Multigrid Tutorial

Multigrid (MG) and Local Refinement for Elliptic Partial Differential Equations

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Overview

- Why multigrid?
- Basic multigrid principles
- Full multigrid (FMG)
- Nonlinear multigrid (FAS)
- Eigenproblems
- Local Refinements

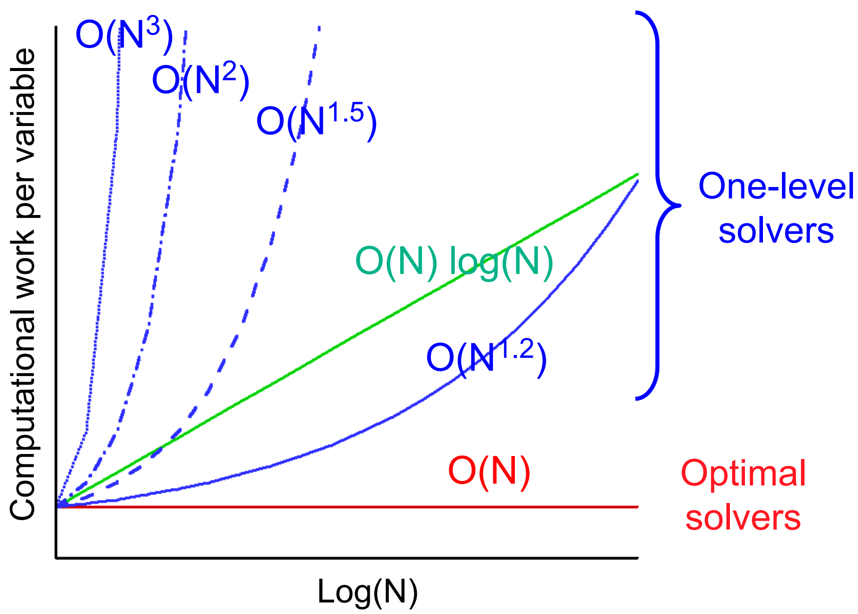
Why Multigrid?

Why Multigrid?

Large problems require
optimal solvers

Optimal solvers require
hierarchical algorithms:

- Multigrid
- Multilevel
- Multiscale



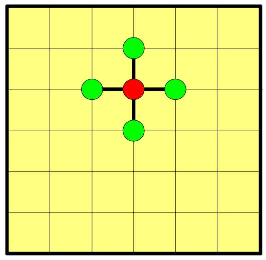
Model Problem: 2D Poisson Equation

$Lu = -\Delta u$ + Dirichlet boundary conditions

$$Lu(x,y) = f(x,y) \quad ((x,y) \in \Omega)$$

$$L_h u_h(x,y) = f_h(x,y) \quad ((x,y) \in \Omega_h)$$

$$L_h = \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_h$$



h

h=1/n, N=(n-1)²

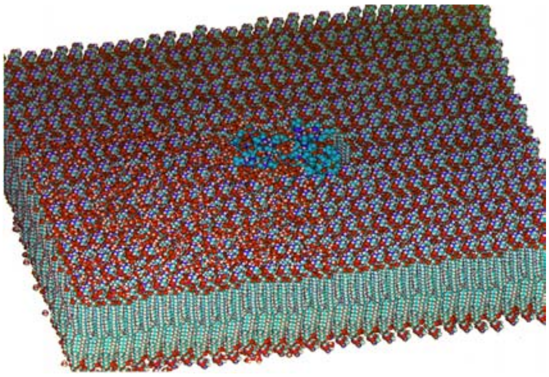
$$\longrightarrow A_h u_h = f_h$$

Model Problem: Solution Methods

Method:	Complexity (accuracy ε):
Gauss Elimination	$O(N^2)$
Jacobi iteration	$O(N^2) \log(\varepsilon)$
Gauss-Seidel iteration	$O(N^2) \log(\varepsilon)$
SOR	$O(N^{3/2}) \log(\varepsilon)$
Conjugate Gradient	$O(N^{3/2}) \log(\varepsilon)$
Nested Dissection	$O(N^{3/2})$
ICCG	$O(N^{5/4}) \log(\varepsilon)$
ADI	$O(N \log(N)) \log(\varepsilon)$
FFT	$O(N \log(N))$
Buneman	$O(N \log(N))$
Total Reduction	$O(N)$
Multigrid (iterative)	$O(N) \log(\varepsilon)$
Multigrid (FMG)	$O(N)$

Quantitative Example: Molecular Dynamics

Analysis of dynamical behavior of biological systems



Time integration:

$$\frac{d^2\vec{x}_i}{dt^2} = \frac{\vec{F}_i}{q_i}$$

Computation of energy/forces:

$$\vec{F}_i = -grad_i(V)$$

Lipid-double layer membrane

Courtesy of Hoechst ZF Scientific Computing

$$\begin{aligned} E \approx V = & \textstyle\sum_b k_b(r_b - r_q^{eq})^2 & \textit{bond lengths} \\ + & \textstyle\sum_v k_v(\vartheta_v - \vartheta_v^{eq})^2 & \textit{valence angles} \\ + & \textstyle\sum_d k_d(1 - cos(n_d\varphi_d - \delta_d)) & \textit{torsion angles} \\ + & \textstyle\sum_{i,j} \frac{q_i \cdot q_j}{4\pi\epsilon r_{i,j}} & \textit{effective charges} \\ + & \textstyle\sum_{i,j} \frac{A_{i,j}}{r_{ij}^{12}} - \frac{B_{i,j}}{r_{i,j}^{6}} & \textit{van der Waals interaction} \end{aligned}$$

Quantitative Example: Molecular Dynamics

MOLMEC: straightforward, $O(N^2)$
MEGADYN: FMM approach, $O(N)$

FMM (Fast Multipole)
Greengard, Rokhlin

Separate short & long range forces:
- Short-range forces are updated in each time step
- Long-range forces are treated on "coarser scales"

P	MOLMEC 7,000 atoms	MEGADYN 550,000 atoms
1	8152 sec	---
2	4481 sec	6305 sec
3	3056 sec	---
4	2427 sec	3295 sec
6	1769 sec	---
8	---	1840 sec

Estim. time for 550,000 atoms: 1.5 years!

