Drift and Volatility Characterization of Progressive Utility:

Stochastic flows method

Or

Investment Performance Process, Part II

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Investment Banking and Utility Theory

Some remarks on martingale theory and utility functions in Investment Banking from M.Musiela, T.Zariphopoulo, C.Rogers +alii (2005-2009)

- No clear idea how to specify the utility function
- Classical or recursive utilities are defined in isolation to the investment opportunities given to an agent.
- Explicit solutions to optimal investment problems can only be derived under very restrictive model and utility assumptions, as Markovian assumption which yields to HJB PDEs.
- In non-Markovian framework, theory is concentrated on the problem of existence and uniqueness of an optimal solution, often via the dual representation of utility.

Shortcomings

Numéraire

The solution is strongly depending of choice of numéraire

Intertemporality

- The investor may want to use intertemporal diversification, i.e., implement short, medium and long term strategies
- Can the same utility function be used for all time horizons ?
- No- in fact the investor gets more value (in terms of the value function) from a longer term investment.
- At the optimum the investor should become indifferent to the investment horizon.

Consistent Dynamic Utility / Investment Performance Process

Consistent Dynamic Utility

Let $\mathcal{X}$ be a convex family of positive portfolios, called Test porfolios

Definition : An $\mathfrak{X}$-Consistent progressive utility $U(t,x)$ process is a positive adapted random field s.t.

- **Concavity assumption :** for $t \geq 0$, $x > 0 \mapsto U(t,x)$ is an increasing concave function, (in short utility function) .
- **Consistency with the class of test portfolios** For any admissible wealth process $X \in \mathfrak{X}$, $\mathbb{E}(U(t,X_t)) < +\infty$ and
$$\mathbb{E}(U(t,X_t)/\mathcal{F}_s) \leq U(s,X_s),\ \forall s \leq t.$$
- **Existence of optimal** For any initial wealth $x > 0$, there exists an optimal wealth process (benchmark) $X^* \in \mathfrak{X}(X_0^* = x)$,
$$U(s,X_s^*) = \mathbb{E}(U(t,X_t^*)/\mathcal{F}_s)\ \forall s \leq t.$$
- **In short** for any admissible wealth $X \in \mathfrak{X}$, $U(t,X_t)$ is a supermartingale, and a martingale for the optimal-benchmark wealth X^*.

Value Function of classical utility problem

Classical problem : Backward point of view

- Given a utility function $u(T, x)$ at time horizon T, the problem at time r is to maximize over all test portfolios starting from (r, x), the conditional expected utility of the terminal wealth

$$V(r, x, U, T)) = \operatorname*{ess\ sup}_{X \in \mathcal{X}(r,x)} \mathbb{E}(u(T, X_T)/\mathcal{F}_r)$$

- **DYNAMIC PROGRAMMING PRINCIPLE**

$$V(t, X_t, (u, T)) = V(t, X_t, (V(t+h, ., U, T), t+h)), a.s.$$

- If an optimal portfolio exists, the process $(V(t, X_t^*, (u, T)))_{t \leq T}$ is a martingale .
- The value function $(V(t, x, (u, T)))_{t \leq T}$ is a consistent utility process, with concave initial value V(r,x, (u,T)).

Transformation by change of probability measure and numéraire : $U(t,x) \stackrel{def}{=} Z_t\, u(x/Y_t)$

- Except the case where u is a power or exponential utility, the process U defined by $U(t,x) = Z_t u(x/Y_t)$ is an $\mathfrak{X}$-consistent stochastic utility iff Z is a martingale, $ZX/Y,\ X \in \mathfrak{X}$ are positive local martingales and Y is an admissible wealth process. Furthermore, Y is the optimal portfolio.
- If u is a power or exponential utility, then this condition is not a necessary condition. For example, if $Y \equiv 1,\ Z \equiv e^{\mu}$, then if u is a power utility with risk aversion a (resp. exponential utility with risk aversion c), it suffices to take Z and Y such that the following equation is satisfied

$$\frac{1}{a}\frac{d\mu_t}{dt} + r_t = -\frac{1}{2(1-a)}\|\eta_t\|^2 \Big(\textit{resp. } r = 0,\ \frac{d\mu_t}{dt} = c^2\|\eta_t\|^2 \Big),\ \forall t \geq 0.$$

where η is proportional to the optimal strategy.

