Risk-Neutral Modeling of Emission Allowance Prices

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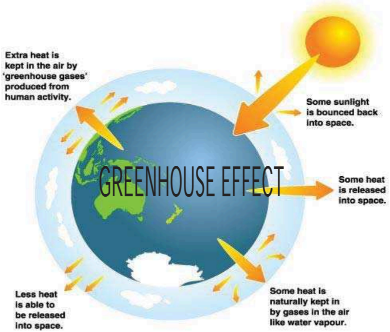
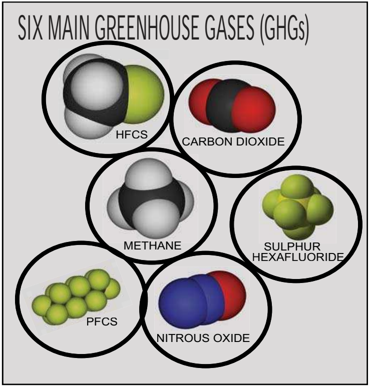
Greenhouse gas effect





Reduction

by cap-and-trade mechanism=emission trading scheme

- central authority
  - allocates credits (allowances) to polluters
  - sets penalty for each unit of pollutant not covered by credits
  - defines compliance dates within a time period
- polluters reduce or avoid penalty by
  - applying abatement measures
    - technological changes
    - replacement of input/output products,
  - trading allowances
    - physically (spot)
    - financially (forwards/futurues)

Example EU ETS Phase I and II credits are called EUA

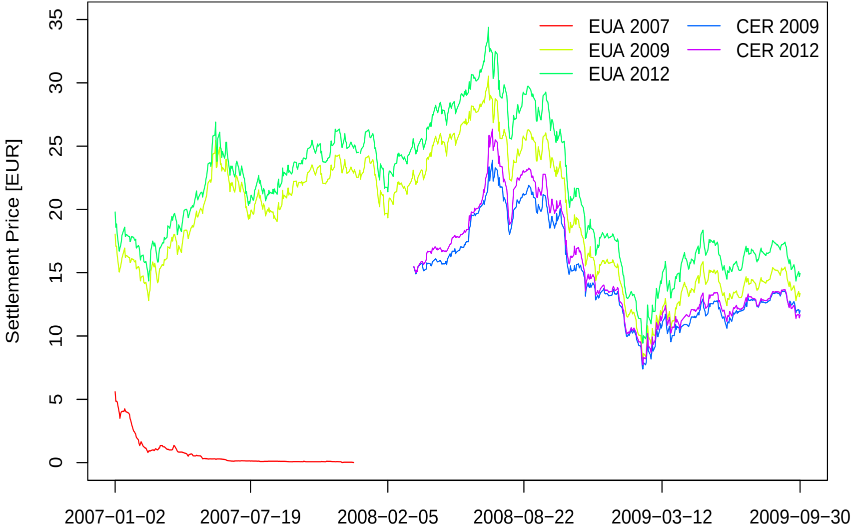
Financial products

- EUA: (European Union Allowance) is the emission certificate which covers the emission of one tonne of carbon dioxide equivalent within EU ETS.
- Futures on EUA's are traded. ECX lists EUA futures with expiry date on the first Monday of March, June, September and December.

Financial products

- CERs: (Certified Emission Reductions) are emissions certificates issued for the successful completion of CDM climate protection projects.
- within EU ETS, installations are allowed to cover their emissions by CERs, (with upper bound on the total number of CERs valid for compliance)
- Futures on CER's are traded. ECX lists CER futures with expiry date on the first Monday of March, June, September and December.

Futures from ECX



Simplest situation

- one period only (no time inter-connection)
- one scheme only (no space inter-connection)
- interest rate zero (spot=future=forward)

Model allowance price $(A_t)_{t\in[0,T]}$ as a digital martingale

$$A_t = \pi \mathbb{E}^Q(1_N \mid \mathcal{F}_t), \qquad t \in [0,T]$$

Distinguish between

- reduced-form model
- hybrid model

Both types of risk neutral models

describe allowance price $(A_t)_{t\in[0,T]}$ as a digital martingale

$$A_t = \pi \mathbb{E}^Q(1_N \,|\, \mathcal{F}_t), \qquad t \in [0,T]$$

- **reduced-form model**: Non-compliance event N is modeled exogenously. (in a flexible way, to match observed price properties, and option prices)
- **hybrid model**: stylized fundamental factors (emission, reduction intensities + market mechanisms) determine N.

