Mean Field Games: Numerical Methods

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I. Finite Horizon: Numerical Methods

The system of PDEs

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) & = & V[m], & \text{in } (0,T) \times \mathbb{T}, \\ \dfrac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u) \right) & = & 0, & \text{in } (0,T) \times \mathbb{T}, \end{cases}$$

with the initial and terminal conditions

$$u(t=0) = V_0[m(t=0)], \quad \text{and} \quad m(T,x) = m_T(x), \quad \text{in } \mathbb{T},$$

and

$$m \geq 0, \qquad \int_{\mathbb{T}} m(t,x)dx = 1.$$

- $\mathbb{T}$ unit torus of $\mathbb{R}^d$
- $\nu \geq 0$
- H is a smooth Hamiltonian (convex):

$$H(x,p) = \sup_{\gamma \in \mathbb{R}^d} \left(p \cdot \gamma - L(x,\gamma) \right), \quad \text{with} \quad \lim_{|\gamma| \to \infty} \inf_x \frac{L(x,\gamma)}{|\gamma|} = +\infty$$

- V and V_0 are operators **from the space of probability measures on** $\mathbb{T}$ **into a bounded set of Lipschitz functions on** $\mathbb{T}$ such that

  $V[m_n]$ converges uniformly on $\mathbb{T}$ to $V[m]$ if m_n weakly converges to m.

  Typical examples for V include nonlocal smoothing operators.

- Alternatively $V[m](t,x) = V(m(t,x))$.
- m_T probability density on $\mathbb{T}$.

The pde systems comes from passing to the limit in a finite horizon Nash-equilibrium, where the cost of the player i at time t is

$$\mathbb{E}\left(\int_t^T \left(L(X_s^i, \gamma_s^i) + V\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_s^j}\right]\right) ds + V_0\left[\frac{1}{N-1}\sum_{j\neq i}\delta_{X_T^j}\right]\right)$$

The N players initial conditions are random, independent, with the same probability distribution $m_\circ$, and

$$dX_t^i = \sqrt{2\nu}dW_t^i - \gamma^i dt, \quad X_0^i = x^i \in \mathbb{T}.$$

Finite difference schemes

Goal: use a (semi-)implicit finite difference scheme, robust when $\nu \to 0$, which guarantees existence, and possibly uniform bounds and uniqueness.

Take $d = 2$:

- Let $\mathbb{T}_h$ be a uniform grid on the torus with mesh step h, and x_{ij} be a generic point in $\mathbb{T}_h$.
- Uniform time grid: $\Delta t = T/N_T$, $t_n = n\Delta t$.
- The values of u and m at $(x_{i,j}, t_n)$ are resp. approximated by $U^n_{i,j}$ and $M^n_{i,j}$.

Notation:

- The discrete Laplace operator:

$$(\Delta_h W)_{i,j} = -\frac{1}{h^2}(4W_{i,j} - W_{i+1,j} - W_{i-1,j} - W_{i,j+1} - W_{i,j-1}).$$

- Right-sided finite difference formulas for $\partial_1 w(x_{i,j})$ and $\partial_2 w(x_{i,j})$:

$$(D_1^+ W)_{i,j} = \frac{W_{i+1,j} - W_{i,j}}{h}, \quad \text{and} \quad (D_2^+ W)_{i,j} = \frac{W_{i,j+1} - W_{i,j}}{h}.$$

- The set of 4 finite difference formulas at $x_{i,j}$:

$$[D_h W]_{i,j} = \Big((D_1^+ W)_{i,j}, (D_1^+ W)_{i-1,j}, (D_2^+ W)_{i,j}, (D_2^+ W)_{i,j-1}\Big).$$

Discrete HJB equation

$$\frac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) = V[m]$$

$$\downarrow$$

$$\frac{U^{n+1}_{i,j} - U^n_{i,j}}{\Delta t} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = (V_h[M^n])_{i,j}$$

- $$\begin{aligned} & g(x_{i,j}, [D_h U^{n+1}]_{i,j}) \\ =& g\Big(x_{i,j}, (D_1^+ U^{n+1})_{i,j}, (D_1^+ U^{n+1})_{i-1,j}, (D_2^+ U^{n+1})_{i,j}, (D_2^+ U^{n+1})_{i,j-1}\Big), \end{aligned}$$

- for instance,

$$(V_h[M])_{i,j} = V[m_h](x_{i,j}),$$

calling m_h the piecewise constant function on $\mathbb{T}$ taking the value $M_{i,j}$ in the square $|x - x_{i,j}|_\infty \leq h/2$.

Classical assumptions on the discrete Hamiltonian g

$$(q_1, q_2, q_3, q_4) \to g\,(x, q_1, q_2, q_3, q_4)\,.$$

- **Monotonicity:** g is nonincreasing with respect to q_1 and q_3 and nondecreasing with respect to to q_2 and q_4.
- **Consistency:**

$$g\,(x, q_1, q_1, q_3, q_3) = H(x, q), \quad \forall x \in \mathbb{T}, \forall q = (q_1, q_3) \in \mathbb{R}^2.$$

- **Differentiability:** g is of class $\mathcal{C}^1$, and

$$\left|\frac{\partial g}{\partial x}\Big(x, (q_1, q_2, q_3, q_4)\Big)\right| \leq C(1 + |q_1| + |q_2| + |q_3| + |q_4|).$$

- **Convexity:** $(q_1, q_2, q_3, q_4) \to g\,(x, q_1, q_2, q_3, q_4)$ is convex.

The discrete version of

$$\frac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \frac{\partial H}{\partial p}(x, \nabla v) \right) = 0. \tag{$\dagger$}$$

It is chosen so that

- each time step leads to a linear system with a matrix
  - whose diagonal coefficients are negative,
  - whose off-diagonal coefficients are nonnegative,

  in order to hopefully use some **discrete maximum principle.**

- The argument for uniqueness should hold in the discrete case, so **the discrete Hamiltonian g should be used for $(\dagger)$ as well.**

Principle

Discretize

$$-\int_{\mathbb{T}} \operatorname{div}\left(m\frac{\partial H}{\partial p}(x,\nabla u)\right) w = \int_{\mathbb{T}} m\frac{\partial H}{\partial p}(x,\nabla u)\cdot\nabla w$$

by

$$h^2\sum_{i,j}\mathcal{B}_{i,j}(U,M)W_{i,j} := h^2\sum_{i,j} M_{i,j}\nabla_q g(x_{i,j},[D_hU]_{i,j})\cdot[D_hW]_{i,j},$$

which leads to

$$\mathcal{B}_{i,j}(U,M)=\frac{1}{h}\left(\begin{array}{c}\left(\begin{array}{c} M_{i,j}\dfrac{\partial g}{\partial q_1}(x_{i,j},[D_hU]_{i,j}) - M_{i-1,j}\dfrac{\partial g}{\partial q_1}(x_{i-1,j},[D_hU]_{i-1,j}) \\ +M_{i+1,j}\dfrac{\partial g}{\partial q_2}(x_{i+1,j},[D_hU]_{i+1,j}) - M_{i,j}\dfrac{\partial g}{\partial q_2}(x_{i,j},[D_hU]_{i,j})\end{array}\right) \\ +\left(\begin{array}{c} M_{i,j}\dfrac{\partial g}{\partial q_3}(x_{i,j},[D_hU]_{i,j}) - M_{i,j-1}\dfrac{\partial g}{\partial q_3}(x_{i,j-1},[D_hU]_{i,j-1}) \\ +M_{i,j+1}\dfrac{\partial g}{\partial q_4}(x_{i,j+1},[D_hU]_{i,j+1}) - M_{i,j}\dfrac{\partial g}{\partial q_4}(x_{i,j},[D_hU]_{i,j})\end{array}\right)\end{array}\right)$$

This yields the scheme:

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = (V_h[M^n])_{i,j}$$

$$0 = \frac{M_{i,j}^{n+1} - M_{i,j}^{n}}{\Delta t} + \nu(\Delta_h M^n)_{i,j}$$

$$+\frac{1}{h}\left(\begin{array}{l} \left(\begin{array}{l} M_{i,j}^{n} \dfrac{\partial g}{\partial q_1}(x_{i,j}, [D_h U^{n+1}]_{i,j}) - M_{i-1,j}^{n} \dfrac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U^{n+1}]_{i-1,j}) \\ +M_{i+1,j}^{n} \dfrac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U^{n+1}]_{i+1,j}) - M_{i,j}^{n} \dfrac{\partial g}{\partial q_2}(x_{i,j}, [D_h U^{n+1}]_{i,j}) \end{array} \right) \\ \\ + \left(\begin{array}{l} M_{i,j}^{n} \dfrac{\partial g}{\partial q_3}(x_{i,j}, [D_h U^{n+1}]_{i,j}) - M_{i,j-1}^{n} \dfrac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U^{n+1}]_{i,j-1}) \\ +M_{i,j+1}^{n} \dfrac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U^{n+1}]_{i,j+1}) - M_{i,j}^{n} \dfrac{\partial g}{\partial q_4}(x_{i,j}, [D_h U^{n+1}]_{i,j}) \end{array} \right) \end{array} \right)$$

Classical discrete Hamiltonians g can be chosen.

For example, if the Hamiltonian is of the form

$$H(x, \nabla u) = \psi(x, |\nabla u|),$$

a possible choice is the **Godunov scheme**

$$g(x, q_1, q_2, q_3, q_4) =$$
$$\psi\left(x, \sqrt{\min(q_1, 0)^2 + \max(q_2, 0)^2 + \min(q_3, 0)^2 + \max(q_4, 0)^2}\right).$$

If $\psi(x, w)$ is convex and nondecreasing w.r.t. w, then g is a convex function of (q_1, q_2, q_3, q_4); g is nonincreasing w.r.t. q_1 and q_3 and nondecreasing w.r.t. q_2 and q_4.

Finally, it can be proven that the global scheme is consistent if H is smooth enough.

Existence for the discrete problem

Theorem Assume that $M^{N_T} \geq 0$ and that $h^2 \sum_{i,j} M_{i,j}^{N_T} = 1$. Under the assumptions above on V, V_0 and g, **the discrete problem has a solution and there is a Lipschitz estimate on U_h^n uniform in n, h and Δt.**

Strategy of proof

$$\mathcal{K} = \left\{ (M_{i,j})_{0 \leq i,j < N} : h^2 \sum_{i,j} M_{i,j} = 1, M_{i,j} \geq 0 \right\}.$$

Apply Brouwer fixed point theorem to a well chosen mapping

$$\begin{array}{rccccc} \chi : & \mathcal{K}^{N_T} & & \longrightarrow & & \mathcal{K}^{N_T}, \\ & (M^n)_n & \rightarrow & (U^n)_n & \rightarrow & (M^n)_n. \end{array}$$

Proof: a fixed point method in $\mathcal{K}^{N_T}$,

Step 1: a map $\Phi: \quad (M^n)_n \to (U^n)_n$**.**

Given $(M_{i,j}^{N_T})$, define the map Φ: $(M^n)_{0\le n<N_T} \in \mathcal{K}^{N_T} \to (U^n)_{0\le n\le N_T}$:

$$\begin{cases} \dfrac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) = (V_h[M^n])_{i,j}\,, \\ U_{i,j}^0 = V_0[m_h^0](x_{i,j}). \end{cases}$$

- Existence is classical: (Leray-Schauder fixed point theorem at each time step, making use of the monotonicity of g, the uniform boundedness assumption on V and of $H(\cdot, 0)$).

- Uniqueness stems from the monotonicity of g.

Step 2: estimates

• There exists a constant C independent of $(M^n)_n$ and h s.t.

$$\|U^n\|_\infty \leq C(1+T).$$

• The map Φ is continuous, from the continuity of V and well known results on continuous dependence on the data for monotone schemes.