Basic Multigrid Principles

Smoothing &
Coarse-grid correction

Model Problem: Gauss-Seidel Relaxation

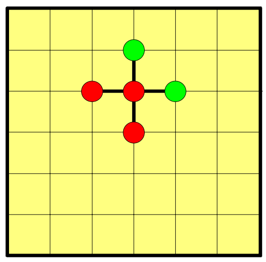
One Gauss-Seidel step:

$$u \to \bar{u}: \quad -\bar{u}_{i-1,j} - \bar{u}_{i,j-1} + 4\bar{u}_{i,j} - u_{i+1,j} - u_{i,j+1} = h^2 f_{i,j}$$

or, in terms of the error:

$$v \to \bar{v}: \quad -\bar{v}_{i-1,j} - \bar{v}_{i,j-1} + 4\bar{v}_{i,j} - v_{i+1,j} - v_{i,j+1} = 0$$

lexicographic order:



h

Very slow convergence: $\rho = 1 - O(h^2)$

Asymptotic convergence factor

Very fast „smoothing" of the error:

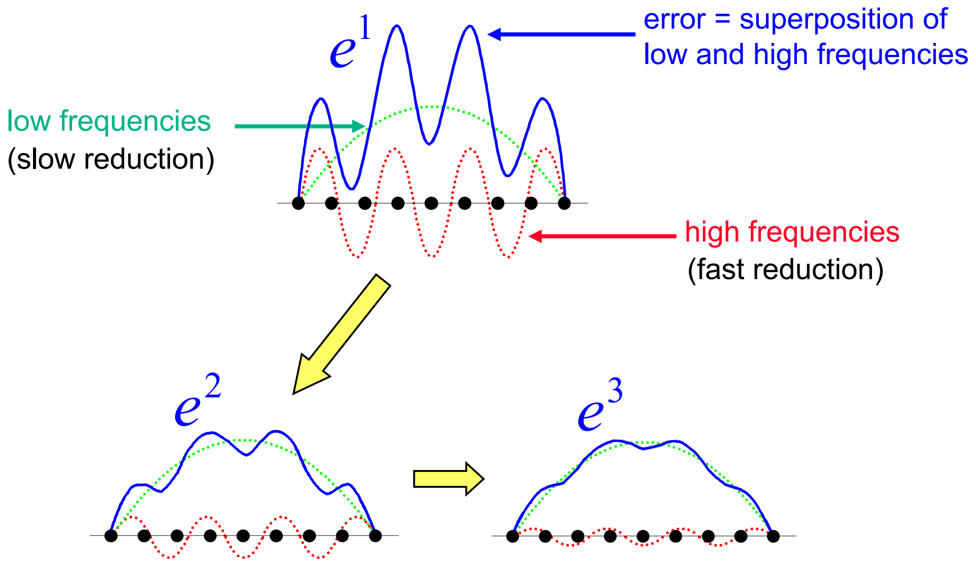
$$\bar{v}_{ij}^h \approx \frac{1}{4}\begin{bmatrix} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{bmatrix} v_{ij}^h$$

Averaging of error

The smoother the error, the less efficient a further error reduction becomes!

Principle of Error Smoothing

Relaxation methods converge slowly
but smooth the error quickly!



Infinite Grid Smoothing Analysis

Fourier components on infinite grid

$$e^{i\Theta x/h} \quad (|\Theta| \le \pi) \quad \begin{cases} |\Theta| := \max(|\Theta_1|,|\Theta_2|) \\ \Theta x := \Theta_1 x_1 + \Theta_2 x_2 \end{cases}$$