The General Market Model

- The security market consists of one riskless asset S^0, $dS^0_t = S^0_t r_t dt$, and d continuous risky assets S^i, $\quad i = 1..d$ defined on a filtred Brownian space $(\Omega, \mathcal{F}_{t\geq 0}, \mathbb{P})$

$$\frac{dS^i_t}{S^i_t} = b^i_t dt + \sigma^i_t dW_t, \qquad 1 \leq i \leq d$$

- volatility matrix σ_t is invertible, with bounded inverse
- Risk premium vector, $\eta(t)$ with $\quad b(t) - r(t)\mathbf{1} = \sigma_t \eta(t)$

Def A positive wealth process is defined as a pair (x, π)

- $x > 0$ is the initial value of the portfolio
- $\pi = (\pi^i)_{1\leq i\leq d}$ is the (predictable) proportion of each asset held in the portfolio, assumed to be S-integrable process

- Thanks to AOA in the market, wealth process with π-strategy is driven by

$$\frac{dX^\pi_t}{X^\pi_t} = r_t dt + \underline{\pi_t}\sigma_t(dW_t + \eta_t dt),$$

Test Portfolios and Convexity

- The Class of Test Portfolios The strategy π is required to lie at any time t in some non empty adapted closed convex set $\mathcal{K}_t(X_t) \subseteq \mathbb{R}^d$
- In general, for simplicity, we assume that
  - $\mathcal{K}_t(X_t)$ is a closed convex cone,
  - or more generally a translated of closed convex cone.
  - Often, $\mathcal{K}_t(X_t)$ is a vector (affine) space. (today for example)

$\Rightarrow$ Then the set of admissible portfolios is convex

Def The class of test wealth processes is denoted

$$\mathfrak{X}(\mathcal{K}) := \{(X_t^\pi) \in \mathbb{X}^+ \ \text{ such that } \pi_t \in \mathcal{K}_t(, X_t^\pi)\}$$

Utility Dynamics

Let U be a dynamic utility (concave, increasing) ,

$$dU(t,x) = \beta(t,x)dt + \Gamma(t,x)dW_t$$

such that $U(t, X_t^\pi)$ is a supermartingale for $X^\pi \in \mathfrak{X}(\mathcal{K})$ and a martingale for the optimal one

Open questions

- What about the drift β of the utility ?
- What about the volatility Γ of the utility ?
- Under which assumptions on (β, Γ) can one be sure that solutions are concave and increasing,
- or verify Inada condition and asymptotic elasticity constraint ?

Main difficulties come from the forward definition

Stochastic calculus depending of a parameter

From Kunita Book, Carmona-Nualart

- Let ϕ be a semimartingale random field satisfying

$$d\phi(t,x) = \mu(t,x)dt + \gamma(t,x)dW_t, \tag{1}$$

- The pair (μ, γ) is called the local characteristic of ϕ, and γ is referred as the volatility random field.
- A semimartingale random field ϕ is said to be Itô-Ventzel regular if
  - ϕ is a continuous $\mathcal{C}^{2+\cdots}$-process in x
  - local characteristic (μ, γ) are $\mathcal{C}^1$ in x
  - additional assumptions as more regularity, uniform integrability are need to guarantee smoothness of ϕ and its derivatives, and the existence of regular version of these random fields

Itô-Ventzel's Formula (Kunita)