• There exists a constant L independent of $(M^n)_n$ and h s.t.

$$\|D_h U^n\|_\infty \leq LT, \quad \forall n,$$

proved by using the assumption

$$\left|\frac{\partial g}{\partial x}(x, q_1, q_2, q_3, q_4)\right| \leq C(1 + |q_1| + |q_2| + |q_3| + |q_4|).$$

Step 3: A map $\chi: \quad (M^n)_n \to (\widetilde{M}^n)_n$

- Choose a positive constant $\mu > 0$ large enough.
- For $(U^n)_n = \Phi((M^n)_n)$, **backward linear** parabolic problem for $\widetilde{M}^n$:

$$\left\{\begin{array}{l}
\widetilde{M}^{N_T} = M^{N_T}, \\[2em]
-\mu M_{i,j}^n = \dfrac{\widetilde{M}_{i,j}^{n+1} - \widetilde{M}_{i,j}^n}{\Delta t} - \nu(\Delta_h \widetilde{M}^n)_{i,j} - \mu \widetilde{M}_{i,j}^n \\
\qquad + \dfrac{1}{h}\left(\begin{array}{l}
\widetilde{M}_{i,j}^n \dfrac{\partial g}{\partial q_1}(x_{i,j}, [D_h U^{n+1}]_{i,j}) - \widetilde{M}_{i-1,j}^n \dfrac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U^{n+1}]_{i-1,j}) \\
+\widetilde{M}_{i+1,j}^n \dfrac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U^{n+1}]_{i+1,j}) - \widetilde{M}_{i,j}^n \dfrac{\partial g}{\partial q_2}(x_{i,j}, [D_h U^{n+1}]_{i,j})
\end{array}\right) \\
\qquad + \dfrac{1}{h}\left(\begin{array}{l}
\widetilde{M}_{i,j}^n \dfrac{\partial g}{\partial q_3}(x_{i,j}, [D_h U^{n+1}]_{i,j}) - \widetilde{M}_{i,j-1}^n \dfrac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U^{n+1}]_{i,j-1}) \\
+\widetilde{M}_{i,j+1}^n \dfrac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U^{n+1}]_{i,j+1}) - \widetilde{M}_{i,j}^n \dfrac{\partial g}{\partial q_4}(x_{i,j}, [D_h U^{n+1}]_{i,j})
\end{array}\right)
\end{array}\right.$$

From the previous estimates on $(U^n)_n$, one can find μ large enough and independent of $(M^n)_n$ such that the iteration matrix is the opposite of a M-matrix, thus **there is a discrete maximum principle.**

Therefore, there exists a unique solution $(\widetilde{M}^n)_n$.
Moreover,

$$\begin{array}{rclr} M^n \geq 0 & \Rightarrow & \widetilde{M}^n \geq 0, & \forall n, \\ h^2 \sum_{i,j} M^n = 1 & \Rightarrow & h^2 \sum_{i,j} \widetilde{M}^n = 1, & \forall n. \end{array}$$

Thus $\widetilde{M}^n \in \mathcal{K}$ for all n. Define the map

$$\begin{array}{rrcl} \chi : & \mathcal{K}^{N_T} & \mapsto & \mathcal{K}^{N_T}, \\ & (M^n)_{0 \leq n < N_T} & \rightarrow & (\widetilde{M}^n)_{0 \leq n < N_T} \end{array}$$

Step 4: existence of a fixed point of χ

From the boundedness and continuity of the mapping Φ, and from the fact that g is $\mathcal{C}^1$, we obtain that $\chi : \mathcal{K}^{N_T} \mapsto \mathcal{K}^{N_T}$ is continuous.

From Brouwer fixed point theorem, χ has a fixed point, which yields a solution of the full system.

Uniqueness

Theorem Same assumptions as above on V, V_0, H and g. Assume also that the operators V_h and $V_{0,h}$ are strictly monotone, i.e.

$$\begin{aligned} \left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M}\right)_2 \leq 0 \quad &\Rightarrow \quad V_h[M] = V_h[\widetilde{M}], \\ \left(V_{0,h}[M] - V_{0,h}[\widetilde{M}], M - \widetilde{M}\right)_2 \leq 0 \quad &\Rightarrow \quad V_{0,h}[M] = V_{0,h}[\widetilde{M}]. \end{aligned}$$

The discrete problem has a unique solution.

Proof The choice of the scheme makes it possible to mimic the proof used in the continuous case: uses the convexity and monotonicity assumptions on g.

II. Infinite Horizon: A numerical method

$$
\begin{cases}
-\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j}\,, \\
\left(\begin{array}{l}
-\nu(\Delta_h M)_{i,j} \\
-\dfrac{1}{h}\left(\begin{array}{l}
M_{i,j}\dfrac{\partial g}{\partial q_1}(x_{i,j}, [D_h U]_{i,j}) - M_{i-1,j}\dfrac{\partial g}{\partial q_1}(x_{i-1,j}, [D_h U]_{i-1,j}) \\
+M_{i+1,j}\dfrac{\partial g}{\partial q_2}(x_{i+1,j}, [D_h U]_{i+1,j}) - M_{i,j}\dfrac{\partial g}{\partial q_2}(x_{i,j}, [D_h U]_{i,j})
\end{array}\right) \\
-\dfrac{1}{h}\left(\begin{array}{l}
M_{i,j}\dfrac{\partial g}{\partial q_3}(x_{i,j}, [D_h U]_{i,j}) - M_{i,j-1}\dfrac{\partial g}{\partial q_3}(x_{i,j-1}, [D_h U]_{i,j-1}) \\
+M_{i,j+1}\dfrac{\partial g}{\partial q_4}(x_{i,j+1}, [D_h U]_{i,j+1}) - M_{i,j}\dfrac{\partial g}{\partial q_4}(x_{i,j}, [D_h U]_{i,j})
\end{array}\right)
\end{array}\right) = 0, \\
M_{i,j} \geq 0,
\end{cases}
$$

and

$$
h^2 \sum_{i,j} M_{i,j} = 1, \quad \text{and} \ \sum_{i,j} U_{i,j} = 0.
$$

Existence for the discrete problem: strategy of proof

- Use Brouwer fixed point theorem in the set of discrete probability measures for a mapping $\chi: \quad M \to U \to M$.

- The map $\Phi: \quad M \to U$ consists of solving

$$\begin{cases} -\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j}, \\ \sum_{i,j} U_{i,j} = 0 \end{cases}$$

- (U, λ) is obtained by considering the ergodic approximation:

$$-\nu(\Delta_h U^{(\rho)})_{i,j} + g(x_{i,j}, [D_h U^{(\rho)}]_{i,j}) + \rho U^{(\rho)}_{i,j} = (V_h[M])_{i,j},$$

  and passing to the limit as $\rho \to 0$.

- We need estimates on $U^{(\rho)} - U^{(\rho)}_{0,0}$ uniform in ρ and h.

Difficulty

The proof of existence for the continuous problem used the estimate $\|\nabla u\|_\infty \le C$, which was obtained with the Bernstein method and the assumption: there exists $\theta \in (0,1)$ such that for $|p|$ large,

$$\inf_{x\in\mathbb{T}} \left(\frac{\partial H}{\partial x} \cdot p + \frac{\theta}{2\nu} H^2 \right) > 0.$$

Discrete case: this argument seems difficult to reproduce.

We had to make more restrictive assumptions on H and g to obtain bounds on $\|D_h u\|_\infty$ uniform in h.

Assumptions on the Hamiltonian

$$H(x,p) = \max_{\alpha \in \mathcal{A}} \Big(p \cdot \alpha - L(x,\alpha) \Big),$$

where

- $\mathcal{A}$ is a compact subset of $\mathbb{R}^2$,
- L is a $\mathcal{C}^0$ function on $\mathbb{T} \times \mathcal{A}$,

For the discrete Hamiltonian $g(x,q)$

- monotocity, consistency.
- continuous with respect to x, $\mathcal{C}^1$ with respect to q
- sublinear with respect to q,
- there exists $g^\infty : \mathbb{R}^4 \to \mathbb{R}$ monotonous and sublinear s.t.
  $$\lim_{\epsilon \to 0} \sup_x \left| \epsilon g(x, \tfrac{q}{\epsilon}) - g^\infty(q) \right| = 0.$$

Estimates on the discrete ergodic approximation

Proposition (using Kuo-Trudinger(1992) and Camilli-Marchi(2008))
Consider a grid function V and make the assumptions:

- as above for H and g
- $\|V\|_\infty$ is bounded uniformly w.r.t. h.

For any real number $\rho > 0$, there exists a unique grid function U^ρ such that

$$\rho U^\rho_{i,j} - \nu(\Delta_h U^\rho)_{i,j} + g(x_{i,j}, [D_h U^\rho]_{i,j}) = V_{i,j},$$

and there exist two constants δ, $\delta \in (0,1)$ and C, $C > 0$, uniform in h and ρ s.t.

$$|U^\rho(\xi) - U^\rho(\xi')| \leq C|\xi - \xi'|^\delta, \qquad \forall \xi, \xi' \in \mathbb{T}_h.$$

Proposition (using Krylov(2007) and Camilli-Marchi(2008))
Same assumptions as before, and furthermore

- $$g(x, q_1, q_2, q_3, q_4) = \sup_{\beta \in \mathcal{B}} \left(\sum_{\ell=1}^{4} (-a_\ell(x,\beta) s_\ell + b_\ell(x,\beta) q_\ell) - f(x,\beta) \right),$$

with $s_1 = s_2 = (q_1 - q_2)/h$, $s_3 = s_4 = (q_3 - q_4)/h$, $a_1 = a_2 \geq 0$ and $a_3 = a_4 \geq 0$, b_ℓ, a_ℓ and f are uniformly Lipschitz continuous w.r.t. x.

- $\|D_h V\|_\infty$ is bounded uniformly w.r.t h.

Then, for any real number $\rho > 0$, there exists a unique grid function U^ρ s.t.

$$\rho U^\rho_{i,j} - \nu(\Delta_h U^\rho)_{i,j} + g(x_{i,j}, [D_h U^\rho]_{i,j}) = V_{i,j},$$

and there exists a constant C, $C > 0$, uniform in h and ρ s.t.

$$|U^\rho(\xi) - U^\rho(\xi')| \leq C|\xi - \xi'|, \qquad \forall \xi, \xi' \in \mathbb{T}_h.$$

The map Φ: $M \to U$

Proposition

Under the first set of assumptions, there exists a unique grid function U and a real number λ such that

$$\begin{cases} -\nu(\Delta_h U)_{i,j} + g(x_{i,j}, [D_h U]_{i,j}) + \lambda = (V_h[M])_{i,j}\,, \\ \sum_{i,j} U_{i,j} = 0, \end{cases}$$

and there exist two constants δ, $\delta \in (0,1)$ and C, $C > 0$, uniform in h s.t.

$$|U(\xi) - U(\xi')| \le C|\xi - \xi'|^{\delta}, \qquad \forall \xi, \xi' \in \mathbb{T}_h.$$

Under the second set of assumptions,

$$|U(\xi) - U(\xi')| \le C|\xi - \xi'|, \qquad \forall \xi, \xi' \in \mathbb{T}_h.$$

Existence and uniqueness for the stationary problem

Theorem Under the above assumptions on V and g, the discrete stationary problem has at least a solution and we have either a uniform Hölder or a Lipschitz estimate on u_h, depending on the assumptions.

