first passage time

$$\tau = \inf\{t \geq 0 : \xi_t = \log c\} = \inf\left\{t \geq 0 : \widetilde{W}_t = \tilde{c}\right\}$$

where

$$\tilde{c} = \frac{\log c - \xi}{\sigma_m}$$

Density of first passage time of $\xi_t = \xi$ to level log c is

$$p(u; \tilde{c}) = \frac{|\tilde{c}|}{\sqrt{2\pi u^3}} \exp\left(-\frac{\tilde{c}^2}{2u}\right), \quad u > 0$$

Including First Passage Time Information

First passage time τ may happen after T, so need to be careful
Can partition time horizon into two pieces:

$$[0, \tau \wedge T] \quad \text{and} \quad [\tau \wedge T, T]$$

If $\xi_t < \log c$, $\quad \tau \wedge T$ counts as occupation time

Including First Passage Time Information

Incorporating this information into X_T yields

$$
\begin{aligned}
X_T &= x + A_1 T + \sigma_m \beta \widetilde{W}_T + \sigma \widetilde{Z}_T \\
&\quad - A_2 (\tau \wedge T)\, \mathbb{I}_{\{\tilde{c}>0\}} - A_2 \int_{\tau \wedge T}^{T} \mathbb{I}_{\left\{\widetilde{W}_t < \tilde{c}\right\}} dt \\
&\quad + \sigma_m \delta\, \widetilde{W}_{\tau \wedge T}\, \mathbb{I}_{\{\tilde{c}>0\}} + \sigma_m \delta \int_{\tau \wedge T}^{T} \mathbb{I}_{\left\{\widetilde{W}_t < \tilde{c}\right\}} d\widetilde{W}_t
\end{aligned}
$$

Working with the Stochastic Integral

Stochastic integral can be re-expressed in terms of local time $\widetilde{L}^{\tilde{c}}$ of $\widetilde{W}$ at level $\tilde{c}$.

Applying Tanaka's formula to $\phi(w) = (w - \tilde{c})\mathbb{I}_{\{w<\tilde{c}\}}$ between $\tau \wedge T$ and T, we get:

$$\int_{\tau\wedge T}^{T} \mathbb{I}_{\{\widetilde{W}_t<\tilde{c}\}} d\widetilde{W}_t = \phi(\widetilde{W}_T) - \phi(\widetilde{W}_{\tau\wedge T}) + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_{\tau\wedge T}^{\tilde{c}}\,.$$

Starting Level of Market: Three Cases

Consider separately the three cases $\xi = \log c$, $\xi > \log c$, and $\xi < \log c$

$$(\text{or equivalently } \tilde{c} = 0,\ \tilde{c} < 0,\ \tilde{c} > 0)$$

Notation for terminal log-stock price, given ξ

Case $\xi = \log c$	terminal log-stock price Ψ_0
Case $\xi > \log c$	terminal log-stock price Ψ^+
Case $\xi < \log c$	terminal log-stock price Ψ^-

Consider Case $\xi < \log c$ as Example

In this case, $\tilde{c} > 0$ and we have

$$
\begin{aligned}
X_T &= x + A_1 T + \sigma_m \beta \widetilde{W}_T + \sigma \widetilde{Z}_T \\
&\quad - A_2(\tau \wedge T) - A_2 \int_{\tau \wedge T}^{T} \mathbb{I}_{\{\widetilde{W}_t < \tilde{c}\}} dt + \sigma_m \delta \widetilde{W}_{\tau \wedge T} \\
&\quad + \sigma_m \delta \left[\left(\widetilde{W}_T - \tilde{c} \right) \mathbb{I}_{\{\widetilde{W}_T < \tilde{c}\}} - \left(\widetilde{W}_{\tau \wedge T} - \tilde{c} \right) \mathbb{I}_{\{\widetilde{W}_{\tau \wedge T} < \tilde{c}\}} + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_{\tau \wedge T}^{\tilde{c}} \right]
\end{aligned}
$$

Treat separately cases $\{\tau < T\}$ and $\{\tau > T\}$

Case $\xi < \log c$, contd.