Amplification factor per relaxation step

$$e^{i\Theta x/h} \longrightarrow \mu(\Theta)\, e^{i\Theta x/h}$$

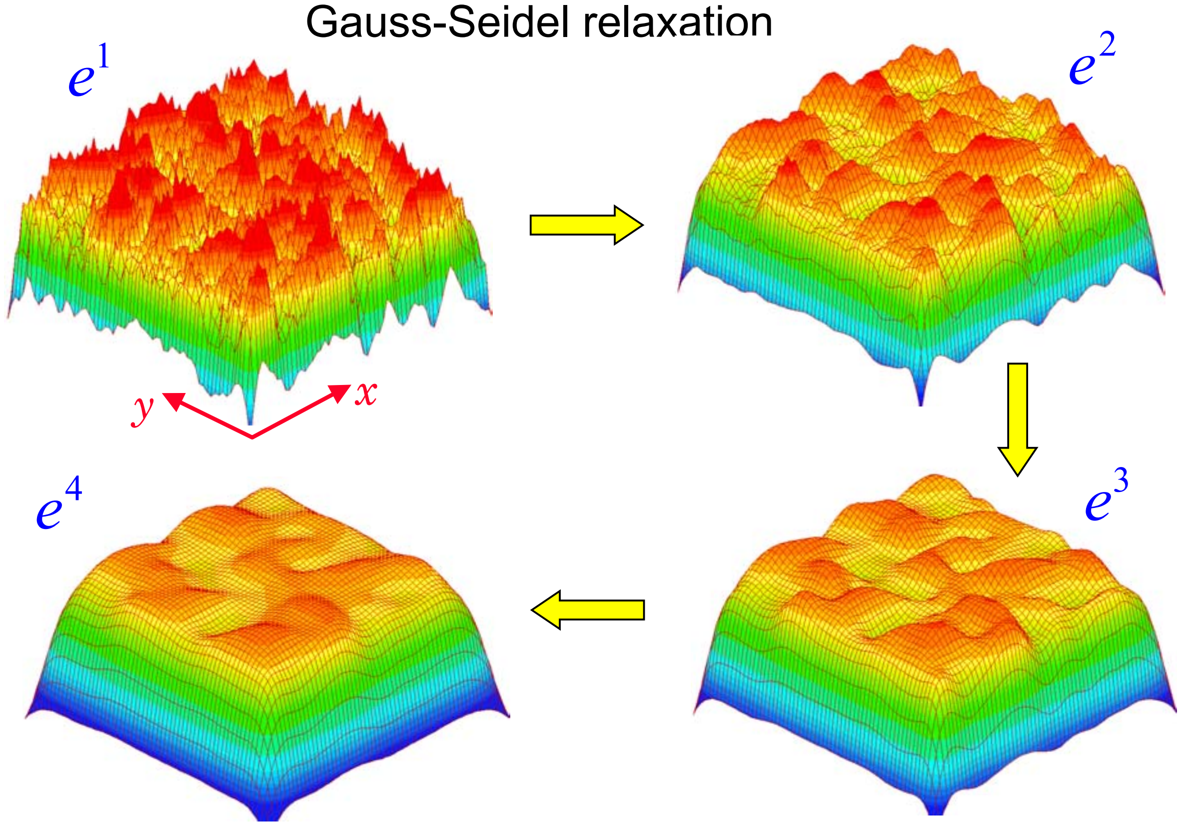
Distinguish low & high frequencies

$$e^{i\Theta x/h} \begin{cases} |\Theta| < \pi/2 & \text{low (smooth) frequencies} \\ |\Theta| \ge \pi/2 & \text{high (non-smooth) frequencies} \end{cases}$$

Smoothing factor	Model problem, Gauss-Seidel
$\mu^* = \max\{\|\mu(\Theta)\|: \ \|\Theta\| \ge \pi/2\}$	$\mu^* = 0.5$, h-independent!

Model Problem: Error Smoothing

Smoothing by
Gauss-Seidel relaxation



Coarse-Grid Correction

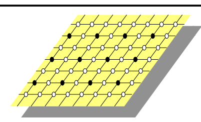
Defect equation:

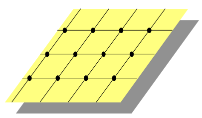
$$d_h^m = f_h - A_h u_h^m \;\Rightarrow\; A_h v_h = d_h^m \;\Rightarrow\; u_h^* = u_h^m + v_h$$

Coarse-grid correction:

$$d_h^m = f_h - A_h u_h^m \qquad\qquad u_h^{m+1} = u_h^m + \hat{v}_h$$

$d_H^m := I_h^H d_h^m$ restriction $\searrow$ $A_H v_H = d_H^m$ $\nearrow$ $\hat{v}_h := I_H^h v_H$ interpolation





$$M_{h,H} = I_h - \underbrace{I_H^h A_H^{-1} I_h^H}_{\text{not full rank}} A_h$$

$\rho(M_{h,H}) \geq 1$!

Two-Grid Cycle

ν_1 smoothing steps

$$d_h^m = f_h - A_h u_h^m$$

$$d_H^m := I_h^H d_h^m$$

restriction

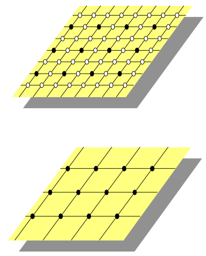
$$A_H v_H = d_H^m$$

ν_2 smoothing steps

$$u_h^{m+1} = u_h^m + \hat{v}_h$$

$$\hat{v}_h := I_H^h v_H$$

interpolation



$$M_{h,H} = S_h^{\nu_2} (I_h - I_H^h A_H^{-1} I_h^H A_h) \, S_h^{\nu_1}$$

$$\rho(M_{h,H}) << 1$$

independent of h!

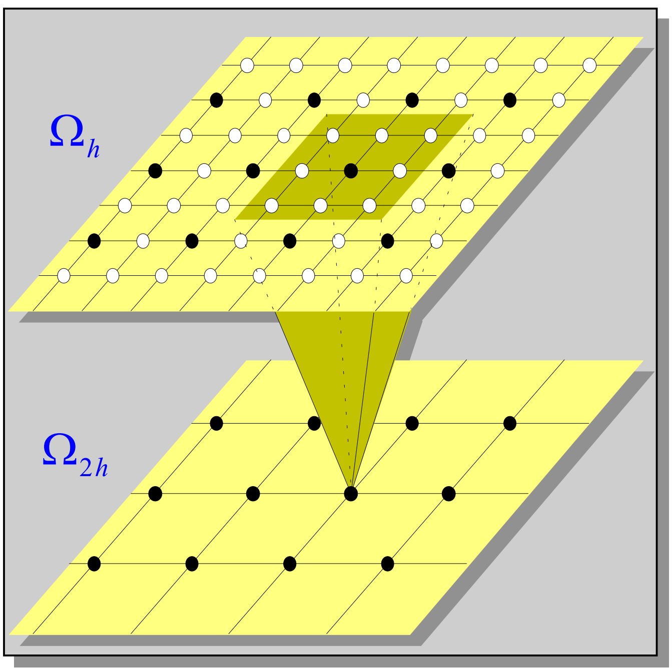
Example: Model Problem

Standard coarsening: $\Omega_h \rightarrow \Omega_{2h}$

Smoothing: Gauss-Seidel relaxation

$$L_h = \frac{1}{h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_h$$

$$L_{2h} = \frac{1}{4h^2}\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}_{2h}$$