- Let ϕ and ψ be Itô-Ventzel's regular one-dimensional stochastic flows

$$d\phi(t,x) = \mu(t,x)dt + \gamma(t,x)dW_t, \quad d\psi(t,x) = \alpha(t,x)dt + \nu(t,x)dW_t.$$

- The compound random field $\phi o \psi(t,x) = \phi(t,\psi(t,x))$ is a regular semimartingale

Itô-Ventzel's Formula

$$\begin{aligned} d(\phi o \psi)(t,x) &= \mu(t,\psi(t,x))dt + \gamma(t,\psi(t,x))dW_t \\ &+ \phi'_x(t,\psi(t,x))d\psi(t,x) + \frac{1}{2}\phi''_{xx}(t,x)(t,\psi(t,x))||\nu(t,x)||^2 dt \\ &+ \langle \gamma'_x(t,\psi(t,x)), \nu(t,x)\rangle dt. \end{aligned}$$

The volatility of $\phi o \psi$ is given by $\nu^{\phi o \psi}(t,x) = \gamma(t,\psi(t,x)) + \phi'_x(t,\psi(t,x))\nu(t,x)$.

Drift Constraint I

Let U be a Itô-Ventzel regular utility (concave, increasing) with drift and volatility $\beta(t,x), \Gamma(t,x)$,

$$dU(t,x) = \beta(t,x)dt + \Gamma(t,x)dW_t$$

Same method as in HJM framework, or Implied Volatility dynamics.

Lemma : For any portfolio X^π,

$$\begin{aligned}
dU(t,X_t^\pi) &= dM_t(x,\pi) + \big(\beta(t,X_t^\pi) + U'_x(t,X_t^\pi) r_t X_t^\pi + \mathcal{P}(t,X_t^\pi,\pi_t)\big)dt,\\
\mathcal{P}(t,x,\pi) &= \Big[< \pi\sigma_t, U'_x(t,x)\eta_t + \Gamma'_x(t,x) > + \frac{1}{2} U''_{xx}(t,x)\|\pi\sigma_t\|^2 \Big]\\
&= \frac{1}{2} x^2\, U''_{xx}(t,x)\Big[\|\pi\sigma_t + \frac{\Psi'_x}{xU''_{xx}}(t,x)\|^2 - \|\frac{\Psi'_x}{xU''_{xx}}\|^2(t,x)\Big]\\
\text{where } \Psi'_x(t,x) &= U'_x(t,x)\eta_t + \Gamma'_x(t,x)\\
dM_t(x,\pi) &= U'_x(t,X^\pi)X^\pi\, \pi_t\sigma_t + \Gamma(t,X^\pi)dW_t
\end{aligned}$$

Let $\mathcal{P}^*(t,x) = \sup_{\pi\sigma_t \in \mathcal{K}_t^\sigma(x)} \mathcal{P}(t,x,\pi) := (-\frac{1}{2}x^2\, U''_{xx}(t,x))\mathcal{Q}^*(t,x)$. Then,

$$\mathcal{Q}^*(t,x) = -\mathrm{dist}^2_{\mathcal{K}_t^\sigma(x)}\Big(-\frac{\Psi'_x}{xU''_{xx}}\Big) + \Big\|\frac{\Psi'_x}{xU''_{xx}}\Big\|^2 = \mathrm{dist}^2_{\mathcal{K}_t^{\sigma,\perp}}\Big(-\frac{\Psi'_x(t,x)}{xU''_{xx}(t,x)}\Big)$$

where $\mathcal{K}_t^{\sigma,\perp}(x)$ is the orthogonal set of the convex set $\mathcal{K}_t^\sigma(x)$