- On $\{\tau > T\}$, we have:

$$\begin{aligned} X_T &= x + (A_1 - A_2)T + \sigma_m(\beta + \delta)\,\widetilde{W}_T + \sigma \widetilde{Z}_T \\ &=: \Psi^-_{T^+}(\widetilde{W}_T, \widetilde{Z}_T), \end{aligned}$$

where lower index T^+ stands for $\tau > T$

Distribution of X_T is given by distn of independent Gaussian r.v. $\widetilde{Z}_T$, and conditional distn of $\widetilde{W}_T$ given $\{\tau > T\}$.

Case $\xi < \log c$, contd.

Conditional distn of $\widetilde{W}_T$ given $\{\tau > T\}$:

From Karatzas and Shreve, one easily obtains:

$$\begin{aligned} I\!\!P\left\{\widetilde{W}_T \in da, \tau > T\right\} &= \frac{1}{\sqrt{2\pi T}}\left(e^{-\frac{a^2}{2T}} - e^{-\frac{(2\tilde{c}-a)^2}{2T}}\right) da, \quad a < \tilde{c}, \\ &=: q_T(a;\tilde{c})\, da \end{aligned}$$

Case $\xi < \log c$, contd.

- On $\{\tau = u\}$ with $u \leq T$, we have $\widetilde{W}_u = \tilde{c}$, and

$$\begin{aligned} X_T &= x + (A_1 - A_2)T + \sigma_m(\beta+\delta)\tilde{c} + \sigma_m \beta(\widetilde{W}_T - \widetilde{W}_u) + \sigma \widetilde{Z}_T \\ &\quad + A_2 \int_u^T \mathbb{I}_{\left\{\widetilde{W}_t - \widetilde{W}_u > 0\right\}} dt \\ &\quad + \sigma_m \delta \left[\left(\widetilde{W}_T - \widetilde{W}_u \right) \mathbb{I}_{\{\widetilde{W}_T - \widetilde{W}_u < 0\}} + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_u^{\tilde{c}} \right] \end{aligned}$$

Distn of X_T given by distn of $\widetilde{Z}_T$ and indep triplet $\left(B_{T-u}, L^0_{T-u}, \Gamma^+_{T-u}\right)$

Triplet comprised of value, local time at 0, and occupation time of positive half-space, at time $T-u$, of standard Brownian motion B.

Case $\xi < \log c$, contd.

In distribution:

$$\begin{aligned} X_T &= x + (A_1 - A_2)T + \sigma_m(\beta+\delta)\tilde{c} + \sigma_m B_{T-u}\left(\beta + \delta\,\mathbb{I}_{\{B_{T-u}<0\}}\right) + \sigma \widetilde{Z}_T \\ &\quad + A_2\,\Gamma^+_{T-u} + \sigma_m \delta L^0_{T-u} \\ &=: \Psi^-_{T-}(B_{T-u}, L^0_{T-u}, \Gamma^+_{T-u}, \widetilde{Z}_T). \end{aligned}$$

Distn of triplet $\left(B_{T-u}, L^0_{T-u}, \Gamma^+_{T-u}\right)$ developed in paper by Karatzas and Shreve.

Karatzas-Shreve Triplet (1984)

$$\mathbb{P}\left\{\widetilde{W}_T \in da,\ \widetilde{L}^0_T \in db,\ \widetilde{\Gamma}^+_T \in d\gamma\right\}$$

$$= \begin{cases} 2p(T-\gamma; b)\, p(\gamma; a+b) & \text{if} \quad a > 0, b > 0, 0 < \gamma < T, \\ 2p(\gamma; b)\, p(T-\gamma; -a+b) & \text{if} \quad a < 0, b > 0, 0 < \gamma < T, \end{cases}$$