Uniqueness: Ok if

$$\left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M}\right)_2 \leq 0 \Rightarrow M = \widetilde{M}.$$

Remark Existence is still OK if for $\gamma > 1$,

$$g(x, q_1, q_2, q_3, q_4) \geq \alpha((q_1)_-^2 + (q_2)_+^2 + (q_3)_-^2 + (q_4)_+^2)^{\gamma/2} - C,$$

but no bounds on u_h uniform in h.

Convergence as $h \to 0$

The same method used for uniqueness can be used for proving convergence of the discrete scheme under some assumptions on consistency and stronger assumptions on V_h.

Example

If there exist $s > 0$ such that

$$h^2 \left(V_h[M] - V_h[\widetilde{M}], M - \widetilde{M} \right)_2 \geq c \| V_h[M] - V_h[\widetilde{M}] \|_\infty^s ,$$

then uniform convergence for u, convergence of λ and a convergence related to V for m.

Uses the Hölder or Lipschitz estimates on U_h uniform w.r.t. h.

The case when V is a local operator

$$V[m](x) = F(m(x), x),$$

Same assumptions on H, g as above.

- Existence for the discrete problem: OK
- If F is a bounded and $\mathcal{C}^1$ function on $\mathbb{R} \times \mathbb{T}$, uniform bounds for some Hölder norm of u_h.

III. Infinite Horizon: long time approximation

Long time approximation (Eductive strategy, see Guéant-Lasry)

$$\begin{cases} \dfrac{\partial \tilde u}{\partial t} - \nu \Delta \tilde u + H(x, \nabla \tilde u) & = & V[\tilde m], \\ \dfrac{\partial \tilde m}{\partial t} - \nu \Delta \tilde m - \operatorname{div}\left(\tilde m \dfrac{\partial H}{\partial p}(x, \nabla \tilde u)\right) & = & 0, \\ \tilde u(0,x) = \tilde u_0(x), \quad \tilde m(0,x) & = & \tilde m_0(x), \end{cases}$$

with $\int_{\mathbb{T}} \tilde m_0 = 1 \quad \text{and} \quad \tilde m_0 \geq 0.$

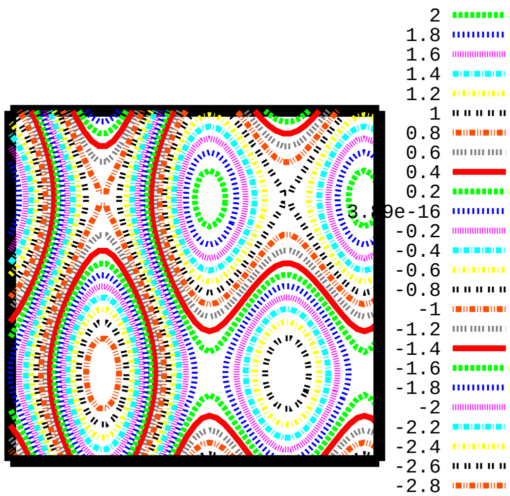
We expect that

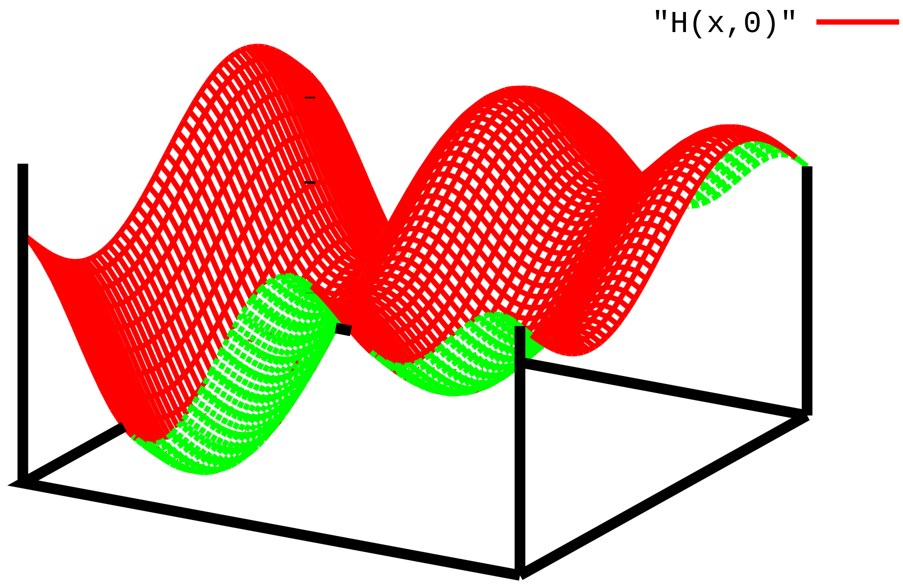
$$\lim_{t\to\infty} (\tilde u(t,x) - \lambda t) = u(x), \qquad \lim_{t\to\infty} \tilde m(t,x) = m(x),$$

Same thing at the discrete level.

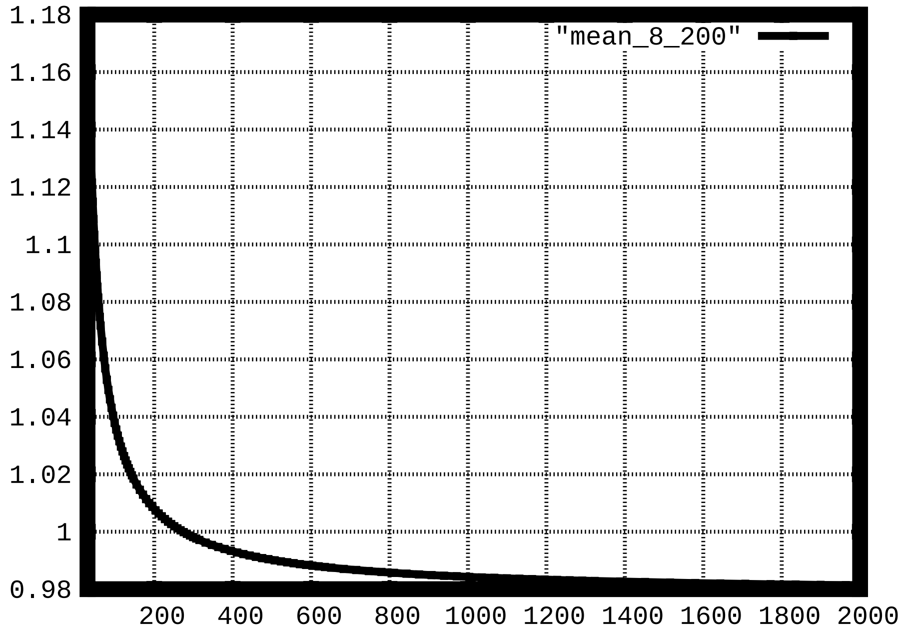
We use a semi-implicit linearized scheme. It requires the numerical solution of a linearized problem. Linearizing must be done carefully and is not always possible. In such cases, an explicit method can be used.

$$\nu = 1, \quad H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2, \quad F(x,m) = m^2$$



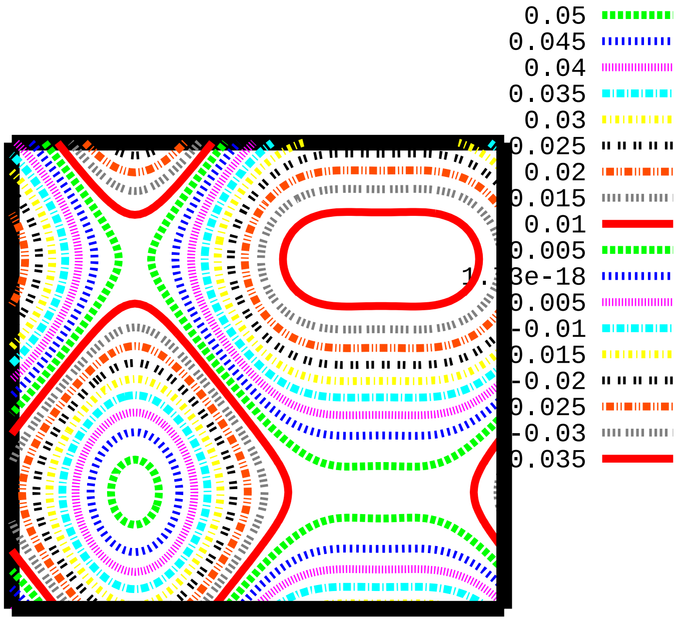


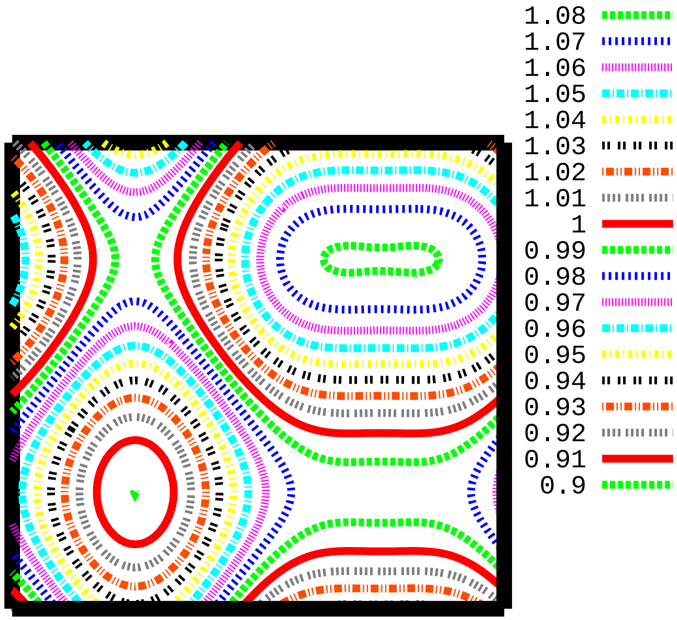
$$H(x,0) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1).$$



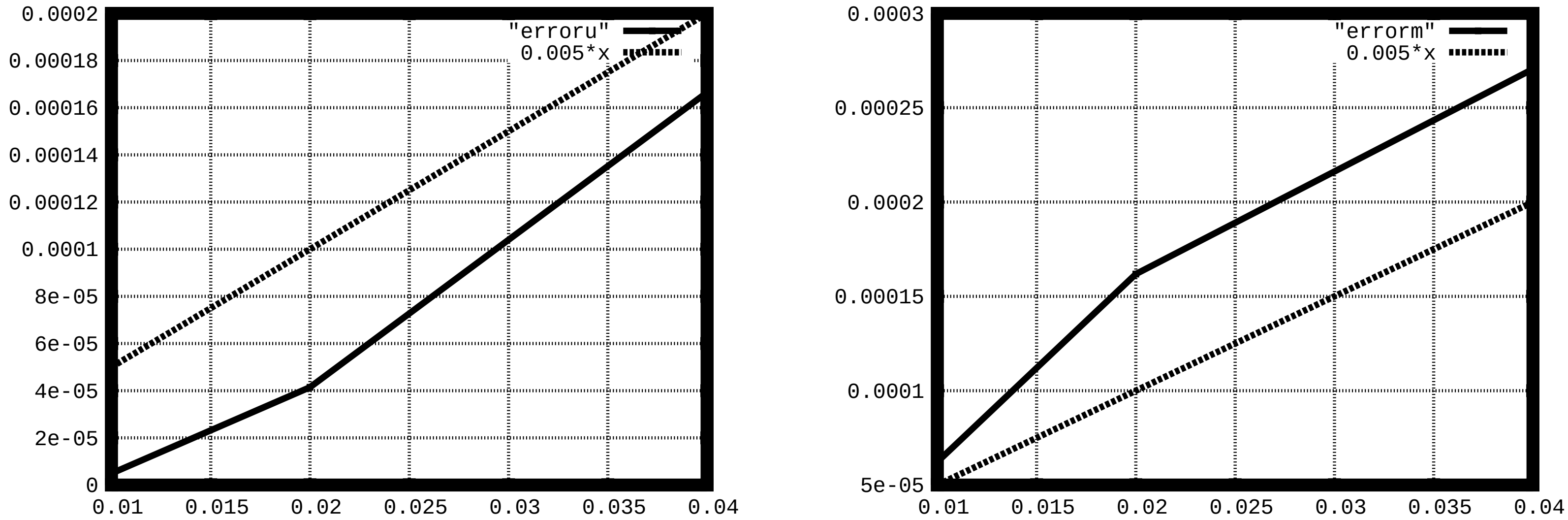
$\nu = 1,$ Convergence $\frac{1}{t}\int_{\mathbb{T}} \tilde{u}(x,t)dx \to \lambda$ as $t \to \infty$.