Standard cases

- If $\mathcal{K}_t^\sigma(x)$ is a cone, $-\mathcal{P}^*(t,x) = \frac{1}{2U''_{xx}(t,x)}\|\prod_{\mathcal{K}_t^\sigma(x)} -\Psi'_x(t,x)\|^2$
- If $\mathcal{K}_t^\sigma(x)$ is a displaced cone $\mathcal{K}_t^\sigma(x) = \mathcal{K}_t^{\sigma,0}(x) - \delta_t$,

$$-\mathcal{P}^*(t,x) = \frac{1}{U''_{xx}}\Big(\|\prod_{\mathcal{K}_t^{\sigma,0}}(-\Psi'_x + xU''_{xx}\delta_t)\|^2 + \|\Psi'_x\|^2 - \|-\Psi'_x + xU''_{xx}\delta_t\|^2\Big)$$

emark Put $\Psi^\eta = (\eta_t U + \Gamma)(t,x)$, and $\Psi^{\eta,\delta} = (\eta_t + \delta_t)U + \Gamma)(t,x) - xU'_x\delta_t$, then

$$\Psi'_x - xU''_{xx}\delta_t = \Psi_x^{\eta,\delta,'}$$

Verification Theorem I

Let U be a Itô-Ventzel regular progressive utility (concave, increasing) with decomposition $dU(t,x) = \beta(t,x)dt + \Gamma(t,x)dW_t$. We assume that :

Hyp The drift β is related to the volatility :

$$\beta(t,x) = -U'_x(t,x)r_t x + \frac{1}{2}x^2\, U''_{xx}(t,x))\, \mathrm{dist}^2_{\mathcal{K}_t^{\sigma,\perp}}\big(-\frac{U'_x(t,x)\eta_t + \Gamma'_x(t,x))}{xU''_{xx}(t,x)}\big)$$

$\Rightarrow$ Then the optimal policy $\pi^*(t,x)\sigma_t$ is

$$\pi^*(t,x)\sigma_t = \mathrm{proj}_{\mathcal{K}_t^{\sigma}(x)}\big(-\frac{U'_x(t,x)\eta_t + \Gamma'_x(t,x)}{xU''_{xx}(t,x)}\big)$$

$\Rightarrow$ The volatility $\Gamma(t,x)$ verifies

$$U'_x(t,x)\eta_t + \Gamma'_x(t,x) = -xU''_{xx}(t,x)\pi^*(t,x)\sigma_t - \nu^{\perp}],\ \text{where}\ \nu^{\perp}(t,x) \in \mathcal{K}_t^{\sigma,\perp}(x)$$

Verification Theorem II

Theorem

Under previous hypothesis,

- Assume that $\pi^*(t,x)$ is sufficiently smooth so that the equation

$$dX_t^* = X_t^*(r_t dt + \pi^*(t, X_t^*)\sigma_t(dW_t + \eta_t dt)$$

has a (unique ? strong ?) positive solution for any initial wealth x>0.

$\Rightarrow$ Then, the progressive increasing utility U is a $\mathcal{X}(\mathcal{K})$-consistent utility, with optimal wealth X_t^*.

Change of numéraire, and Martingale Market

Let $Y > 0$ be a new numéraire such that

$$\frac{dY_t}{Y_t} = (r_t - \mu_t)dt + \delta_t^\top (dW_t + \eta_t dt)$$

- In the new market, prices are given by $\hat{X}_t := \frac{X_t^\pi}{Y_t}$,
- Portfolio dynamics is now constraint by

$$\frac{d\hat{\mathcal{X}}_t^\pi}{\hat{X}_t^\pi} = \mu_t dt + (\pi_t\sigma_t - \delta_t)^\top)(dW_t + (\eta_t - \delta_t)dt)$$

with $\hat{\pi}_t\sigma_t = \pi_t\sigma_t - \delta_t \in \hat{\mathcal{K}}(\hat{\mathcal{X}}^\pi) = \mathcal{K}^\sigma(Y_t\hat{\mathcal{X}}_t^\pi) - \delta_t$

- In the new market, $\hat{r}_t = \mu_t$, $\hat{\eta}_t = \eta_t - \delta_t$, $\hat{\mathcal{K}}(x) = \mathcal{K}^\sigma(Y_t x)\delta_t$.