In this talk: hybrid model

Idea Analyze equilibrium of a stylized market. Derive a relation between allowance price, stochastic drivers, abatement activity Conclude implications for risk-neutral dynamics

- non-risk averse setting: optimal stochastic control
- risk-averse setting: fixed-point equalities for martingales

References With co-workers:

1. *Optimal stochastic control and carbon price formation.* SIAM Journal on Control and Optimization, 2009.
2. Market designs for emissions trading schemes. SIAM Review (to appear)
3. *On fair pricing of emission-related derivatives* Bernoulli (to appear)
4. *Jump-diffusion modeling in emission markets* Preprint

Dynamical model

- compliance date T
- action times $t = 0, \dots, T$
- all processes on $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t=0}^T)$ are adapted
- finite number of agents $i \in I$

abstract from production, focus on abatement activity

Revenue of agent i for strategy (ξ^i, ϑ^i), given prices $A = (A_t)_{t=0}^T$

$$
\begin{aligned}
L^{A,i}(\vartheta^i, \xi^i) \quad = \quad & -\sum_{t=0}^{T-1}(\vartheta_t^i A_t + C_t^i(\xi_t^i)) - \vartheta_T^i A_T \\
& - \underbrace{\pi}_{\text{penalty}} \Big(\sum_{t=0}^{T-1}(E_t^i - \xi_t^i - \vartheta_t^i) - \vartheta_T^i - \gamma^i\Big)^+
\end{aligned}
$$

Model ingredients

$$
\begin{aligned}
L^{A,i}(\vartheta^i, \xi^i) \;=\; & -\sum_{t=0}^{T-1}(\vartheta^i_t A_t + C^i_t(\xi^i_t)) - \vartheta^i_T A_T \\
& - \underbrace{\pi}_{\text{penalty}}\Big(\sum_{t=0}^{T-1}(E^i_t - \xi^i_t - \vartheta^i_t) - \vartheta^i_T - \gamma^i\Big)^+
\end{aligned}
$$

- Business-as-usual emissions $(E^i_t)_{t=0}^{T-1}$ of the agents $i \in I$
- Abatement policy $\xi^i = (\xi^i_t)_{t=0}^{T-1}$ of the agent $i \in I$
- Costs of abatement policy $(\xi^i_t)_{t=0}^{T-1}$ are $\sum_{t=0}^{T-1} C^i_t(\xi^i_t)$
- ϑ^i_t change of allowance number by trade at time t
- $\sum_{t=0}^{T-1} A_t \vartheta^i_t$ costs of trading at allowance prices $(A_t)_{t=0}^{T}$
- $\gamma^i \in [0, \infty[$ endowment less unpredictable emission

Model ingredients

- **Costs** can be random

$$(\omega, x) \mapsto C_t(x)(\omega) \quad \mathcal{F}_t \otimes \mathcal{B}\text{-measurable}$$

  due to stochastic fuel prices
- **Abatement** activity $\xi^i = (\xi^i_t)_{t=0}^{T-1}$ must be feasible

$$\xi^i \in \mathcal{U}^i := \{(\vartheta^i, \xi^i) \,:\, 0 \leq \xi^i_t \leq E^i_t \quad t = 0, \ldots, T-1\}.$$

  since abatement can not exceed emission.

Model ingredients

Risk aversion of agent $i \in I$

is described by agent-specific utility function U^i

Rational behavior

Given prices $A = (A_t)_{t=0}^T$, each agent $i \in I$ maximizes

$$(\vartheta^i, \xi^i) \mapsto \underbrace{\mathbb{E}\left(U^i(L^{A,i}(\vartheta^i, \xi^i))\right)}_{=u^i\left(L^{A,i}(\vartheta^i, \xi^i)\right)}$$

over all admissible policies $\mathcal{U}^i$.

Equilibrium state

Definition

$A^* = (A^*_t)_{t=0}^T$ is an equilibrium allowance price process, if there exist agent's policies $(\vartheta^{*i}, \xi^{*i}) \in \mathcal{U}^i$, $i \in I$ such that:

(i) Each agent $i \in I$ is satisfied by the own policy

$$(\vartheta^{*i}, \xi^{*i}) \text{ is maximizer to } (\vartheta^i, \xi^i) \mapsto u^i(L^{A^*,i}(\vartheta^i, \xi^i))$$

on $\mathcal{U}^i$, furthermore $u^i(L^{A^*,i}(\vartheta^i, \xi^i)) < \infty$.