Restriction:

„full weighting“

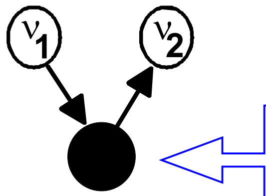
$$I_h^{2h} = \frac{1}{16}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}_h$$

Interpolation:

bilinear

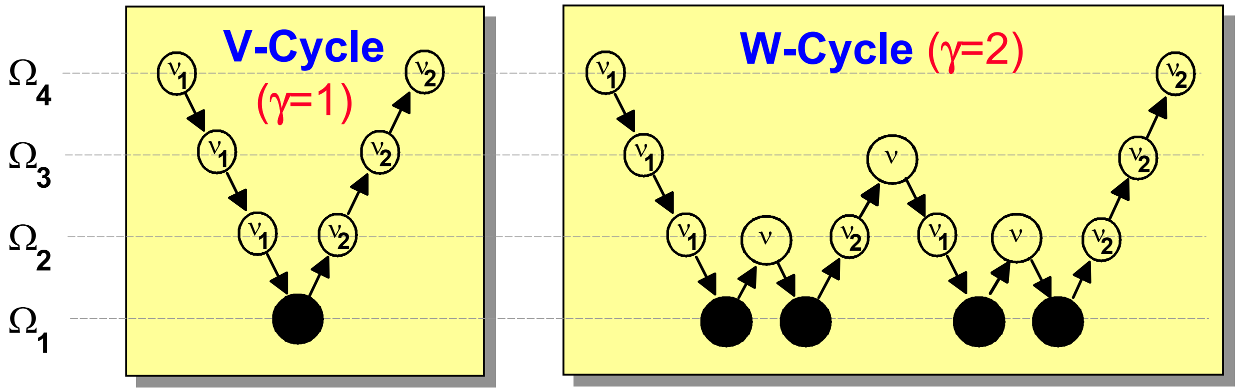
$$I_{2h}^h = \frac{1}{4}\left]\begin{matrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{matrix}\right[_h$$

Multigrid Cycle



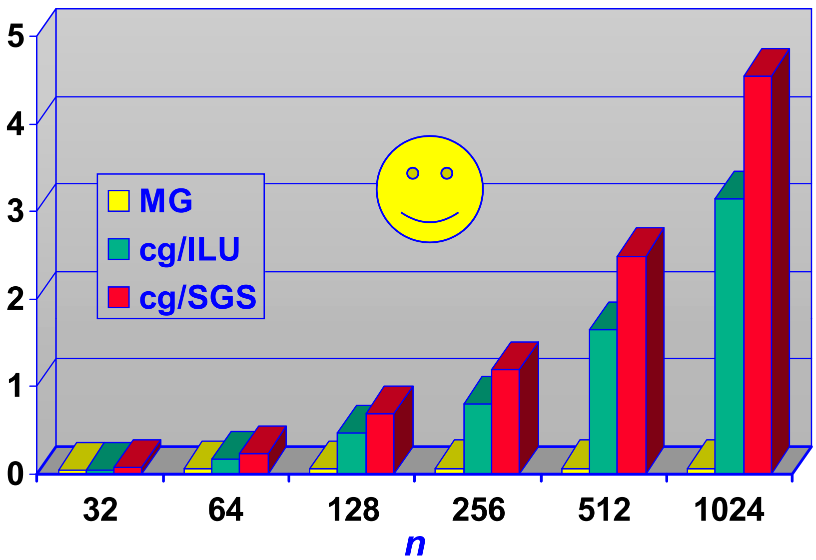
Recursive extension of two-grid cycle

Approximate solution by γ two-grid cycles using still coarser grids, ...



MG Performance for Model Problem

Time per variable (msec)



Residual reduction: $\varepsilon = 10^{-12}$

General Remarks

MG Components

What is multigrid not?

A particular solver

What is multigrid?

A general strategy for constructing hierarchical solvers

MG components

- Smoothing process (type, number of steps, ...)
- Coarsening process (type of hierarchy, speed of coarsening, ...)
- Intergrid transfer processes (interpolation, restriction)
- Coarser-level operators
- Coarsest-level solver
- More advanced techniques
- ......