Stability by change of numéraire

Theorem Let $U(t,x)$ be a $\mathfrak{X}$-consistent Itô-Ventzel regular utility and let Y be a positive numéraire. Denote by $\mathfrak{X}^Y$ the class of processes defined by $\mathfrak{X}^Y = \{\frac{X}{Y},\ X \in \mathfrak{X}\}$, then

- $V(t,x) = U(t, xY_t)$ is a $\mathfrak{X}^Y$-consistent utility in the market of numeraire Y,

$$
\begin{aligned}
dV(t,x) &= (\hat{\Gamma}(t,x)dW_t - xV'_x(t,x)\mu_t dt \\
&+ \frac{1}{2V''_{xx}}(\|V'_x\hat{\eta}_t + \Gamma'_x\|^2 - \|\prod_{(\mathcal{K}_t\sigma_t)^\perp)}(V'_x\hat{\eta}_t + \hat{\Gamma}'_x - xV''_{xx}\delta_t)\|^2)dt
\end{aligned}
$$

- When the market numéraire is used as numéraire, $\mu = 0, \delta = \eta$, the market has no risk premium (martingale), and the ratio $\frac{\Gamma'_x}{V'_x}$ has the same impact than a risk premium, but depending on the level of the wealth x at time t.

Inverse flows

Proposition

Let ϕ be a strictly monotone Itô-Ventzel regular flow with inverse process $\xi(t,y)=\phi(t,.)^{-1}(y)$. Assume $d\phi(t,x)=\mu(t,x)dt+\gamma(t,x)dW_t$,

i) *The inverse flow $\xi(t,y)$ has as dynamics in old variables*

$$d\xi(t,y)=-\xi'_y(t,y)(\mu(t,\xi)dt+\gamma(t,\xi)dW_t)+\frac{1}{2}\partial_y\frac{\|\gamma(t,\xi)\|^2}{\phi'_x(t,\xi)}dt$$

ii) *In terms of new variable, with $\nu^{\xi}(t,y)=-\xi'_y\gamma(t,\xi)$*

$$d\xi(t,y)=\nu^{\xi}(t,y)dW_t+\Big(\frac{1}{2}\partial_y\big(\frac{\|\nu^{\xi}(t,y)\|^2}{\xi'_y}\big)-\mu(t,\xi)\xi'_y(t,y)\Big)dt$$

iii) *If $\phi=\Phi'_x(t,x)$with $d\Phi(t,x)=M(t,x)dt+C(t,x)dW_t$, then $\xi=\Xi'_y(t,y)$*

$$d\Xi(t,y)=-C(t,\xi)dW_t-M(t,\xi)dt+\frac{1}{2}\frac{\|C'_x(t,\xi)\|^2}{\Phi''_{xx}(t,\xi)}dt$$

Convex conjugate SPDE

Let Inada conditions $U'_x(t,0) = +\infty,\; U'_x(t,\infty) = 0$ hold true.

- $(\tilde{U}(t,y); y \geq 0)$ is the convex decreasing Fenchel transform of U

$$\tilde{U}(t,y) = \sup_{x>0, x\in Q^+} \big(U(t,x) - x\,y\big)$$

- The optimum is achieved at $U'_x(t,x*) = y$, and $-\tilde{U}'_y = (U'_x(t,.))^{-1}$

Dual SPDE

Let U be a $\mathcal{X}(\mathcal{K})$-consistent utility, with Itô-Ventzel regularity, then

$$\begin{aligned} d\tilde{U}(t,y) &= \beta^1(t,-\tilde{U}'_y)dt + \Gamma(t,-\tilde{U}'_y)dW_t \quad \text{where} \\ \beta^1(t,x) &= \beta(t,x) - \frac{1}{2}\frac{\|\Gamma'_x(t,x)\|^2}{\mathcal{U}''_{xx}(t,x)} \\ &= -r_t x U'_x + \frac{-1}{2U''_{xx}}\Big(\|\Gamma'_x(t,x)\|^2 - \|\prod_{\mathcal{K}\sigma} s - \eta_t U'_x - \Gamma'_x\|^2\Big) \end{aligned}$$