Very long time steps are used near convergence.





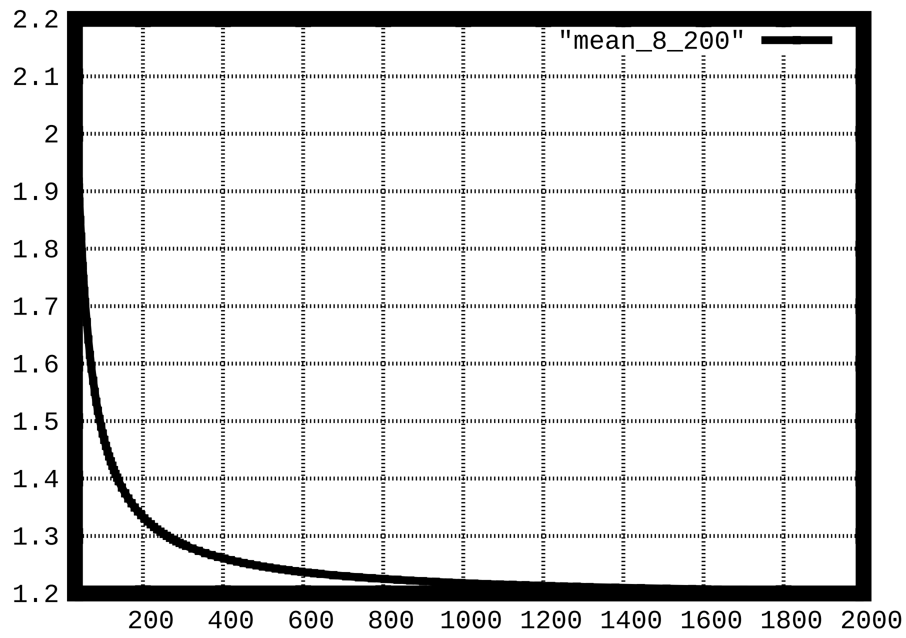
$\nu = 1$, left: u, right m.



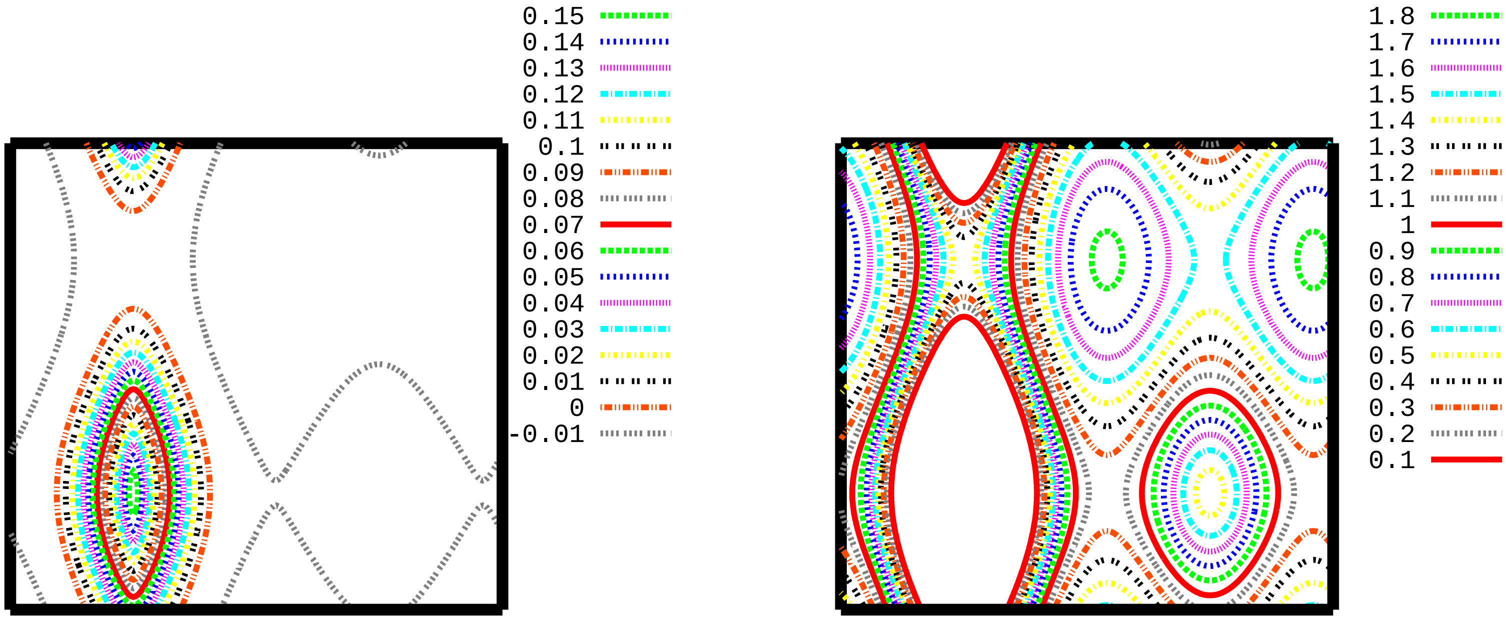
Convergence as $h \to 0$

$$\nu = 0.01,$$

$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2, \quad F(x,m) = m^2.$$



$\nu = 0.01,$ Convergence $\dfrac{1}{t}\displaystyle\int_{\mathbb{T}} \tilde{u}(x,t)dx \to \lambda$ as $t \to \infty$

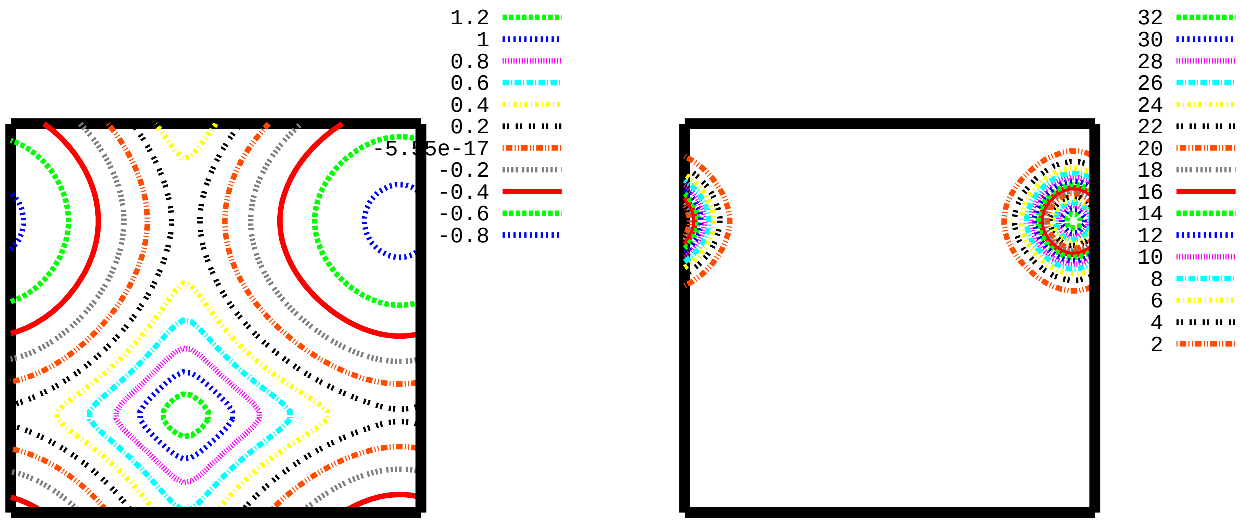


$\nu = 0.01$, left: u, right m.

Note that the supports of ∇u and of m tend to be disjoint as $\nu \to 0$.

$$V[m](x) = F(m(x)) = -\log(m(x)).$$

Same Hamiltonian as before. We now take $\nu = 0.1$.



left: u, right m.

The measure m_h concentrates near the minimum of u_h.

Deterministic limit $\nu \to 0$

Theorem (Lasry-Lions)
If

- $H(x,p) \geq H(x,0) = 0$,
- $V[m] = F(m) + f_0(x)$ where $F' > 0$,

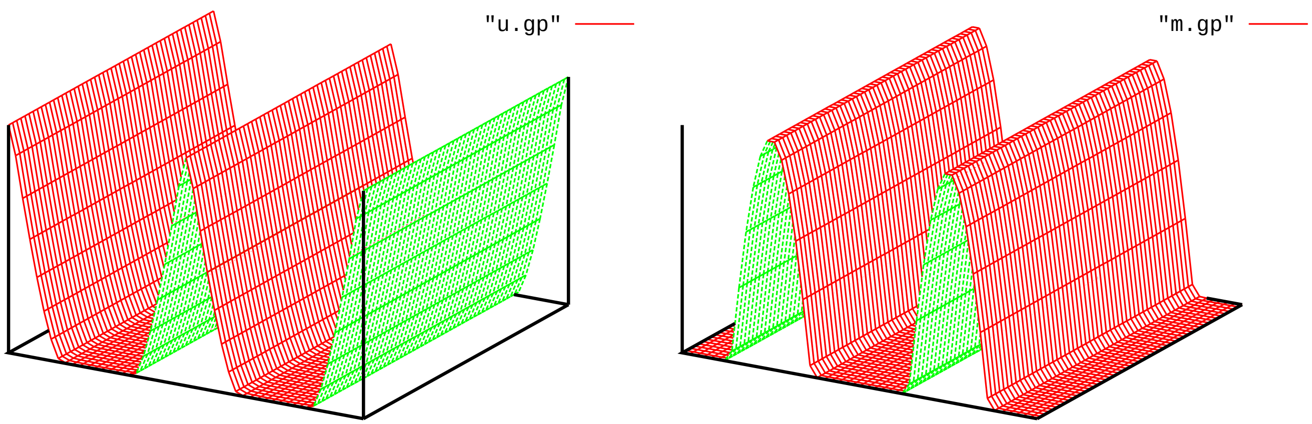
then

$$\lim_{\nu \to 0} (\lambda_\nu, m_\nu) = (\lambda, m),$$

where

$$m(x) = \left(F^{-1}(\lambda - f_0(x))\right)^+ \qquad \text{and} \qquad \int_{\mathbb{T}} m dx = 1.$$

$$\nu = 0.001,$$
$$H(x,p) = |p|^2,$$
$$V[m](x) = 4\cos(4\pi x) + m(x)$$

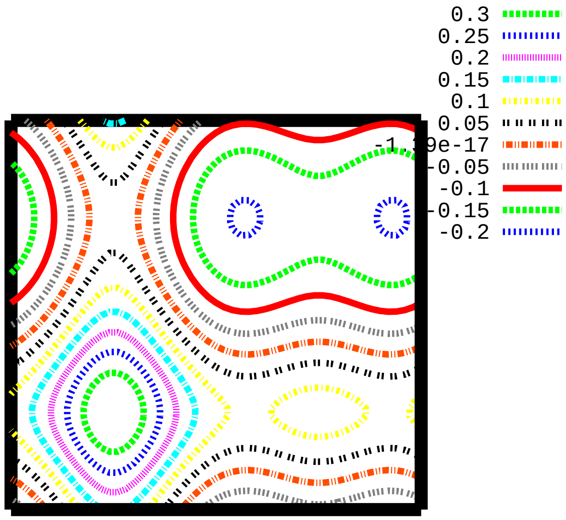


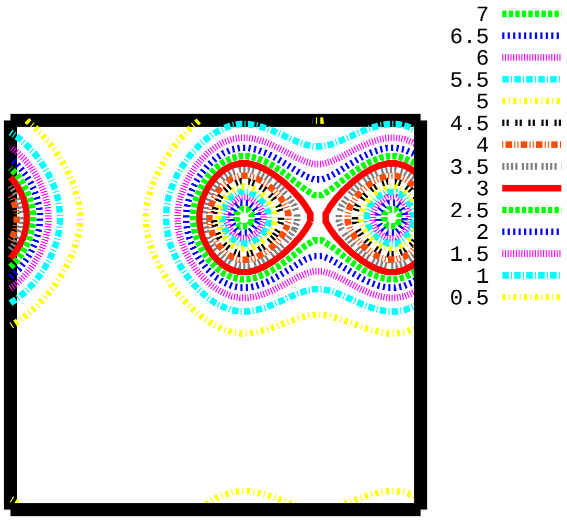
left: u, right m.