Advanced MG Techniques/Applications

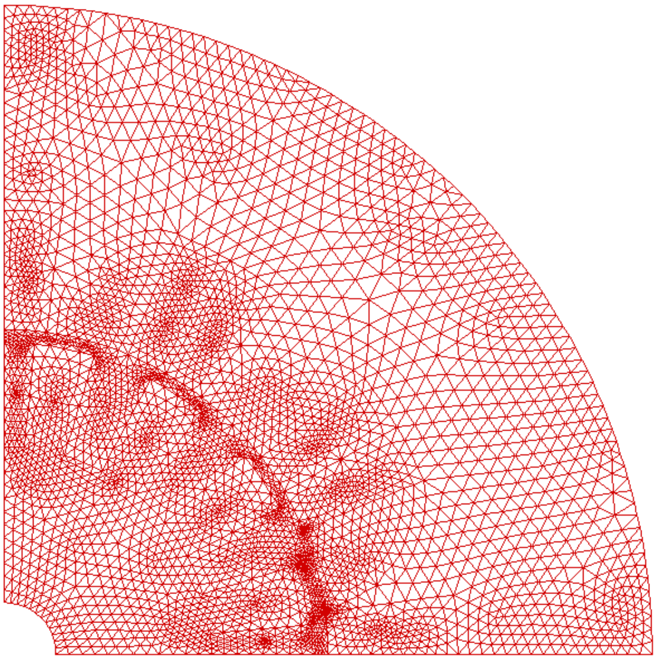
Advanced techniques

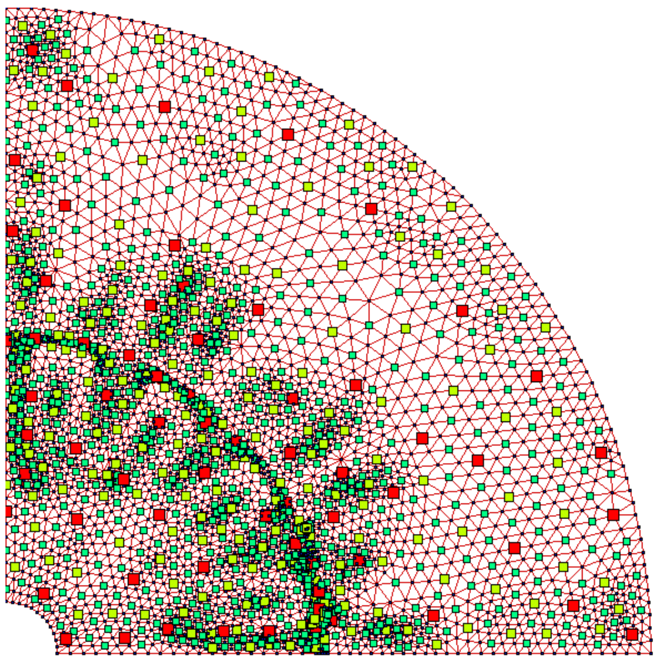
- Full multigrid (FMG)
- Nonlinear multigrid (FAS)
- Local refinements
- Multigrid for eigenproblems
- Parallel multigrid
- Algebraic multigrid (AMG)

Applications

- General domains and BCs
- Variable coefficients
- Singular perturbed problems
- Discontinuous coefficients
- Systems of PDEs
- Non-elliptic PDEs
- Algebraic problems
- ....

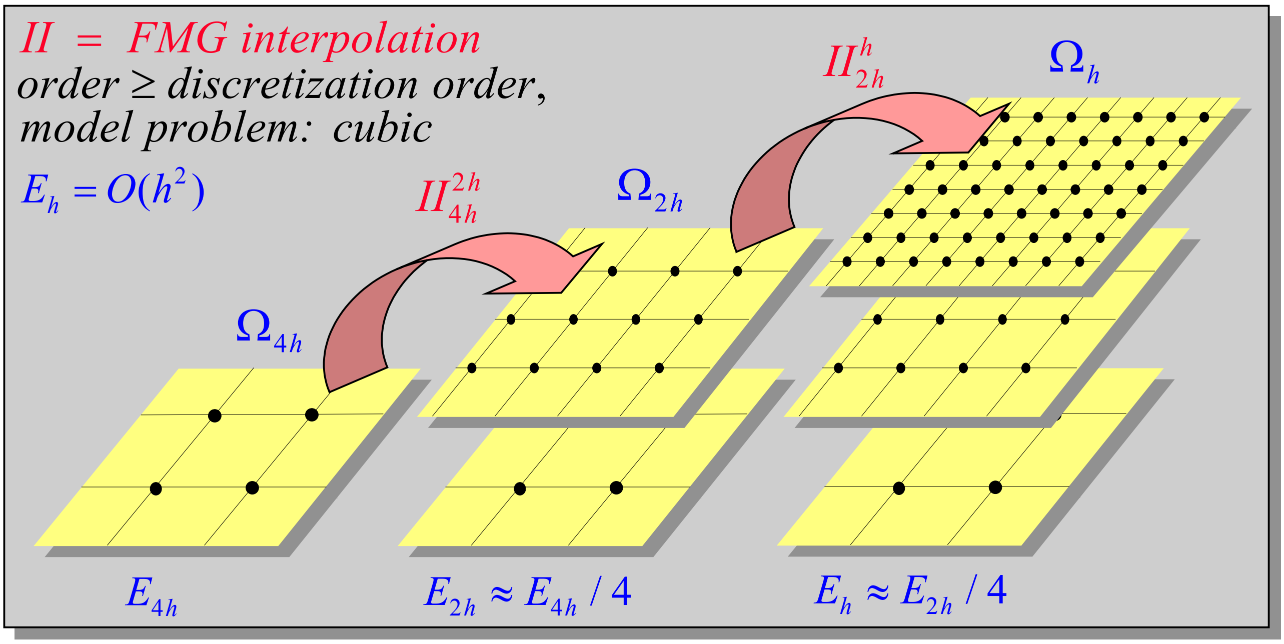
Unstructured Grids





Full Multigrid (FMG)

Nested Iteration + MG = Full Multigrid (FMG)



On each level, a fixed number of cycles, κ, is sufficient (typically, κ=*1* or κ=*2*) ➡ *O(N)* complexity!