(ii) Changes in allowance positions are in zero net supply

$$\sum_{i \in I} \vartheta^{*i}_t = 0, \text{ for all } t = 0, \ldots, T.$$

Three equilibrium properties (under additional assumptions)

It turns out that in the equilibrium:

a) No arbitrage opportunities for allowance trading
b) Allowance price instantaneously triggers all abatement measures whose costs are below allowance price
c) There are merely two final outcomes for allowance price
   - $A_T^* = 0$ in the case of allowance excess
   - $A_T^* = \pi$ in the case of allowance shortage

Formal characterization (under slight assumptions)

Theorem

If $(A_t^*)_{t=0}^T$ is an equilibrium price and $(\xi_t^{i*})_{t=0}^{T-1}$ for $i \in I$ are corresponding abatement policies, then

(a) $(A_t^*)_{t=0}^T$ is a martingale with respect to some $Q \sim P$

(b) For each $i \in I$ holds

$$\xi_t^{i*} = c_t^i(A_t^*), \quad t = 0, \dots, T-1,$$

with abatement volume function

$$c_t^i(a) = \mathrm{argmax}(x \mapsto -C_t^i(x) + ax)$$

(c) The terminal allowance price is given by

$$A_T^* = \pi 1_{\{\sum_{i \in I}(\sum_{t=0}^{T-1}(E_t^i - \xi_t^{i*}) - \gamma^i) \geq 0\}}$$

From risk-neutral perspective, allowance price

is a Q-martingale, whose terminal value

$$\boxed{A^*_T = \pi 1_{\{\mathcal{E}_T - \sum_{t=0}^{T-1} c_t(A^*_t) \geq 0\}}}$$

depends on the intermediate values through

Cap-adjusted BAU emission

$$\mathcal{E}_T = \sum_{i \in I} \sum_{t=0}^{T-1} E^i_t - \sum_{i \in I} \gamma^i$$

and market abatement volume function

$$c_t(a) := \sum_{i \in I} c^i_t(a), \qquad t = 0, \ldots, T-1$$

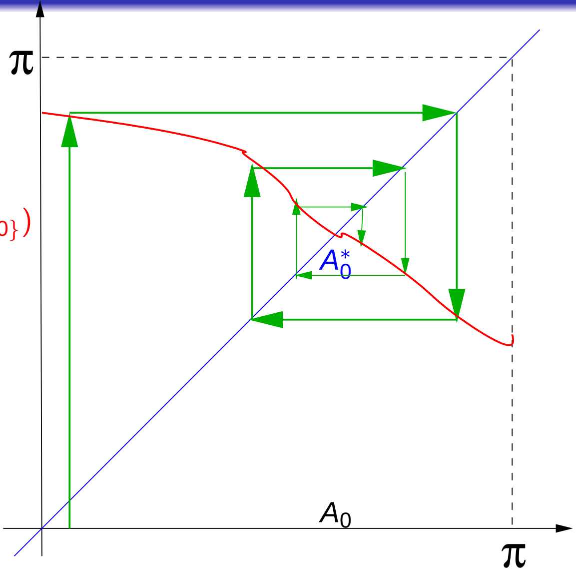
Hybrid modeling

given measure $\mathbb{Q} \sim \mathbb{P}$, random variable $\mathcal{E}_T$, and abatement volume functions $(c_t)_{t=0}^{T-1}$, determine a $\mathbb{Q}$-martingale $(A_t^*)_{t=0}^{T}$ with

$$A_T^* = \pi 1_{\{\mathcal{E}_T - \sum_{t=0}^{T-1} c_t(A_t^*) \geq 0\}}.$$

Illustration for one time step from 0 to $T = 1$

$$A_0^* = \pi \mathbb{E}_0^Q(\mathbf{1}_{\{\mathcal{E}_1 - c_0(A_0^*) \geq 0\}})$$



Solution to the problem of hybrid modeling

To obtain $(A^*_t)_{t=0}^T$ from Q, $\mathcal{E}_T$, and $(c_t)_{t=0}^{T-1}$ follow the intuition that the allowance price is a function of

$$A^*_t(\omega) = \alpha_t(G_t(\omega))(\omega)$$

- recent time t
- current situation ω
- reduction demand $G_t = \underbrace{E^Q_t(\mathcal{E}_T)}_{\mathcal{E}_t} - \sum_{s=0}^{t-1} c_s(A^*_s)$

Reduced-form approach

Current allowance price is a function of

$$\left.\begin{array}{l}\text{time to maturity}\\ \text{current situation}\\ \text{saved pollutant}\end{array}\right\} \Rightarrow \begin{array}{l} A_t^*(\omega) = \alpha_t(G_t(\omega))(\omega) \\ G_t(\omega) = \mathcal{E}_t(\omega) - \sum_{s=0}^{t-1} c_s(A_s^*)(\omega) \end{array}$$