Convex conjugate SPDE

- $\beta^1(t,x)$ is the solution of an optimization program achieved by the projection of $-\eta_t U'_x - \Gamma'_y$ on $(\mathcal{K}\sigma)^\perp)$, defined before as $\nu^\perp(t,x)$
- $\tilde{\Gamma}(t,y) = \Gamma(t,-\tilde{U}'_y)$ and $\tilde{\beta}(t,y) = \beta^1(t,-\tilde{U}'_y))$

$$\tilde{\beta}(t,y) = -r_t y\tilde{U}'_y + \frac{-1}{2\tilde{U}''_{yy}}\Big(\| \prod_{\mathcal{K}\sigma}(-\eta_t y\tilde{U}''_{yy} + \tilde{\Gamma}'_y\|^2 - \|\tilde{\Gamma}'_y\|^2\Big)$$

Convex conjugate forward Utility I

Under previous assumption,

- The conjugate Utility $\tilde{U}(t,y)$ is a convex decreasing stochastic flows,
- consistent with the family $\mathcal{Y}$ of semimartingales Y^{ν}, defined from

$$\frac{dY_t}{Y_t} = (-r_t + |\eta_t|^2)dt + (\nu_t - \eta_t)(dW_t - \eta_t dt), \quad \nu_t \in (\mathcal{K}\sigma)^{\perp})(-\tilde{U}'_y)$$

- There exists a dual optimal choice $\tilde{\nu}^*(t,y) = \nu^{\perp}(t, -\tilde{U}'_y)$

From any $y > 0$, the optimal dual process $Y^*_t = Y^{\tilde{\nu}^*}_t$ is given by

$$Y^*(t,y) = U'_x(t, X^*(t, -\tilde{U}'_y)), \quad \mathcal{Y}(t,x) := U'_x(t, X^*(t,x))$$

Remark : If $X^*(t,x)$ is strictly monotone in x, by taking the inverse $\mathcal{X}(t,x)$, we can obtain $U'_x(t,x)$ in terms of $\mathcal{Y}(t,x)$ and $\xi(t,x)$

$\mathfrak{X}$-Consistent Utilities with given optimal portfolio

Approach by Stochastic Flows

Flows Assumption

Monotony assumption

Let $X_t^*(x)$ be a wealth process assumed to be continuous and increasing in x from 0 to $+\infty$.

- true in a lot examples,
- may be a consequence of no arbitrage opportunity.
- from flows point of view, it is implied by coefficient regularity

Hyp Moreover, $X_t^*(x)$ is assumed to be a Itô-Ventzel stochastic flow

$$dX_t^*(x) = X_t^*(x)(r_t + \pi(t, X_t^*(x))\sigma_t dW_t = \mu(t, X_t^*(x))dt + \delta(t, X_t^*(x))dW_t$$

Not Denote by $\mathcal{X}(t,z)$ the inverse of the flow $\mathcal{X}(t,z) = (X_t^*(.))^{-1}(z)$.

Proposition (Basic example)

Let $\mathcal{X}(t,z)$ be the inverse flow of $X^*(t,z)$, then for any utility function u such that $u'(\mathcal{X}(t,z))$ is locally integrable near $z=0$, the stochastic process U defined by

$$U(t,x) = \int_0^x u'(\mathcal{X}(t,z))H_t^{r,\eta}dz, \quad U(t,0)=0 \tag{2}$$

is a $\mathfrak{X}$-Consistent utility constrained by $\mathcal{K}$.