where $p(u; \cdot)$ is first passage time density

Back to Option Pricing Formula

Given final expression for X_T, option price at time $t = 0$ is

$$
\begin{aligned}
P_0 &= \mathbb{E}^\star \left\{ e^{-rT} h(S_T) \right\} \\
&= \widetilde{\mathbb{E}} \left\{ e^{-rT} h(e^{X_T}) \frac{d\mathbb{P}^\star}{d\widetilde{\mathbb{P}}} \right\} \\
&= \widetilde{\mathbb{E}} \left\{ e^{-rT} h(e^{X_T}) e^{\theta \widetilde{W}_T - \frac{1}{2}\theta^2 T} \right\} \\
&= e^{-rT} e^{-\frac{1}{2}\theta^2 T} \widetilde{\mathbb{E}} \left\{ h(e^{X_T}) e^{\theta \widetilde{W}_T} \right\}
\end{aligned}
$$

Option Pricing Formula, contd.

Decompose expectation on $\{\tau \leq T\}$ and $\{\tau > T\}$,

Denote by $n_T(z)$ the $\mathcal{N}(0,T)$ density,

Define the following convolution relation involving the K-S triplet:

$$
\begin{aligned}
&\int_0^{T-\gamma} g(a,b,\gamma;T-u)p(u;\tilde{c})du \\
&= \begin{cases} 2p(\gamma;a+b)\,p(T-\gamma;b+|\tilde{c}|) & \text{if} \quad a>0 \\ 2p(\gamma;b)\,p(T-\gamma;-a+b+|\tilde{c}|) & \text{if} \quad a<0 \end{cases} \\
&=: G(a,b,\gamma;T)
\end{aligned}
$$

Option Pricing Formula, contd.

The option pricing formula becomes

$$P_0 = e^{-(r+\frac{1}{2}\theta^2)T}\Bigg[e^{\theta\tilde{c}}\int_{-\infty}^{\infty}\int_0^T\int_0^{\infty}\int_{-\infty}^{\infty} h(e^{\Psi^{\pm}_{T-}(a,b,\gamma,z)})e^{\theta a}$$

$$\times G(a,b,\gamma;T)\,da\,db\,d\gamma\,n_T(z)dz$$

$$+\left(\int_{-\infty}^{\infty}\int_{D^{\pm}} h(e^{\Psi^{\pm}_{T+}(a,z)})e^{\theta a}q_T(a;\tilde{c})da\,n_T(z)dz\right)\Bigg]$$

where

$$D^{\pm} = \begin{cases} (-\infty,\tilde{c}) & \text{if} \quad \tilde{c} > 0 \\ (\tilde{c},\infty) & \text{if} \quad \tilde{c} < 0 \end{cases}$$

Note About Market Stochastic Volatility (SV)

- Assumption of constant market volatility σ_m not realistic
- Let market volatility be driven by fast mean-reverting factor
- Introducing market SV in model has effect on asset price dynamics
- To leading order, these prices are given by risk-neutral dynamics with σ_m replaced by *adjusted effective volatility* σ^* (see Fouque, Kollman (2009) for details)
- One could derive a formula for first-order correction, but formula is quite complicated and numerically involved

Market Implied Volatilities

Following Fouque, Papanicolaou, Sircar (2000) and Fouque, Kollman (2009), introduce *Log-Moneyness to Maturity Ratio* ($LMMR$)