- for $t = T$ obviously

$$\alpha_T(g) = \pi 1_{[0,\infty[}(g), \quad \text{for all } g \in \mathbb{R}$$

- for $t = T-1, \ldots, 0$ hypothetically

$$\alpha_t : \mathbb{R} \times \Omega \to [0, \pi], \quad \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t\text{-measurable}$$

Guess a recursion from martingale property

Idea

$$\alpha_t(g)(\omega) = \mathbb{E}_t^{\mathrm{Q}}[\alpha_{t+1}(g - c_t(\alpha_t(g)) + \varepsilon_{t+1})](\omega),$$
for all $g \in \mathbb{R}$, $\omega \in \Omega$ where $\varepsilon_{t+1} = \mathcal{E}_{t+1} - \mathcal{E}_t$

Indeed:

$$\begin{aligned}\alpha_t(G_t) = A_t^*(\omega) = \mathbb{E}_t^{\mathrm{Q}}[A_{t+1}^*] &= \mathbb{E}_t^{\mathrm{Q}}[\alpha_{t+1}(G_{t+1})] \\ &= \mathbb{E}_t^{\mathrm{Q}}(\alpha_{t+1}(G_t - c_t(A_t^*) + \varepsilon_{t+1})) \\ &= \mathbb{E}_t^{\mathrm{Q}}[\alpha_{t+1}(G_t - c_t(\alpha_t(G_t)) + \varepsilon_{t+1})]\end{aligned}$$

Recursion for $(\alpha_t)_{t=0}^T$

Idea

$$\alpha_t(g)(\omega) = \mathbb{E}_t^{\mathrm{Q}}(\alpha_{t+1}(g - c_t(\alpha_t(g)(\omega)) + \varepsilon_{t+1}))(\omega), \\ \text{for all } g \in \mathbb{R}, \omega \in \Omega$$

- start with $\alpha_T(g) = \pi 1_{[0,\infty[}(g)$, for all $g \in \mathbb{R}$
- proceed recursively for $t = T-1, \dots, 1$, determining $\alpha_t(g)(\omega)$ as the unique solution to the fix point equation

$$a = \mathbb{E}_t^{\mathrm{Q}}(\alpha_{t+1}(g - c_t(a) + \varepsilon_{t+1}))(\omega)$$

Formal result

Theorem

i) Given measure $\mathrm{Q} \sim P$ there exist functionals

$$\alpha_t : \mathbb{R} \times \Omega \to [0, \pi], \quad \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t\text{-measurable, for } t = 0, \dots T$$

which fulfill for all $g \in \mathbb{R}$

$$\begin{aligned} \alpha_T(g) &= \pi 1_{[0,\infty[}(g), \\ \alpha_t(g) &= \mathbb{E}_t^{\mathrm{Q}}(\alpha_{t+1}(g - c_t(\alpha_t(g)) + \varepsilon_{t+1})), \quad t = 0, .., T-1 \end{aligned}$$

ii) There exists a Q–martingale $(A_t^*)_{t=0}^T$ which satisfies

$$A_T^* = \pi 1_{\{\mathcal{E}_T - \sum_{t=0}^{T-1} c_t(A_t^*) \geq 0\}} \quad \text{recursively obtained by}$$

$$A_t^* := \alpha_t(\mathcal{E}_t - \sum_{s=0}^{t-1} c_s(A_s^*)), \quad t = 0, .., T$$

A numerical example: constant and deterministic $c_t = c$

Suppose that

$$\varepsilon_{t+1} \text{ and } \mathcal{F}_t \text{ are independent under } Q \text{ for all } t = 0, \dots, T-1.$$

which makes calculations easier, since the randomness enters allowance price through the present up-to-day emissions only. More precisely one verifies (recursively!) that

$$\omega \mapsto \alpha_t(g)(\omega) = \alpha_t(g) \quad \text{is constant on } \Omega.$$

Hence, allowance price A^*_{t+1} is just Borel function of the present up-to-day emission G_{t+1} and the condition $\mathcal{F}_t$ can be replaced by the condition $\sigma(G_t)$:

$$\alpha_t(G_t) = \mathbb{E}^Q(\alpha_{t+1}(G_t - c_t(\alpha_t(G_t)) + \varepsilon_{t+1}) \,|\, \sigma(G_t)).$$