The supports of ∇u and of m tend to be disjoint.

$$m(x) \approx (\lambda - 4\cos(4\pi x))^+$$

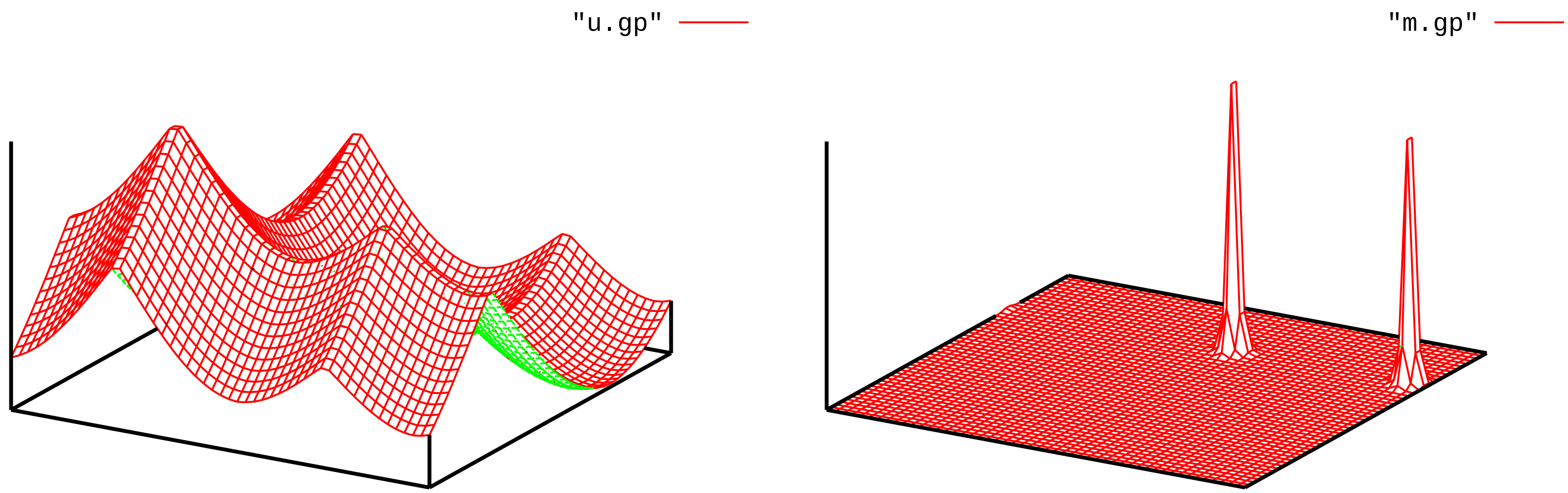
A nonlocal operator V





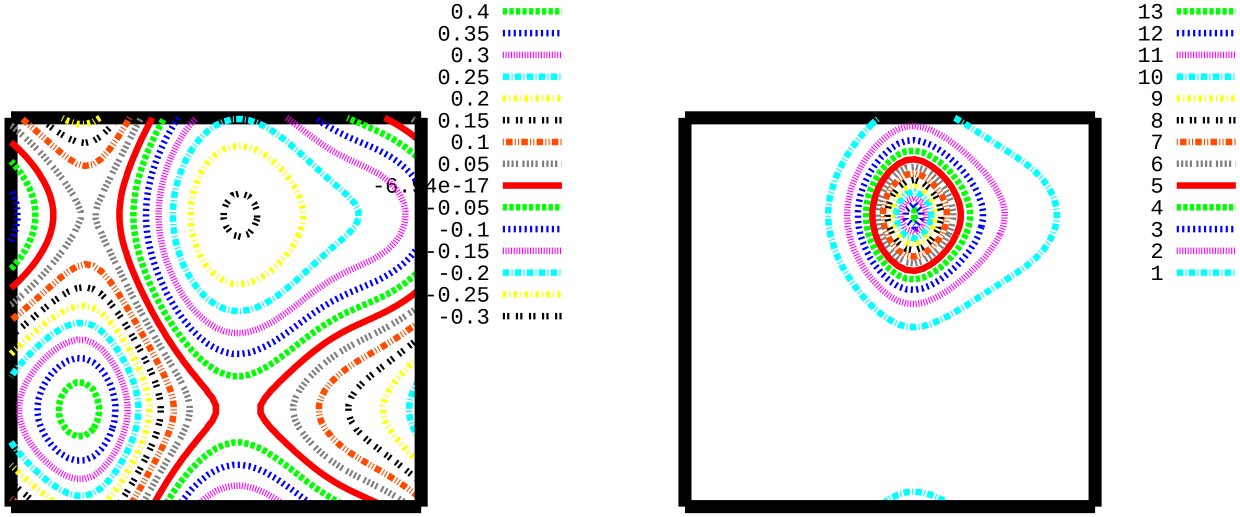
$$\nu = 0.1,$$
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^{3/2},$$
$$F(x,m) = 200(1-\Delta)^{-1}(1-\Delta)^{-1}m$$

left: u, right m.



$$\nu = 0.001,$$
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2,$$
$$F(x,m) = (1-\Delta)^{-1}(1-\Delta)^{-1} m$$

left: u, right m.



$$\nu = 0.1,$$
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + (0.6 + 0.59\cos(2\pi x))|p|^{3/2},$$
$$F(x,m) = 200(1-\Delta)^{-1}(1-\Delta)^{-1}m$$

left: u, right m.

IV. Finite Horizon: a Newton method

Difficulty: time dependent problem with conditions at both initial and final times

$$\begin{cases} \mathcal{F}_U(\mathcal{U},\mathcal{M}) &= & 0,\\ \mathcal{F}_M(\mathcal{U},\mathcal{M}) &= & 0, \end{cases}$$

Solution procedure: Newton method

$$\begin{pmatrix} \mathcal{U}^{i+1} \\ \mathcal{M}^{i+1} \end{pmatrix} = \begin{pmatrix} \mathcal{U}^{i} \\ \mathcal{M}^{i} \end{pmatrix} - \begin{pmatrix} A_{U,U}(\mathcal{U}^i,\mathcal{M}^i) & A_{U,M}(\mathcal{U}^i,\mathcal{M}^i) \\ A_{M,U}(\mathcal{U}^i,\mathcal{M}^i) & A_{M,M}(\mathcal{U}^i,\mathcal{M}^i) \end{pmatrix}^{-1} \begin{pmatrix} \mathcal{F}_U(\mathcal{U}^i,\mathcal{M}^i) \\ \mathcal{F}_M(\mathcal{U}^i,\mathcal{M}^i) \end{pmatrix}$$

where

$$\begin{aligned} A_{U,U}(\mathcal{U},\mathcal{M}) &= D_{\mathcal{U}}\mathcal{F}_{\mathcal{U}}(\mathcal{U},\mathcal{M}), & A_{U,M}(\mathcal{U},\mathcal{M}) &= D_{\mathcal{M}}\mathcal{F}_{\mathcal{U}}(\mathcal{U},\mathcal{M}),\\ A_{M,U}(\mathcal{U},\mathcal{M}) &= D_{\mathcal{U}}\mathcal{F}_{\mathcal{M}}(\mathcal{U},\mathcal{M}), & A_{M,M}(\mathcal{U},\mathcal{M}) &= D_{\mathcal{M}}\mathcal{F}_{\mathcal{M}}(\mathcal{U},\mathcal{M}). \end{aligned}$$

The linear systems

The most time consuming part of the procedure lies in solving the systems

$$\begin{pmatrix} A_{U,U} & A_{U,M} \\ A_{M,U} & A_{M,M} \end{pmatrix} \begin{pmatrix} \mathcal{U} \\ \mathcal{M} \end{pmatrix} = \begin{pmatrix} G_U \\ G_M \end{pmatrix}.$$

The matrices A_{UU} and A_{UM} have the form

$$A_{UU} = \begin{pmatrix} I & 0 & \dots & \dots & 0 \\ -\frac{1}{\Delta t} I & D_1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -\frac{1}{\Delta t} I & D_{N_T - 1} & 0 \\ 0 & \dots & 0 & -\frac{1}{\Delta t} I & D_{N_T} \end{pmatrix} \quad A_{UM} = \begin{pmatrix} E_0 & 0 & \dots & \dots & 0 \\ E_1 & 0 & & & \vdots \\ 0 & E_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & E_{N_T} & 0 \end{pmatrix}.$$

The block D_n correponds to the discrete operator

$$(Z_{i,j}) \mapsto (Z_{i,j}/\Delta t - \nu(\Delta_h Z)_{i,j} + [D_h Z]_{i,j} \cdot \nabla g(x_{i,j}, [D_h U^n]_{i,j})).$$

Monotonicity $\Rightarrow D_n$ is a M-matrix, thus A_{UU} is invertible.

The matrices A_{MM} and A_{MU} have the form

$$A_{MM} = \begin{pmatrix} D_1^T & -\frac{1}{\Delta t} I & 0 & \dots & 0 \\ 0 & D_2^T & -\frac{1}{\Delta t} I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & D_{N_T}^T & -\frac{1}{\Delta t} I \\ 0 & \dots & \dots & 0 & I \end{pmatrix} \quad A_{MU} = \begin{pmatrix} 0 & \widetilde{E}_1 & 0 & \dots & 0 \\ \vdots & \ddots & \widetilde{E}_2 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \widetilde{E}_{N_T} \\ 0 & \dots & & \dots & 0 \end{pmatrix}.$$

Note that

$$\mathcal{V}^T \widetilde{E}_n \mathcal{W} = \sum_{i,j} M_{i,j}^{n-1} [D_h V]_{i,j} \cdot D_{q,q}^2 g(x_{i,j}, [D_h U^n]_{i,j}) [D_h W]_{i,j}.$$

From the convexity of g, $\widetilde{E}_n$ is positive if $M^{n-1} \geq 0$.

Th. If V is strictly monotone and if $M^{n-1} \geq 0$, then the Jacobian matrix

$$\begin{pmatrix} A_{U,U} & A_{U,M} \\ A_{M,U} & A_{M,M} \end{pmatrix} \text{ is invertible.}$$

The chosen procedure is as follows:

1. solve first $A_{U,U}\widetilde{\mathcal{U}} = G_U$. This is done by sequentially solving

$$D_k \widetilde{U}^k = -L_k \widetilde{U}^{k-1} + G_U^k, \tag{1}$$

i.e. marching in time in the forward direction. (1) are solved with efficient direct solvers.