- *The associated optimal wealth process is X^**
- *and the optimal dual choice $\nu^\perp$ is 0.*
- $\Gamma'_x = -\eta_t H_t\, u'(\mathcal{X}(t,z)) - H_t u''(\mathcal{X}(t,z))\mathcal{X}'_z\delta(t,x) = -U'_x\eta - U''_{xx}\delta(t,x)$

Furthermore, the conjugate process of U denoted by $\tilde{U}$, is given by

$$\tilde{U}(t,y) = \int_y^{+\infty} X^*(t, -\tilde{u}'_0(z(H_t^{r,\eta})^{-1})dz, \tag{3}$$

Hypothesis (C3)

Suppose that the optimal portfolio X^* and the optimal dual process Y^* are Ito-Ventzel regular processes

$$dX_t^*(x) = X_t^*(r_t + \pi(t, X^*)\sigma(t, X^*)(dW_t + \eta_t dt), \quad \pi(t, X_t^*(x)) \in \mathcal{K}_t(X_t(x))$$

$$dY^*(t, y) = Y^*\{(-r_t + |\eta|^2)dt + (\nu(t, Y^*) - \eta)(dW_t - \eta dt)\},$$

$$\nu(t, \mathcal{Y}(t, x)) \in ((\mathcal{K}\sigma)^\perp)(t, \tilde{u}'(X_t^*(x)))$$

where u is a deterministe $\mathcal{C}^2$ concave function.

Interpretation of SPDE Utility

Theorem

Let $(X_t^*(x))$, and $Y*(t,y)$ two regular stochastic flows as above.
Put $\mathcal{Y}(t,x) = Y^*(t,u'(x))$ and $\mathcal{X}(t,z) = (X^*(t,.))^{-1}$.

i) *Then the process U defined by*

$$U(t,x) = \int_0^x \mathcal{Y}(t,\mathcal{X}(t,z))dz$$

is a $\mathfrak{X}$-Consistent stochastic utility satisfying the HJB type SPDE, with volatility vector Γ'_x

$$\eta_t \mathcal{Y}(t,\mathcal{X}(t,x)) + \Gamma'_x(t,x) = -xU''_{xx}\pi(t,x)\sigma_t - \nu(\mathcal{Y}(t,\mathcal{X}(t,x))$$

Converse point of view I

Martingale market is assumed for simplicity

Let (β, Γ) be the regular characteristic of Stochastic HJB PDE, where Γ'_x has the following decomposition

$$\Gamma'_x(t,x) = \nu(t, U'_x(t,x)) - U''_{xx}(t,x)\delta(t,x) \tag{4}$$

with $\delta(t,x) \in \mathcal{K}\sigma_t(x))$ and $\nu(t, U'_x(t,x))(\mathcal{K}\sigma_t(x)))^{\perp}$

Converse point of view II

If the solution of the system

$$dX_t^*(x) = \delta(t, X_t^*(x))dW_t, \quad \gamma(t, X_t^*(x)) \in \mathcal{K}_t(X_t^*(x))$$

$$d\mathcal{Y}(t,x) = \nu(t, \mathcal{Y}(t,x))dW_t, \quad \nu(t, \mathcal{Y}(t,x)) \in ((\mathcal{K}\sigma)^* \cap \gamma^{\perp})(t, X_t^*(x))$$

exist, and is a regular flow, then same ideas may be used to show the existence, uniqueness, regularity of solution of

$$dU(t,x) = \frac{1}{2}\Big[\frac{\|(\Gamma_x')^{\mathcal{K}}(t,x)\|^2}{U_{xx}''(t,x)}\Big]dt + \Gamma(t,x)dW_t$$

Conclusion

- All Consistent dynamic utilities with continuous strictly increasing optimal portfolio may be generated as above
- Valid also for classical optimization problem.
- Work in progress

The main assumption is that the optimal portfolio is increasing in x, because we have the same characterization in more abstract form, based on the properties of the optimum.

Thank you for your attention