A numerical example: least-square Monte-Carlo method

Given the fixed point equation for Borel measurable function α_t

$$\alpha_t(G_t) = \mathbb{E}^Q(\alpha_{t+1}(G_t - c_t(\alpha_t(G_t)) + \varepsilon_{t+1}) \,|\, \sigma(G_t)),$$

try to obtain a solution as limit $\alpha_t = \lim_{n\to\infty} \alpha_t^n$ of iterations

$$\alpha_t^{n+1}(G_t) = \mathbb{E}^Q(\alpha_{t+1}(G_t - c_t(\alpha_t^n(G_t)) + \varepsilon_{t+1}) \,|\, \sigma(G_t)), \quad n \in \mathbb{N}$$

started at $\alpha_t^0 = \alpha_{t+1}$.

For numerical calculations, we suggest to use the least-square Monte-Carlo method. The idea here is to consider functions within a linear space spanned by basis functions and to replace the integration by a sum over finite sample.

A numerical example – $(\varepsilon_t)_{t=1}^T$ are i. i. d.

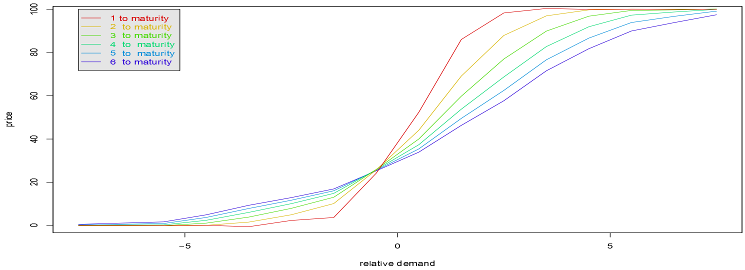
1. *Initialization:* Given sample $S = (e_k, g_k)_{k=1}^K \subset \mathbb{R}^2$ (of i.i.d realizations of (ε_{t+1}, G_t)) and a set of basis functions $\Psi = (\psi_i)_{j=1}^J$ on $\mathbb{R}$, define

$$M = (\psi_j(g_k))_{k=1,j=1}^{K,J}$$

Set $\alpha_T(g) = 1_{[0,\infty]}(g)$ for all $g \in \mathbb{R}$, and proceed in the next step with $t := T - 1$.

2. *Iteration:* Define $\alpha_t^0 = \alpha_t$, and proceed in the next step with $n := 0$.
   - **2a)** Calculate $\phi^{n+1}(S) := (\alpha_{t+1}(g_k - c_t(\alpha_t^n(g_k)) + e_k))_{k=1}^K$
   - **2b)** Determine a solution $q^{n+1} \in \mathbb{R}^J$ to $M^\top M q^{n+1} = M^\top \phi^{n+1}(S)$.
   - **2c)** Define $\alpha_t^{n+1} := \sum_{j=1}^J q_j^{n+1} \psi_j$.
   - **2d)** If $\max_{k=1}^K |\alpha_t^{n+1}(g_k) - \alpha_t^n(g_k)| \geq \varepsilon$, then put $n := n + 1$ and continue with the step 2a).
     If $\max_{k=1}^K |\alpha_t^{n+1}(g_k) - \alpha_t^n(g_k)| < \varepsilon$ then set $t := t - 1$. If $t > 0$, go to the step 2, otherwise finish.

Illustration



Parameters

- penalty $\pi = 100$,
- martingale increments $(\varepsilon_t)_{t=1}^{T}$ i.i.d, $\varepsilon_t = \mathcal{N}(0.5, 1)$, $K = 1000$
- basis functions $(\Psi_j)_{j=1}^{J}$ piecewise linear, $J = 16$
- abatement volume function $c : \mathbb{R} \to \mathbb{R}, a \mapsto 0.1\sqrt{(a)^+}$

Pricing European Call

1. Given maturity time $\tau \in \{1, \dots, T\}$ of the European call, determine its payoff $f^{\tau}_{\tau} := (\alpha_{\tau} - K)^{+}$. Calculate least-square projections recursively processing for $u = \tau, \dots, t$
2. Calculate least-square projections recursively processing for $u = \tau, \dots, t$
   - **a)** put $\phi(S) = (f^{\tau}_{u}(g_k - c_u(\alpha_u(g_k)) + e_k))_{k=1}^{K}$
   - **b)** obtain q as solution to $M^{\top} M q = M\phi$
   - **c)** set $f^{\tau}_{u-1} = \sum_{j=1}^{J} q_j \psi_j$
   - **d)** if $u - 1 = t$ finish, else set $u := u - 1$ and go to a).
3. Given recent allowance price a, calculate the state variable g as solution to $a = \alpha_t(g)$
4. Plug in the state variable g and into function $f^{\tau}(t, \cdot)$ to obtain the price of the European call as as $f^{\tau}(t, g)$.