Nonlinear Multigrid (FAS)

$$L[u](\vec{x}) = f(\vec{x})$$

discrete: $\begin{cases} L_h[u_h](\vec{x}) = f_h(\vec{x}) \\ A_h[u_h] = f_h \end{cases}$

Nonlinear Problems: Two Approaches

Straightforward approach

1. Global linearization (outer iteration)
2. Linear multigrid (inner iteration)

⬇

- Global Jacobian needs to be computed (and stored!)
- Inner and outer iterations have to be matched

Full approximation scheme (FAS)

1. Nonlinear relaxation (local linearization)
2. Nonlinear defect equation

⬇

- No global linearization (except for coarsest level?)
- Same cycle structure as in the linear case
- No matching of different iterations required
- Advantages also for linear problems

Nonlinear Relaxation (Local Linearization)

$$i = 1, \ldots, N: \quad u_i \to \bar{u}_i \quad \begin{cases} a_i[u_1, \ldots, u_{i-1}, \bar{u}_i, u_{i+1}, \ldots, u_N] = f_i & \text{Jacobi} \\ a_i[\bar{u}_1, \ldots, \bar{u}_{i-1}, \bar{u}_i, u_{i+1}, \ldots, u_N] = f_i & \text{Gauss-Seidel (GS)} \end{cases}$$

Example for nonlinear GS

$$L[u] = -\Delta u + g(x,y,u) = f \qquad (g_u(x,y,u) \geq 0)$$

$$-\bar{u}_{i-1,j} - \bar{u}_{i,j-1} + 4\bar{u}_{i,j} - u_{i+1,j} - u_{i,j+1} + h^2 g(x_i, y_j, \bar{u}_{i,j}) = h^2 f_{i,j}$$

GS-Picard: $\longrightarrow g(x_i, y_j, u_{i,j}) \quad (\text{reasonable if } h^2 g_u < 1)$

GS-Newton: $\longrightarrow g(x_i, y_j, u_{i,j}) + g_u(x_i, y_j, u_{i,j})(\bar{u}_{i,j} - u_{i,j})$

Nonlinear Defect Equation

Linear defect equation: $A_h v_h = d_h^m \quad (= f_h - A_h u_h^m)$

Nonlinear analog: $A_h[v_h + u_h^m] - A_h[u_h^m] = d_h^m$

Coarse-grid approximation

$$A_H[v_H + \bar{I}_h^H u_h^m] - A_H[\bar{I}_h^H u_h^m] = I_h^H d_h^m$$

$\bar{I}_h^H$ and I_h^H not necessarily the same!

Standard coarsening

Typical choice: $\bar{I}_h^{2h}$ = straight injection

Full Approximation Scheme (FAS)

ν_1 smoothing steps

$$d_h^m = f_h - A_h[u_h^m]$$

$$d_H^m := I_h^H d_h^m$$

$$A_H[v_H + \bar{I}_h^H u_h^m] = d_H^m + A_H[\bar{I}_h^H u_h^m]$$

$$v_H + \bar{I}_h^H u_h^m =: u_H \qquad d_H^m + A_H[\bar{I}_h^H u_h^m] =: f_H^m$$

$$\hat{v}_h := I_H^h v_H$$

ν_2 smoothing steps

$$u_h^{m+1} = u_h^m + \hat{v}_h$$

Full Approximation Scheme (FAS)

ν_1 smoothing steps

$$d_h^m = f_h - A_h[u_h^m]$$

$$d_H^m := I_h^H d_h^m$$

$$f_H^m := d_H^m + A_H[\bar{I}_h^H u_h^m]$$

$$A_H[u_H] = f_H^m$$

$$\hat{v}_h := I_H^h v_H$$

$$v_H := u_H - \bar{I}_h^H u_h^m$$

ν_2 smoothing steps

$$u_h^{m+1} = u_h^m + \hat{v}_h$$

$$u_h^m \rightarrow u_h^* \quad \Rightarrow \quad u_H \rightarrow \bar{I}_h^H u_h^*$$

Standard coarsening

$\bar{I}_h^{2h}$ = straight injection

$$\Rightarrow \quad u_{2h} \rightarrow u_h^*$$

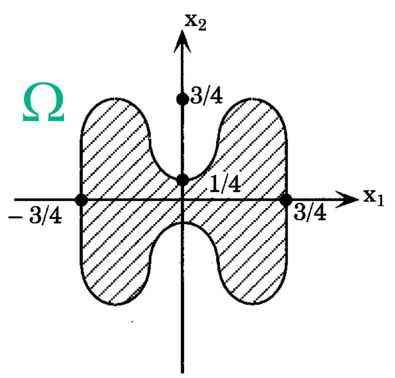
Full Approximation Scheme (FAS)

Remarks

- If A <u>linear</u>:
  - FAS identical to linear cycle („correction scheme“)
  - however: different point of view
- Recursive extension to <u>multi</u>grid as in the linear case
- In general, continuation required
  - approach the range of attraction
  - most natural: combination with FMG
- Natural for solving eigenvalue problems

Example

$$-\Delta u + e^u = f \quad (\Omega)$$
$$u = g \quad (\Gamma)$$



m	*Method I*	*Method II*	*FAS*
1	.18(+2)	.18(+2)	.14(+2)
2	.29	.20	.20
3	.86(-2)	.55(-2)	.54(-2)
4	.14(-3)	.14(-3)	.14(-3)
5	.43(-5)	.42(-5)	.42(-5)
6	.13(-6)	.13(-6)	.13(-6)
7	.47(-8)	.39(-8)	.38(-8)
8	.13(-9)	.12(-9)	.12(-9)
9	.42(-11)	.40(-11)	.39(-11)