$$LMMR = \frac{\log(K/x)}{T}$$

and for calibration purposes, we use *affine LMMR formula*

$$I \sim b^* + a^\epsilon \, LMMR$$

with intercept b^* and slope a^ϵ to be fitted to skew of options data

Then estimate adjusted effective volatility as

$$\sigma^* \sim b^* + a^\epsilon \left(r - \frac{b^{*2}}{2} \right)$$

Numerical Results and Calibration

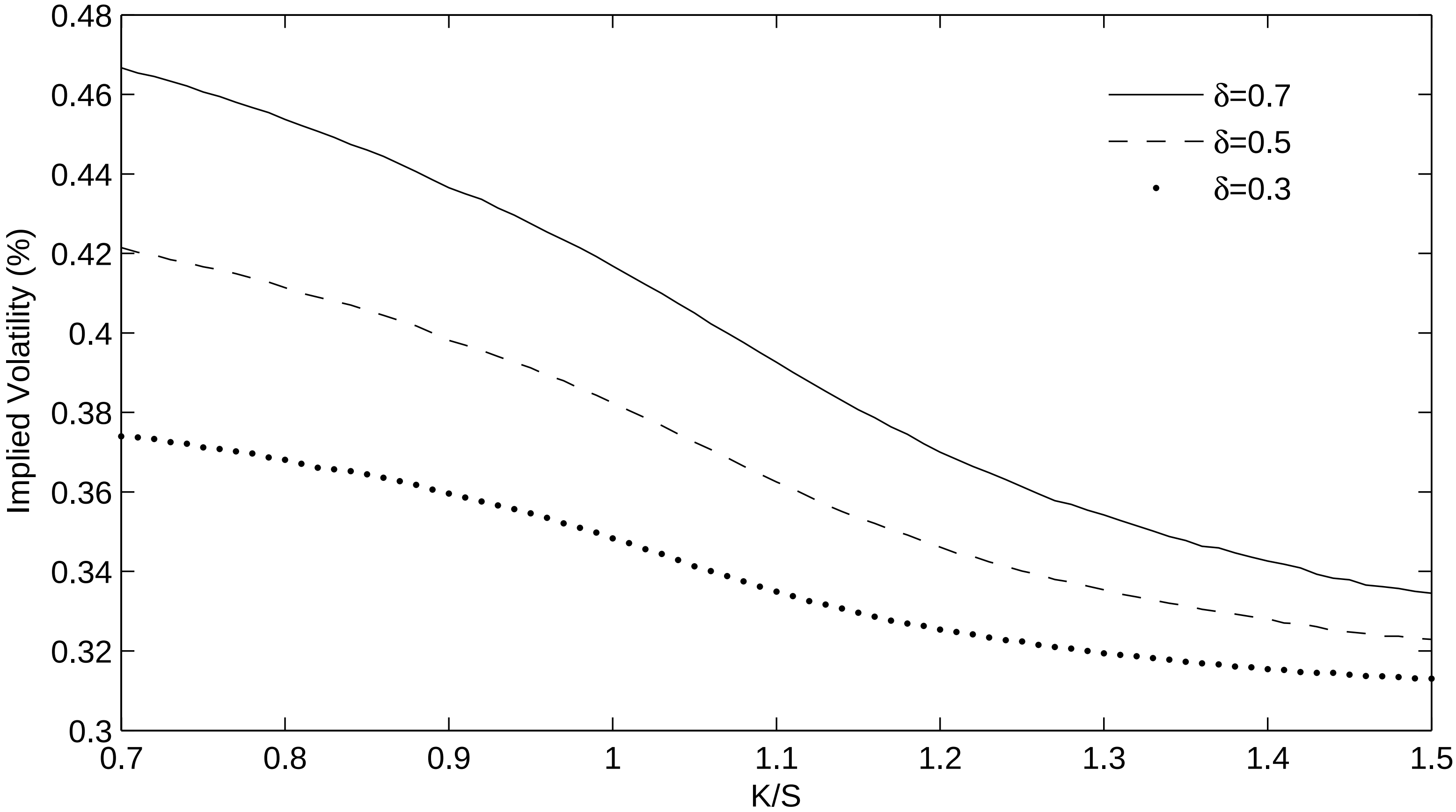
Asset Skews of Implied Volatilities

Using Stressed-Beta model, price European call option

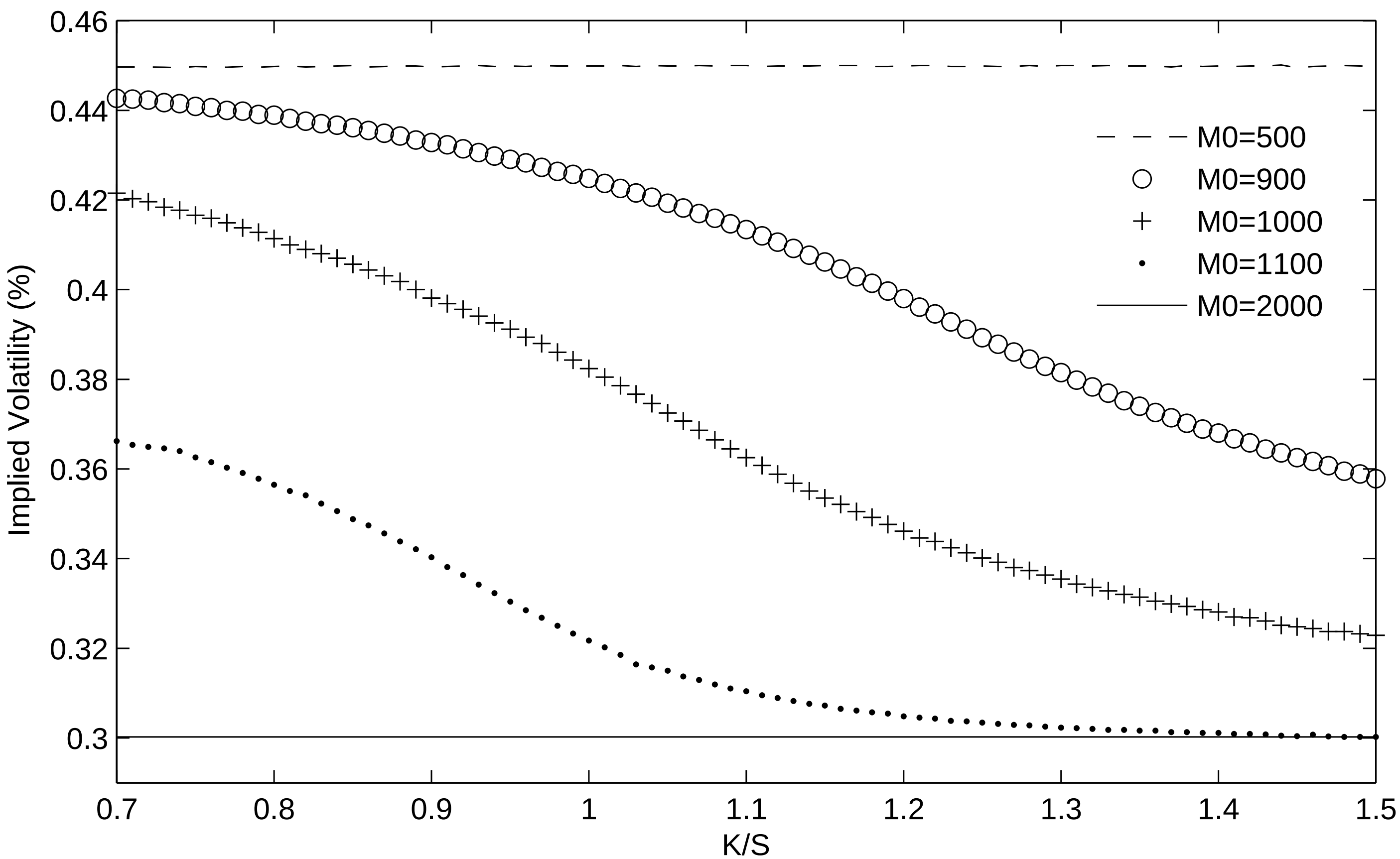
Use following parameter settings:

c	S_0	r	β	σ_m	σ	T
1000	100	0.01	1.0	0.30	0.01	1.0

$K = 70, 71, \ldots, 150$ to build implied volatility curves



Implied Volatility Skew vs. $\delta \quad (M_0 = c)$

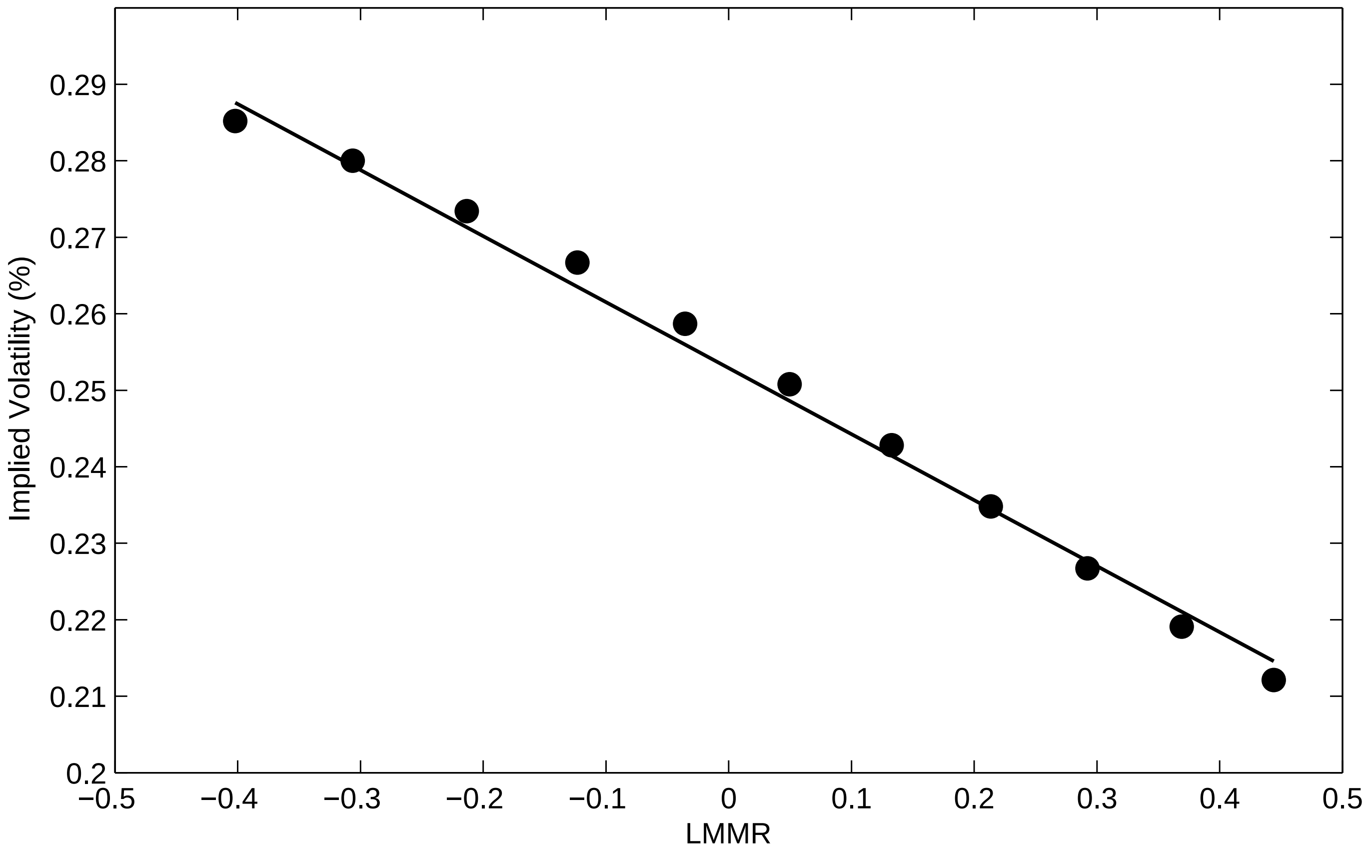


Implied Volatility Versus Starting Market $(\delta = 0.5)$

Calibration to Data: Amgen

- Consider Amgen call options with October 2009 expiry
- Strikes: Take options with $LMMR$ between -1 and 1, using closing mid-prices as of May 26, 2009
- For simplicity, asset-specific volatility $\sigma = 0$
- Market volatility σ^* estimated from call option data on S&P 500 Index (closest expiry Sep09)