Hybrid modeling $\Longrightarrow$ continuous time

- Allowance price is martingale

$$A_t = \mathbb{E}_t^Q(A_T), \qquad t \in [0, T]$$

- Allowance price is digital at compliance date T

$$A_T = \pi 1_N$$

- Allowance price triggers abatement

$$N = \{\mathcal{E}_T - \int_0^T c_u(A_u)du\}$$

Problem

Given on a probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in[0,T]})$ an equivalent measure $\mathrm{Q} \sim \mathrm{P}$, random variable $\mathcal{E}_T$, and a family of abatement functions $(c_t)_{t\in[0,T]}$, determine a Q-martingale $(A^*_t)_{t\in[0,T]}$ with

$$A^*_T = \pi 1_{\{\mathcal{E}_T - \int_0^T c_t(A^*_t)dt \geq 0\}}.$$

If increments of $(\mathcal{E}_t)_{t\in[0,T]}$ are independent then solution can be claimed (?) as

$$A_t = \alpha(t, G_t) \qquad t \in [0, T]$$

with expected current allowance shortage

$$G_t = \underbrace{\mathbb{E}^Q_t(\mathcal{E}_T)}_{\mathcal{E}_t} - \int_0^t c_u(A_u)du$$

In the diffusion framework ($c_u = c$)

$$d\mathcal{E}_t = \sigma_t dW_t \qquad (\sigma_t)_{t\in[0,T]} \text{ is deterministic}$$

we have

$$dA_t = d\alpha(t, G_t)$$

$$= \underbrace{\partial_{(1,0)}\alpha(t,G_t)dt - \partial_{(0,1)}\alpha(t,G_t)c(\alpha(t,G_t))dt + \frac{1}{2}\partial_{(0,2)}\alpha(t,G_t)\sigma_t^2 dt}_{=0}$$

$$+\partial_{(0,1)}\alpha(t,G_t)\sigma_t dW_t$$

which yields PDE on $[0,T] \times \mathbb{R}$

$$\partial_{(1,0)}\alpha(t,g) - \partial_{(0,1)}\alpha(t,g)c(\alpha(t,g)) + \frac{1}{2}\partial_{(0,2)}\alpha(t,g)\sigma_t^2 = 0$$

with boundary condition

$$\alpha(T,g) = \pi 1_{[0,\infty[}(g) \text{ for all } g \in \mathbb{R}$$

Call on allowance price

with strike price K and maturity $\tau \in [0, T]$

$$C^\tau(t) = E_t^Q((A_\tau - K)^+) = f^\tau(t, G_t) \qquad t \in [0, \tau]$$

is obtained by solution of almost the same PDE, on $[0, \tau] \times \mathbb{R}$

$$\partial_{(1,0)} f^\tau(t,g) - \partial_{(0,1)} f^\tau(t,g) c(\alpha(t,g)) + \frac{1}{2}\partial_{(0,2)} f^\tau(t,g)\sigma_t^2 = 0$$

but with other boundary condition

$$f^\tau(\tau, g) = (\alpha(\tau, g) - K)^+ \text{ for all } g \in \mathbb{R}$$

For linear abatement function

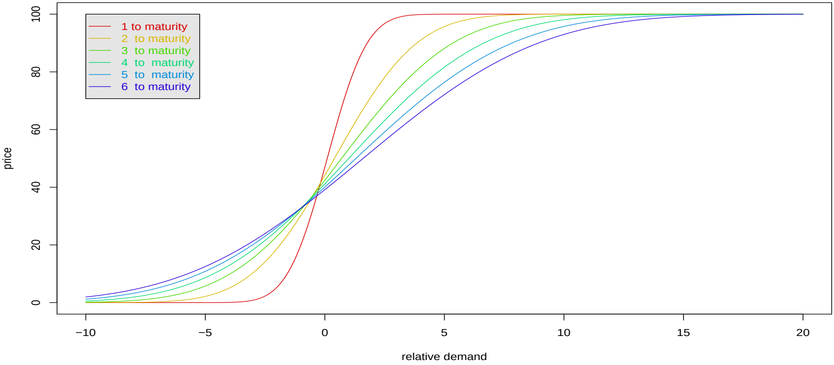
$$a \mapsto c \cdot a$$

there is solution, which is almost explicit, obtained by numerical integrations.