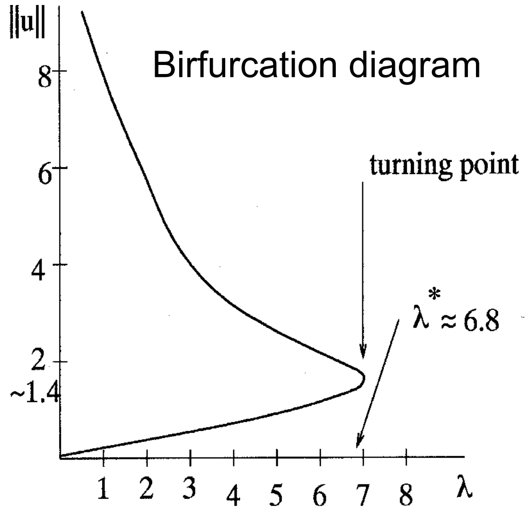
$||u_h^* - u_h^m||_2$ as a function of m

<u>Method I:</u> Global Newton linearization, number of MG cycles doubled
<u>Method II:</u> Global Newton linearization, one MG cycle per Newton step

Continuation Methods

Bratu's problem (bifurcation)

$$-\Delta u - \lambda e^{u} = 0 \quad (\Omega = (0,1)^2)$$
$$u = 0 \quad (\Gamma)$$



Continuation via λ:
gives only lower branch
solutions (up to λ~6.5)

Other solutions require
more sophisticated
continuation techniques!

Continuation Methods

Augmented system: continuation parameter „s"

$$\left.\begin{aligned} L[u,\lambda] &= 0 \quad (\Omega) \\ u &= 0 \quad (\Gamma) \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} L[u(s),\lambda(s)] &= 0 \quad (\Omega) \\ u(s) &= 0 \quad (\Gamma) \\ C[u(s),\lambda(s),s] &= 0 \quad \text{(constraint)} \end{aligned}\right.$$

The constraint has also to be transferred
to coarser grids in the sense of FAS

For example: „arclength continuation"

$$C[u,\lambda,s] = \| du/ds \|_2^2 + |d\lambda/ds|^2 - 1 = 0$$

Simpler choices for Bratu:

$$C[u,\lambda,s] = \|u\| - s$$
$$C[u,\lambda,s] = u(0.5,0.5) - s$$

Eigenproblems

Here: computation of *smallest*

EV/EF for elliptic s.p.d. problems

Eigenproblems + FAS

Problem: Find *smallest* EV/EF of

$$L_h u_h - \lambda_h u_h = 0 \ (=: f_h)$$
$$\eta_h(u_h) = \sigma_h$$

normalization, e.g. $\| u_h \| = 1$

One step of a relaxation method (inefficient as solver!):

(1) GS relaxation step w.r.t. u_h (λ_h fixed) : $L_h u_h - \lambda_h u_h = f_h$

(2) Scale u_h : $\eta_h(u_h) = \sigma_h$

(3) Update of λ_h (u_h fixed) : $(L_h u_h, u_h) - \lambda_h (u_h, u_h) = (f_h, u_h)$

Two multigrid approaches

(a) Replace (1) by *linear* MG cycle ➡ inefficient

(b) Use (1-3) as smoother in (non-linear) FAS process
(can be simplified, see later)

Eigenproblems + FAS

ν_1 smoothing steps

$$d_h^{(L)} = f_h - (L_h u_h - \lambda_h u_h)$$
$$d_h^{(\eta)} = \sigma_h - \eta_h(u_h)$$

$$d_H^{(L)} := I_h^H d_h^{(L)},\ d_H^{(\eta)} := d_h^{(\eta)}$$
$$f_H := d_H^{(L)} + (L_H \bar{I}_h^H u_h - \lambda_h \bar{I}_h^H u_h)$$
$$\sigma_H := d_H^{(\eta)} + \eta_H(\bar{I}_h^H u_h)$$

$$L_H u_H - \lambda_H u_H = f_H$$
$$\eta_H(u_H) = \sigma_H$$

$$v_H := u_H - \bar{I}_h^H u_h$$
$$\hat{v}_h := I_H^h v_H$$

ν_2 smoothing steps

$$u_h \leftarrow u_h + \hat{v}_h$$
$$\lambda_h \leftarrow \lambda_H$$

At convergence:

$$u_H = \bar{I}_h^H u_h,\ \lambda_H = \lambda_h$$

If coarse enough, e.g., solution by relaxation (1-3)

Eigenproblems + FAS

Remarks

- Straightforward: recursive extension to more levels
- Solution on coarsest level: e.g. by relaxation (1-3)
- Generally sufficient:
  - λ-update only on coarsest level
  - normalization only on coarsest level
- Natural combination with FMG
- No more expensive than MG for regular problems
- Generalization to several (many) eigenvalues

Local Refinement

(MLAT)

Adaptivity

Adaptivity

grid resolution,
order of discretization,
type of discretization.

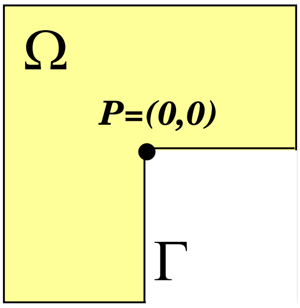
Adaptive grids

pre-defined (static),
self-adapting (dynamic).

Applications

locally non-smooth solutions
(boundary or interior layers, shocks, turbulence, ...),
non-smooth domains,
singularities / discontinuities in the differential problem.