From affine LMMR, $\sigma^* = 0.2549$



Affine $LMMR$ Fit to S&P 500 Index Options

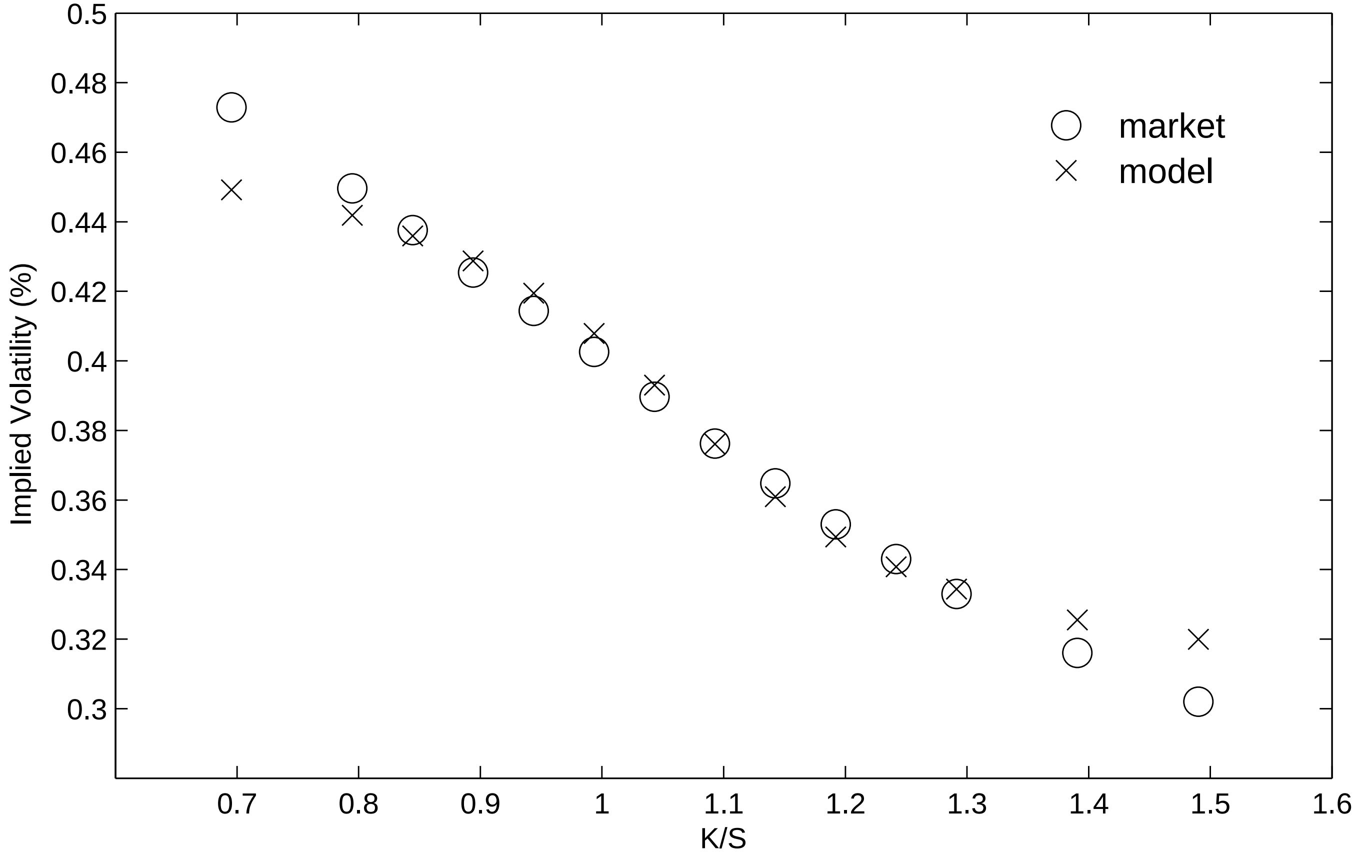
Calibration to Data: Amgen, contd.

- Need c, β, and δ
- Select params which min SSE between option model prices, market prices

For context, closing level of S&P 500 Index as of May 26, 2009 was 910.33

Estimated parameters: $\hat{c} = 925$, $\hat{\beta} = 1.17$, and $\hat{\delta} = 0.65$.

So market is below threshold



Volatility Skews for Amgen Call Options