Here an example: for

- initial allowance price $a = A_0 = 25$
- time compliance date $T = 2$
- diffusion coefficient $\sigma = 4$
- penalty $\pi = 100$
- linear abatement function with proportionality $c = 0.02$.

The functions $\alpha(t,\cdot)$ displayed for $t = 1.9, 1.6, 1.3, 1.0, 0.7, 0.4$.



In the jump-diffusion settings

The allowance price is modeled in the same way,

$$A_t = \alpha(t, G_t), \qquad t \in [0, T]$$

but now we assume that the martingale

$$\mathcal{E}_t := \mathbb{E}^Q \left[\mathcal{E}_T \mid \mathcal{F}_t \right], \qquad t \in [0, T]$$

may have jumps. In view of

$$dG_t = -c(A_t)d_t + d\mathcal{E}_t$$

modeled by

$$dG_t = -c(\alpha(t, G_t))dt + \underbrace{\sigma(t, G_t)dW_t + \int_{\mathbb{R}_0} a(t-, G_{t-}, y)(p_\nu - q_\nu)(dy \times dt)}_{d\mathcal{E}_t},$$

Modeling with jumps

Expected allowance shortage $(G_t)_{t\in[0,T]}$ follows

$$\begin{aligned} dG_t &= -c(\alpha(t,G_t))dt + \sigma(t,G_t)dW_t \\ &\quad + \int_{\mathbb{R}_0} a(t-,G_{t-},y)(p_\nu - q_\nu)(dy \times dt), \end{aligned}$$

with Poisson random measure p_ν and its compensator q_ν.

$q_\nu(dt,dy) = \lambda dt \nu(dy)$,

λ is the jump intensity
ν jump distribution on $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$

Jumps size state and time dependent through $a(\cdot,\cdot)$
volatility is state and time dependent through $\sigma(\cdot,\cdot)$

Integro-PDE instead of PDE

Due to martingale property of $(A_t)_{t\in[0,T]}$ conditions of α are

$$-\partial_{(1,0)}\alpha(t,g) = -c(\alpha(t,g))\partial_{(0,1)}\beta(t,g) + \frac{1}{2}\sigma^2(t,x)\partial_{(0,2)}\alpha(t,g)$$
$$+\lambda\int\left[\alpha(\tau,x+a(t,g,y)) - \beta(\tau,x) - a(t,g,y)\partial_{(0,1)}\alpha(t,g)\right]\nu(dy)$$
$$\text{for all } (t,g)\in]0,T[\times\mathbb{R}$$

subject to

$$\alpha(T,g) = \pi 1_{[0,\infty[}, \qquad g\in\mathbb{R}$$

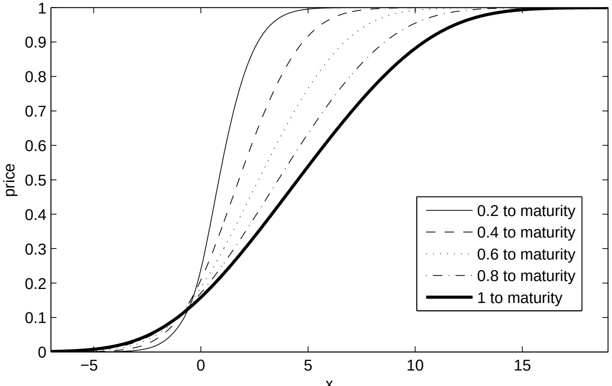
Numerical solution

- Appropriate discretization of integro-pde is available.
- Boils down to numerical integral approximation and solutions of tri-diagonal linear equations
- Conditions for existence and uniqueness of discretized solutions are determined
- Numerics is implemented

Example

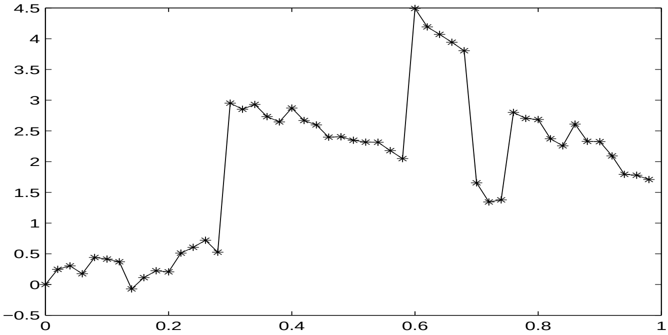
- time $T = 1$, penalty $\pi = 1$, volatility $\sigma(\cdot, \cdot) = 1$,
- jump intensity: $\lambda = 5$
- jump size: $N(0, 1)$-distributed