2. Introducing $\overline{\mathcal{U}} = \mathcal{U} - \widetilde{\mathcal{U}}$,

$$\begin{pmatrix} A_{U,U} & A_{U,M} \\ A_{M,U} & A_{M,M} \end{pmatrix} \begin{pmatrix} \overline{\mathcal{U}} \\ \mathcal{M} \end{pmatrix} = \begin{pmatrix} 0 \\ G_M - A_{M,U}\widetilde{\mathcal{U}} \end{pmatrix},$$

which implies

$$\left(A_{M,M} - A_{M,U} A_{U,U}^{-1} A_{U,M}\right) \mathcal{M} = G_M - A_{M,U}\widetilde{\mathcal{U}}. \tag{2}$$

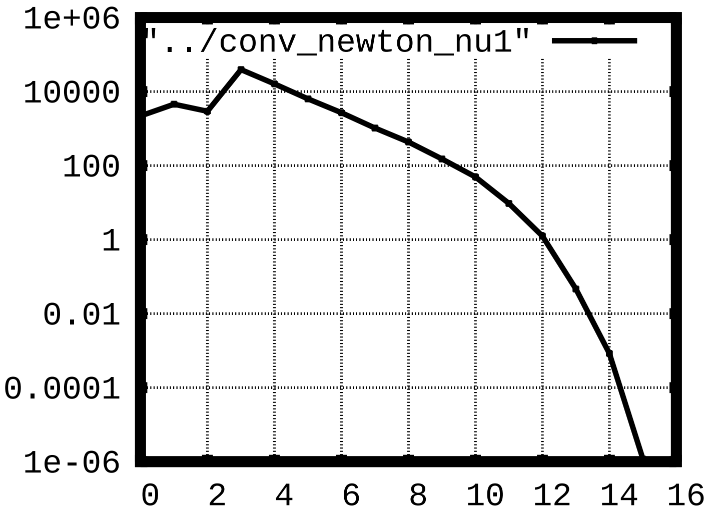
(2) is solved by an iterative method, e.g. BiCGStab.

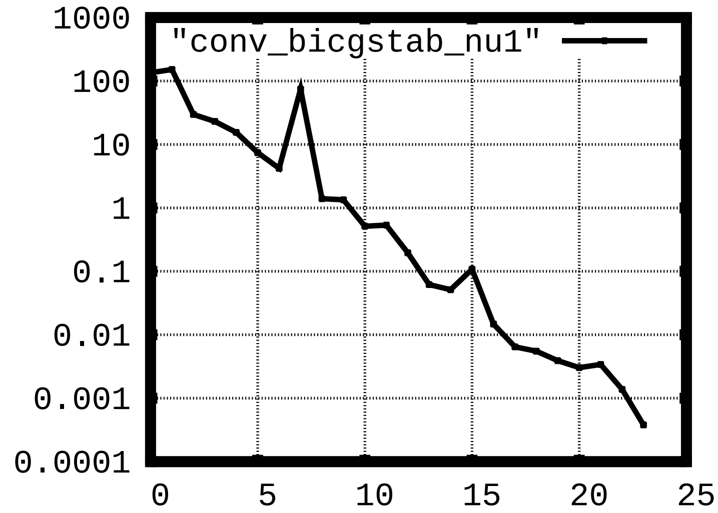
$$\nu = 1, \quad T = 1, \quad \Delta t = h = 1/50,$$

$$m(T) = 1$$

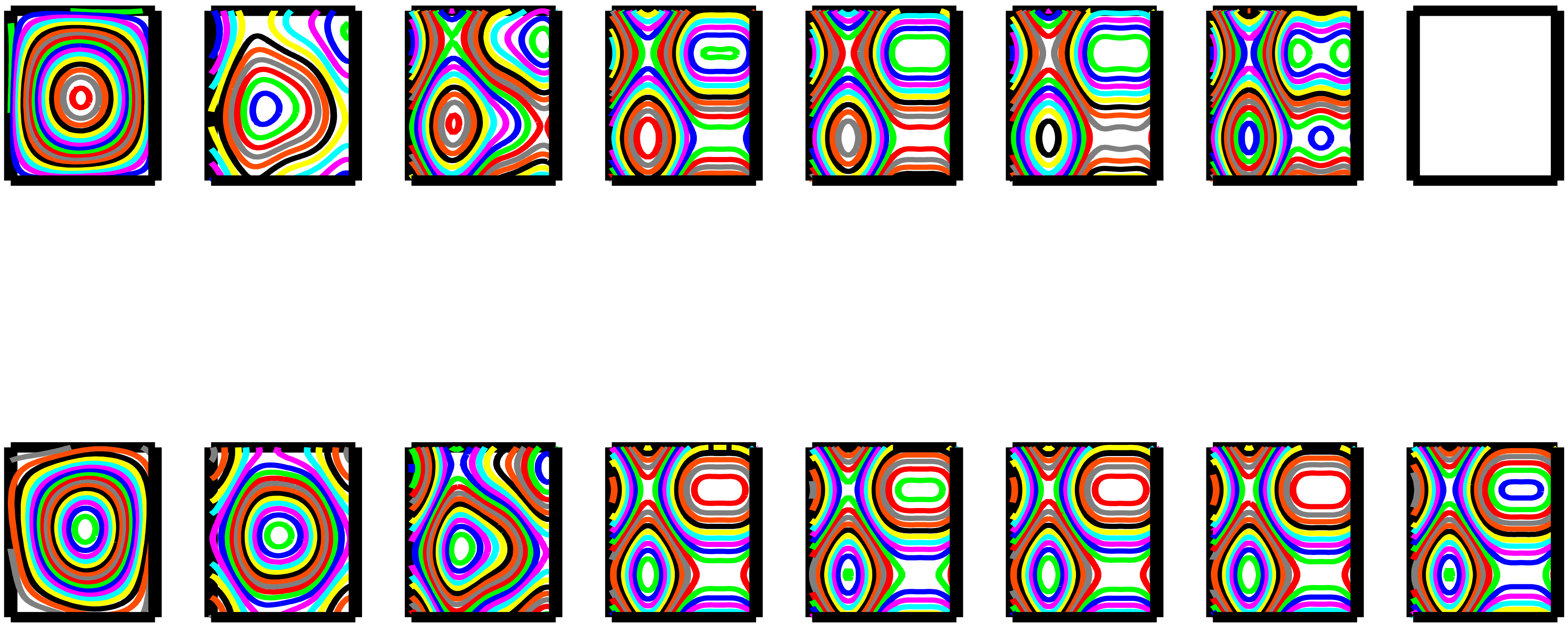
$$H(x,p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2,$$

$$F(x,m) = m^2, \qquad V_0[m](x) = m^2 + \cos(\pi x_1)\cos(\pi x_2).$$



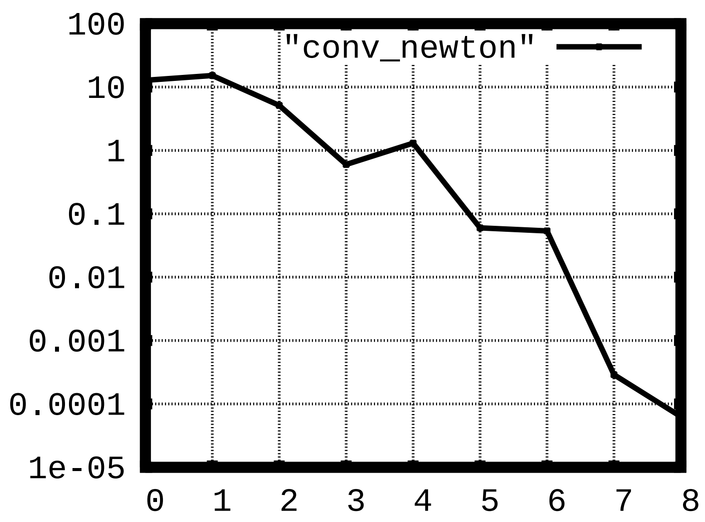


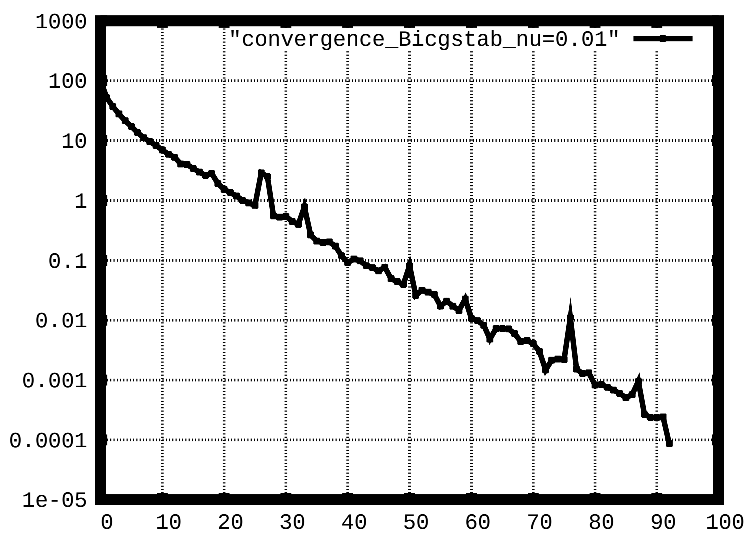
Convergence of the Newton method(left) and of a linear solver (right)