Functions $\alpha(t, \cdot)$: for $t = 0.8, 0.6, 0.4, 0.2, 0$



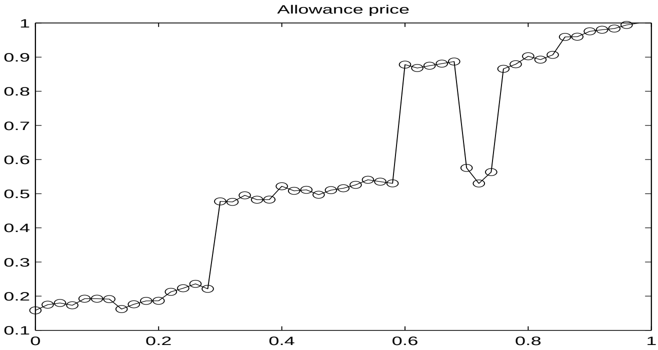
Example

A typical realization of $(G_t)_{t\in[0,T]}$:



Example

The corresponding realization of $(A_t = \alpha(t, G_t))_{t\in[0,T]}$:



Multi periods markets

So far, we focused on one compliance period.

This is a simplification since real-world markets are operating in a multi-period framework

Usually, periods are connected by regulations.

Three regulatory mechanisms

- Import
- Banking
- Withdrawal

Three regulatory mechanisms

- Import rule opens the national market to the international credits. For instance, within EU ETS, CER are accepted;
- Banking allows for the transfer of a (limited) number of (unused) allowances from the present period into the next;
- Withdrawal penalizes firms which fail to comply in two ways: by penalty payment for each unit of pollutant which is not covered by credits and by withdrawal of the missing allowances from their allocation for the next period.

In this realistic setting

Back in the discrete time

Question

How to model allowance dynamics?

Answer:

Study the Markovian case

$$\alpha_t(G_t(\omega))(\omega) \longrightarrow \alpha_t(G_t(\omega), Z_t(\omega))$$

Seemingly,

if there is a state process $(Z_t)_{t=0}^T$ with

$$(\mathcal{E}_s)_{s=0}^{t-1} \perp_{Z_t} (\mathcal{E}_s)_{s=t+1}^{T} \qquad t = 1, \ldots, T-1$$

then the solution to an even more complicated problem

$$A_t = \mathbb{E}^{\mathbb{Q}}(f(\mathcal{E}_T - \sum_{s=0}^{T-1} c_s(A_s), Z_T) \,|\, \mathcal{F}_t) \qquad t = 0, \ldots, T$$

can be claimed in the form

$$A_t = \alpha_t(G_t, Z_t) \qquad t = 0, \ldots, T$$

where $(\alpha_t)_{t=0}^T$ are true functions!

Example: two period model

Assumptions:

- two periods $[0, T]$ and $[T, T']$
- two processes $(A_t)_{t\in[0,T]}$, $({A'}_t)_{t\in[0,T']}$ for allowance prices from the first and the second period respectively.
- $({A'}_t)_{t\in[0,T']}$ is given exogenously
- $(Z_t = (\mathcal{E}_t, A'_t))_{t=0}^{T}$ is Markovian
- Banking+ Withdrawal is applied

$$f(G_T, \underbrace{\mathcal{E}_T, A'_T}_{=Z_T}) = \pi 1_{\{G_T \geq 0\}} + A'_T$$

Passage to continuous time is straight forward

Passage to continuous time

Specify the martingale dynamics

$$dZ_t = (d\mathcal{E}_t, dA'_t) = ???$$

and claim with Ito formula martingale dynamics for

$$dA_t = d\alpha(t, G_t, Z_t) = \ldots = 0$$

which gives a PDE with boundary condition

$$\partial_t \alpha = ????, \qquad \alpha(T, g, z) = f(g, z)$$

finally giving allowance price as function of SDE solution

$$A_t = \alpha(t, G_t, Z_t), \qquad dG_t = d\mathcal{E}_t - c_t(\alpha(t, G_t, Z_t))dt$$

Option pricing schemes boil down to PDEs!

Conclusion

- within EU ETS, beyond physical allowances, a large volume of allowance futures is traded.
- trading volume of European options written on these futures is increasing, although there is no clear pricing principle
- we present merely basic cornerstones of EUA option pricing
- more work is required to reflect the reality:
  - multi-period scheme operation
  - connection to international emission markets