Same test except

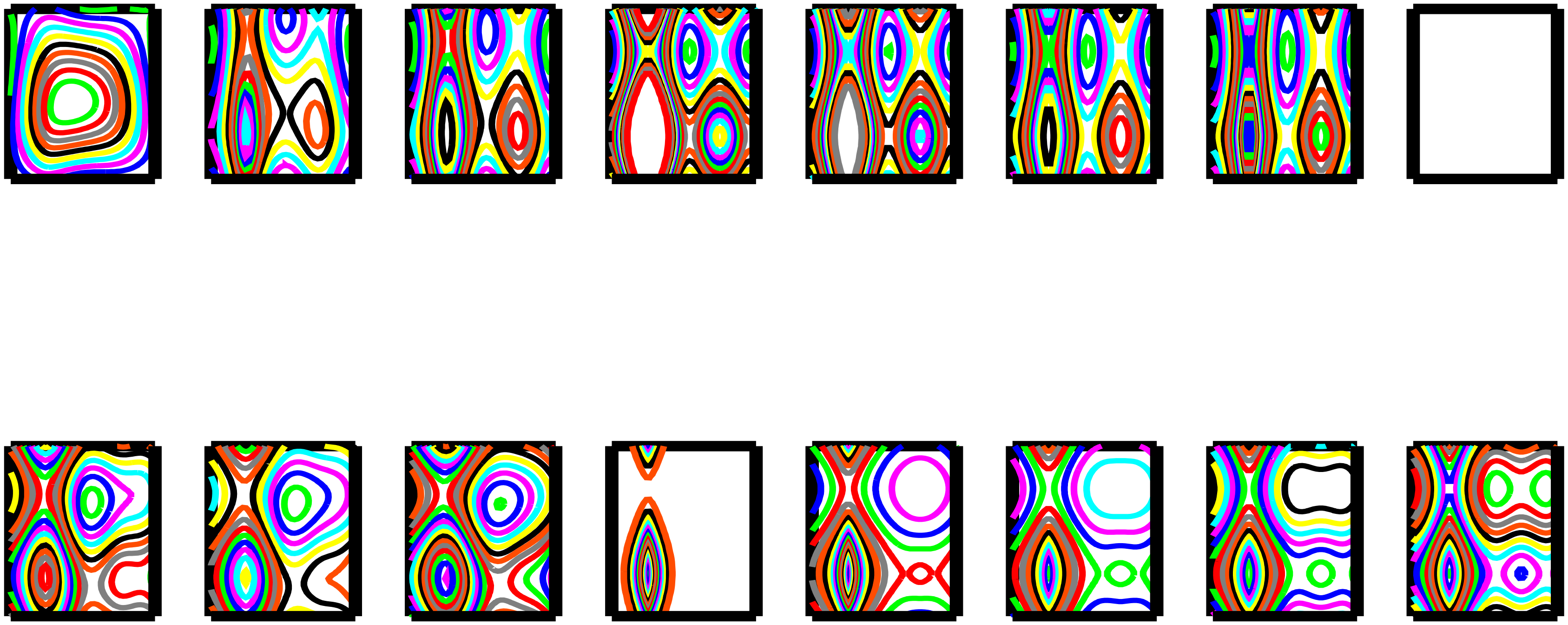
$$\nu = 0.01, \quad \Delta t = 1/200.$$



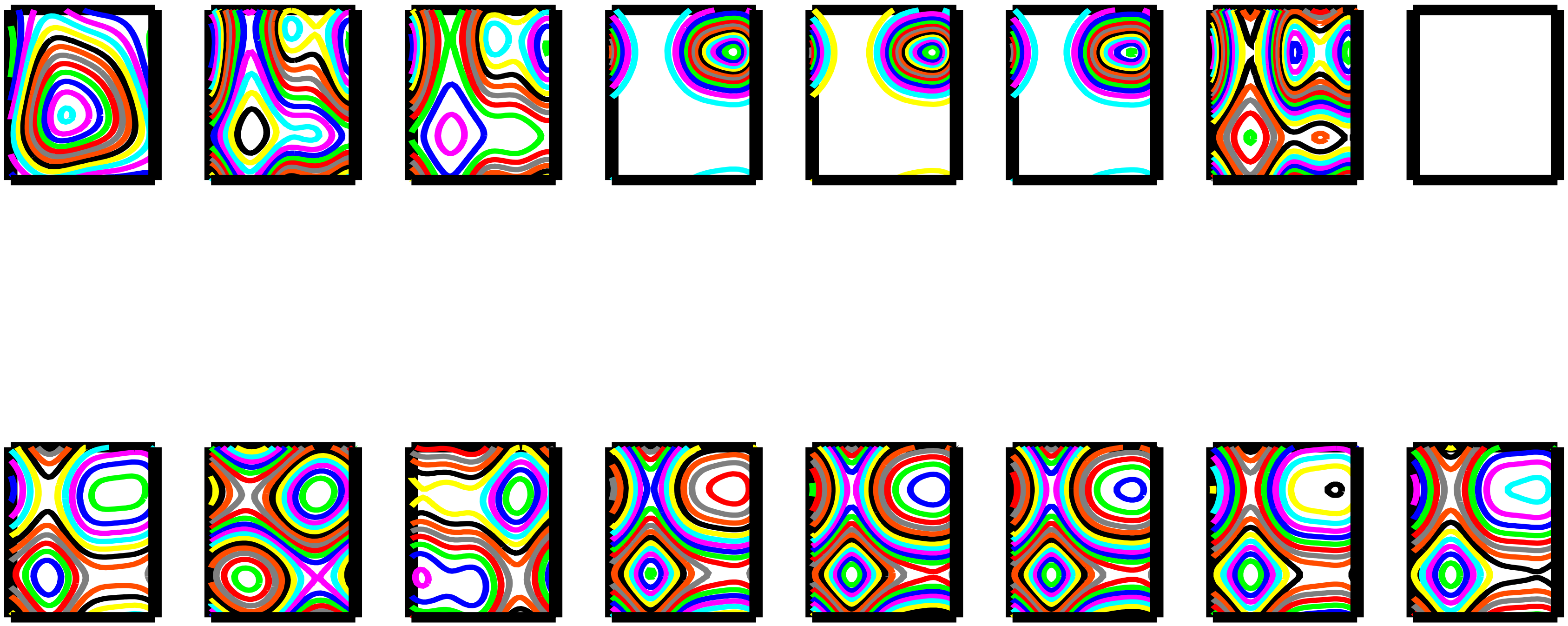


Convergence of the Newton method(left) and of a linear solver (right)

$500,000\ m$ unknowns in the nonlinear system.



$\nu = 0.2, V(m) = -\log(m)$



V. Optimal planning problem

$$\left\{ \begin{array}{rcll} \dfrac{\partial u}{\partial t} - \nu \Delta u + H(x, \nabla u) & = & V(m(x)), & \text{in } (0,T) \times \mathbb{T}, \\ \dfrac{\partial m}{\partial t} + \nu \Delta m + \operatorname{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u) \right) & = & 0, & \text{in } (0,T) \times \mathbb{T}, \end{array} \right.$$

with the initial and terminal conditions

$$m(0,x) = m_0(x), \quad m(T,x) = m_T(x), \quad \text{in } \mathbb{T},$$

and

$$m \geq 0, \qquad \int_{\mathbb{T}} m(t,x)\,dx = 1.$$

- $\mathbb{T}$ unit torus of $\mathbb{R}^d$
- $\nu \geq 0$
- H is a smooth Hamiltonian (convex):

$$H(x,p) = \sup_{\gamma \in \mathbb{R}^d} \left(p \cdot \gamma - L(x,\gamma)\right), \quad \text{with} \quad \lim_{|\gamma| \to \infty} \inf_x \frac{L(x,\gamma)}{|\gamma|} = +\infty$$

- $V : \mathbb{R} \to \mathbb{R}$ is a smooth function
- m_0 and m_T are probability densities on $\mathbb{T}$.

Existence results (P-L. Lions)

- **Ok if $\nu = 0$,** if H coercive, if V is a strictly increasing function and if m_0 and m_T are smooth positive functions.

- OK if $\nu = 0$, if $V = 0$ (optimal transport) and if m_0 and m_T are smooth positive functions.

- **Ok if $\nu > 0$ and if $H(p) = \nu|p|^2$,** if V is a strictly increasing function and if m_0 and m_T are smooth positive functions.

- **If $\nu > 0$ and $H(p) \neq \nu|p|^2$?**

- Non-existence if H is sublinear, $m_0 \neq m_T$ and T small enough.

Optimal control approach
Assumption: $V = W'$ where $W : \mathbb{R} \to \mathbb{R}$ is a strictly convex function.

A weak form of the MFG system can be found by considering the optimal control problem:

minimize $\qquad (m,\gamma) \to \int_0^T \int_{\mathbb{T}} \big[m(t,x)\, L(x,\gamma(t,x)) + W(m(t,x)) \big]\, dt\, dx,$

subject to the constraints

$$\begin{cases} \partial_t m + \nu \Delta m + \mathrm{div}(m\,\gamma) &=& 0, \quad \text{in } (0,T)\times\mathbb{T}, \\ m(T,x) &=& m_T(x) \quad \text{in } \mathbb{T}, \\ m(0,x) &=& m_0(x) \quad \text{in } \mathbb{T}. \end{cases}$$

This approach does not work completely, i.e. one may find the existence of a pair $(m, z \sim \gamma m)$, but the question is to recover the MFG system.

The semi-implicit scheme

$$\begin{cases} \dfrac{U_{i,j}^{n+1} - U_{i,j}^{n}}{\Delta t} - \nu(\Delta_h U^{n+1})_{i,j} + g(x_{i,j}, [D_h U^{n+1}]_{i,j}) & = & V(M_{i,j}^{n}), \\ \dfrac{M_{i,j}^{n+1} - M_{i,j}^{n}}{\Delta t} + \nu(\Delta_h M^{n})_{i,j} + \mathcal{B}_{i,j}(U^{n+1}, M^{n}) & = & 0, \\ M^{n} & \in & \mathcal{K}, \\ M_{i,j}^{N_T} = (m_T)_{i,j}, \qquad M_{i,j}^{0} & = & (m_0)_{i,j}. \end{cases}$$

We will see that convex programming yields the existence of (M, U) under rather general assumptions.

Existence of (M, U) via convex programming

Theorem:

If

- $V = W'$ where W is a strictly convex and coercive C^2 function.
- The discrete Hamiltonian is convex and coercive:

$$
\begin{aligned}
\lim_{q_1 \to -\infty} \frac{g(x, q_1, q_2, q_3, q_4)}{|q_1|} &= \lim_{q_2 \to +\infty} \frac{g(x, q_1, q_2, q_3, q_4)}{q_2} &= +\infty, \\
\lim_{q_3 \to -\infty} \frac{g(x, q_1, q_2, q_3, q_4)}{|q_3|} &= \lim_{q_4 \to +\infty} \frac{g(x, q_1, q_2, q_3, q_4)}{q_4} &= +\infty.
\end{aligned}
$$

- $m_0, m_T \in \mathcal{K}$ with $(m_0)_{i,j} > 0$,

then a solution of the discrete MFG system can be found by solving a saddle-point problem. The primal problem is the discrete analogue of the optimal control of pdes problem above.

Moreover M is unique (same usual proof).

Proof

- Call χ the indicator function of the set $\mathbb{R}_+$.
- Call Θ the convex and continuous functional:

$$\Theta(\alpha, \beta) = \sum_{n=1}^{N_T} \sum_{i,j} (W + \chi)^* \big(\alpha_{i,j}^n + g(x_{i,j}, [\beta^n]_{i,j})\big),$$

where $\alpha = (\alpha_{i,j}^n)$, $\beta = ([\beta^n]_{i,j})$ and $[\beta^n]_{i,j} = (\beta_{i,j}^{1,n}, \beta_{i,j}^{2,n}, \beta_{i,j}^{3,n}, \beta_{i,j}^{4,n})$.

- The cost function of the primal problem is defined as

$$\begin{aligned}\Theta^*(M, Z) = \quad & \sum_{n=0}^{N_T-1} \sum_{i,j} (W + \chi)(M_{i,j}^n) \\ & + \sup_{\beta} \left\{ \sum_{n=0}^{N_T-1} \sum_{i,j} \langle [Z_{i,j}^n], [\beta^{n+1}]_{i,j} \rangle - M_{i,j}^n g(x_{i,j}, [\beta^{n+1}]_{i,j}) \right\}.\end{aligned}$$

- Here $\quad \alpha \sim \partial_t u - \nu \Delta u, \quad \beta \sim \nabla u, \quad Z \sim m\gamma.$

The primal problem is to

minimize $\Theta^*(M, Z)$ subject to the constraint

$$\begin{cases} \dfrac{M_{i,j}^n - M_{i,j}^{n-1}}{\Delta t} + \nu(\Delta_h M^{n-1})_{i,j} + \mathrm{div_h}(Z^{n-1})_{i,j} & = & 0, \quad 1 \le n \le N_T, \\ M_{i,j}^{N_T} & = & (m_T)_{i,j}, \\ M_{i,j}^0 & = & (m_0)_{i,j}, \end{cases}$$

where

$$\mathrm{div_h}(Z)_{i,j} = (D_1^+ Z^1)_{i-1,j} + (D_1^+ Z^2)_{i,j} + (D_2^+ Z^3)_{i,j-1} + (D_2^+ Z^4)_{i,j}.$$

This is an optimal control problem for a discrete density driven by a discrete Fokker-Planck equation.

Convex programming

The primal problem can be written: minimize $\quad \Theta^*(M, Z) + \Sigma^*(-M, -Z)$,

where :

$$\Sigma(\alpha, \beta) = \begin{cases} \mathcal{F}(\Psi) & \text{if } \exists \Psi \quad \text{s.t.} \quad (\alpha, \beta) = \Lambda(\Psi) \text{ and } \sum_{i,j} \Psi^0_{i,j} = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

with

$$(\alpha, \beta) = \Lambda(\Psi) \quad \Leftrightarrow \quad \begin{cases} \alpha^{n+1}_{i,j} & = & \dfrac{\Psi^{n+1}_{i,j} - \Psi^n_{i,j}}{\Delta t} - \nu(\Delta_h \Psi^{n+1})_{i,j}, \\ [\beta^{n+1}]_{i,j} & = & [D_h \Psi^{n+1}]_{i,j}, \quad 0 \leq n < N_T, \end{cases}$$

and

$$\mathcal{F}(\Psi) = \frac{1}{\Delta t} \left(\sum_{i,j} m_{0,i,j} \Psi^0_{i,j} - \sum_{i,j} m_{T,i,j} \Psi^{N_T}_{i,j} \right).$$

Lemma: constraint qualifications

• Θ and Σ are convex and resp. continuous, LSC and there exists $(\tilde{\alpha}, \tilde{\beta})$ such that

$$\Sigma(\tilde{\alpha}, \tilde{\beta}) < +\infty.$$

• Θ^* and Σ^* are convex and LSC and there exists $(\widetilde{M}, \widetilde{Z})$ such that

$$\begin{cases} \Theta^*(\widetilde{M}, \widetilde{Z}) < +\infty, \quad \Sigma^*(-\widetilde{M}, -\widetilde{Z}) < +\infty, \\ \Theta^* \text{ is continuous near } \widetilde{M}, \widetilde{Z}. \end{cases}$$

Fenchel-Rockafeller duality theorem There exists a saddle point:

$$\min\,(\Theta + \Sigma) = -\min\big(\Theta^*(M, Z) + \Sigma^*(-M, -Z)\big).$$

Optimality conditions for the saddle point$\longrightarrow$ the discrete MFG.

Open question Find bounds on M and U independent of h, Δt.

A penalized scheme

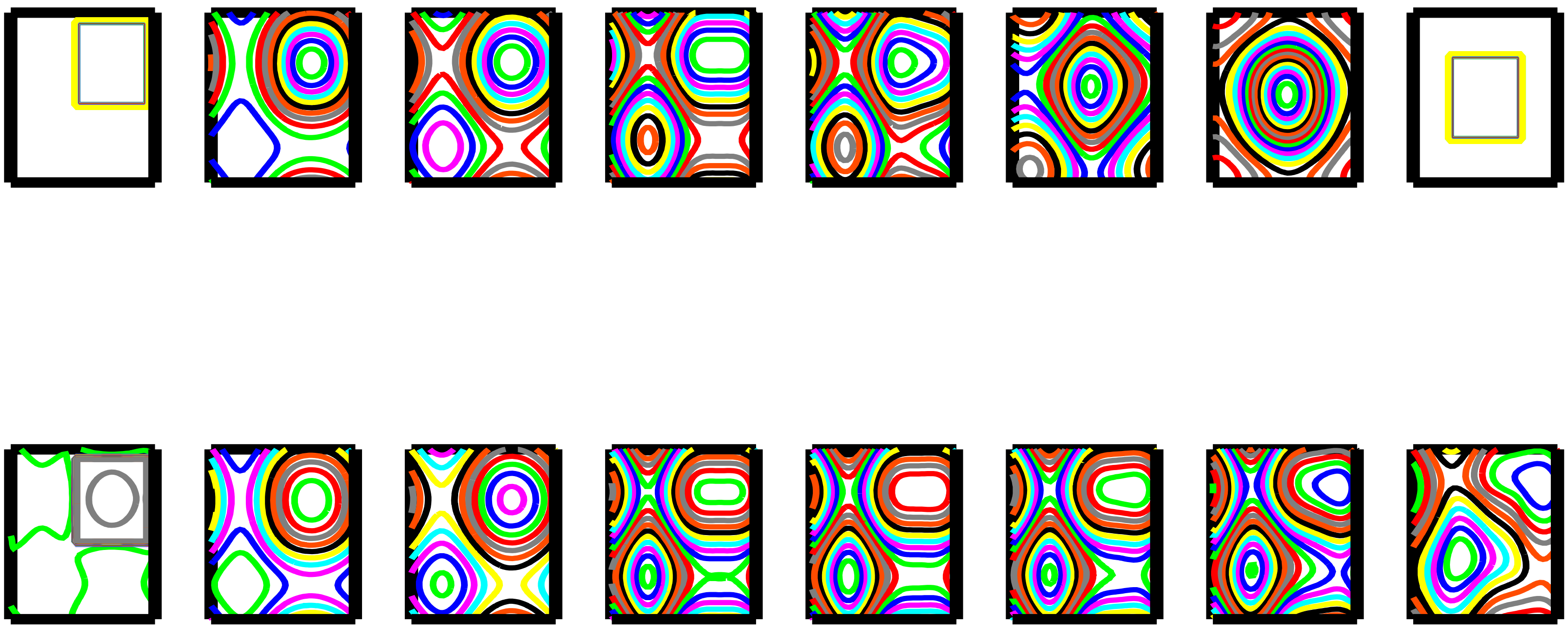
$$\begin{cases} \dfrac{U_{i,j}^{\epsilon,n+1} - U_{i,j}^{\epsilon,n}}{\Delta t} - \nu(\Delta_h U^{\epsilon,n+1})_{i,j} + g(x_{i,j}, [D_h U^{\epsilon,n+1}]_{i,j}) & = & V\left(M_{i,j}^{\epsilon,n}\right), \\ \dfrac{M_{i,j}^{\epsilon,n+1} - M_{i,j}^{\epsilon,n}}{\Delta t} + \nu(\Delta_h M^{\epsilon,n})_{i,j} + \mathcal{B}_{i,j}(U^{\epsilon,n+1}, M^{\epsilon,n}) & = & 0, \\ M^{\epsilon,n} & \in & \mathcal{K}, \end{cases}$$

with the final time and initial time conditions

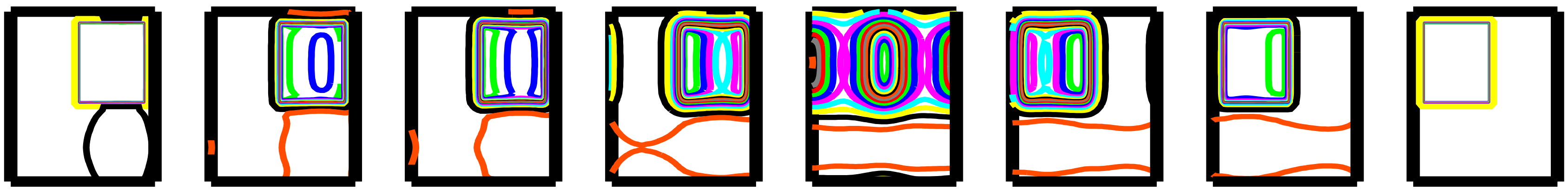
$$U_{i,j}^{\epsilon,0} = \frac{1}{\epsilon}(M_{i,j}^{\epsilon,0} - (m_0)_{i,j}), \qquad M_{i,j}^{\epsilon,N_T} = (m_T)_{i,j}, \quad \forall\, 0 \le i,j < N_h.$$

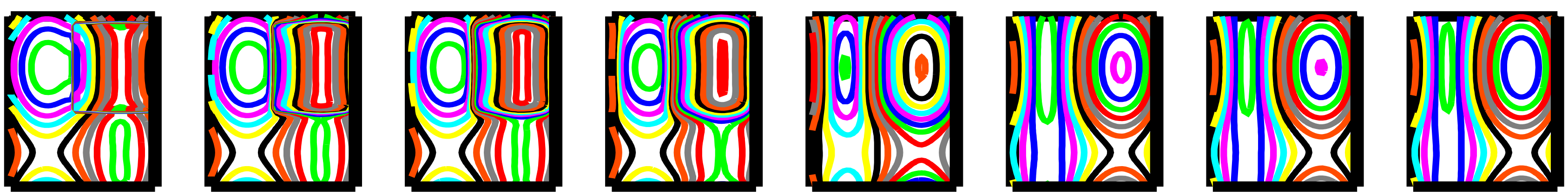
Theorem As $\epsilon \to 0$, $M^\epsilon \to M$, given by the discrete MFG system.

$$T = 1, \nu = 1, V(m) = m^2, H(p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^2$$

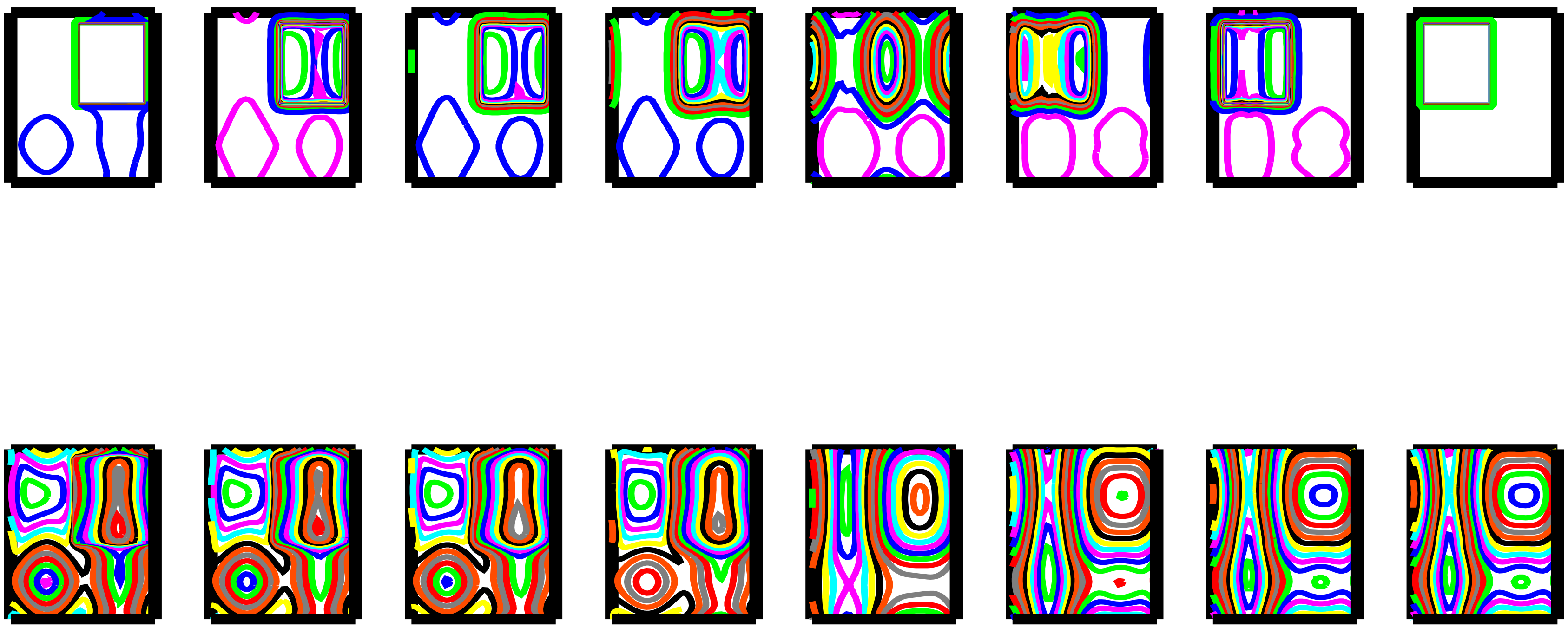


$T = 0.01$





$$T = 0.01, \nu = 0.1, H(p) = \sin(2\pi x_2) + \sin(2\pi x_1) + \cos(4\pi x_1) + |p|^3$$



$T = 0.1, \nu = 0.125, V(m) = -\log(m)$