Boundary-Driven Singularity

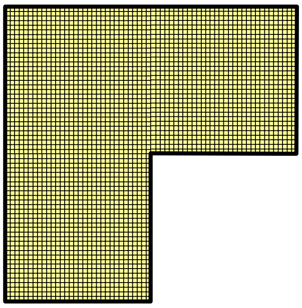


$$-\Delta u = f \quad (\Omega)$$
$$u = 0 \quad (\Gamma)$$

Singular behavior near ***P***:

$u_s(r,\phi) = r^{2/3} \sin(\tfrac{2}{3}\phi)$

$(r,\phi) =$ polar coordinates



Discretization error:

$O(h^{4/3})$ at fixed distance from ***P***

$O(h^{2/3})$ at distance $O(h)$ from ***P***

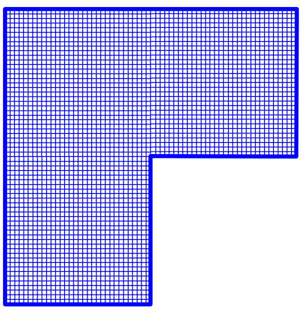
Remedies:

(1) $u^* = \tilde{u} + u_s$, $\tilde{u}$ *smooth*

(2) local refinements

Global grid vs. adaptive grid

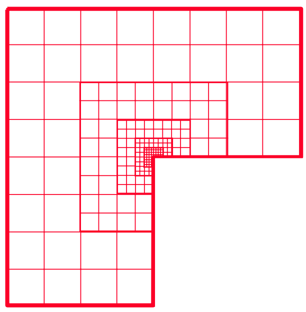
Global fine grid



49 665 points

$||u^* - u_h||_\infty$ = 3.3(-3)

Adaptive grid



657 points

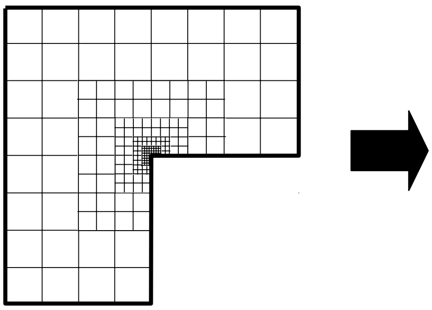
$||u^* - u_h||_\infty$ = 3.8(-3)

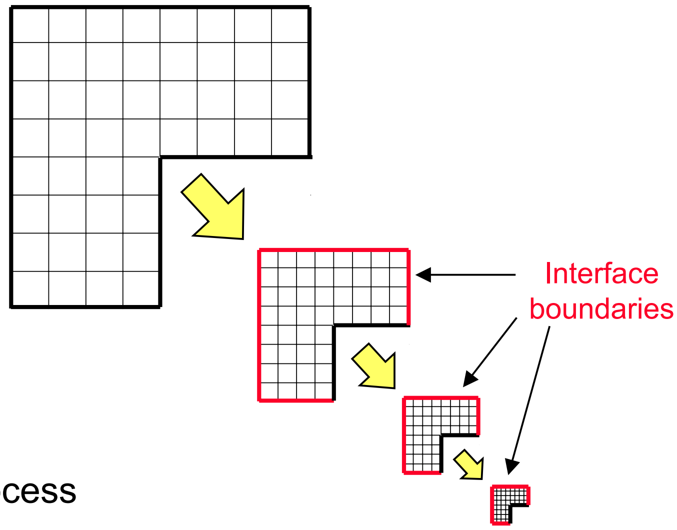
Local Refinement

Natural integration into the MG approach

Composite grid

Multigrid hierarchy of grids





Important aspects:

Dynamic local refinement
- Combination with FMG process

New aspects in MG cycling
- Treatment of interface boundaries
- Grids covering only subdomains ⇒ FAS, even for linear problems!

Local Refinement + FAS

Recall: FAS correction equation

$$L_{2h}[u_{2h}] = I_h^{2h} d_h^m + L_{2h}[\bar{I}_h^{2h} u_h^m]$$

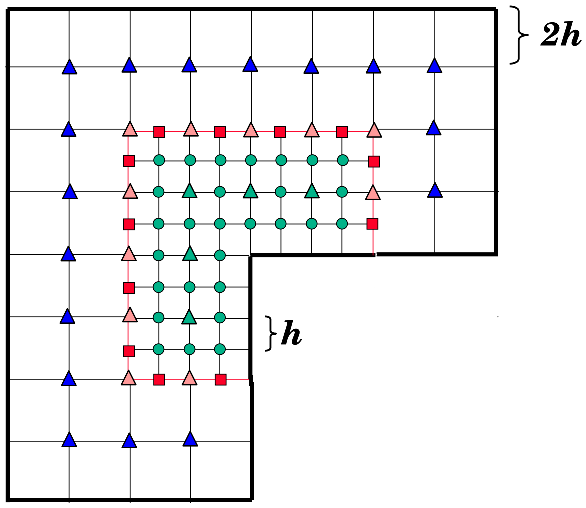
$$= I_h^{2h} f_h + L_{2h}[\bar{I}_h^{2h} u_h^m] - I_h^{2h} L_h[u_h^m]$$

$$= I_h^{2h} f_h + \tau_h^{2h}$$ *(h,2h)*-relative truncation error

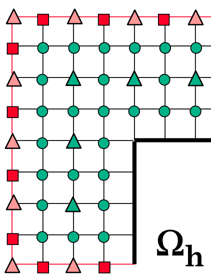
$\bar{I}_h^{2h}$ = straight injection:

$$u_h^m \to u_h^* \Rightarrow u_{2h} \to \bar{I}_h^{2h} u_h^* = u_h^*$$

Local Refinement + FAS

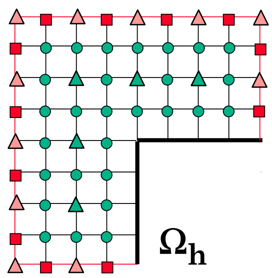


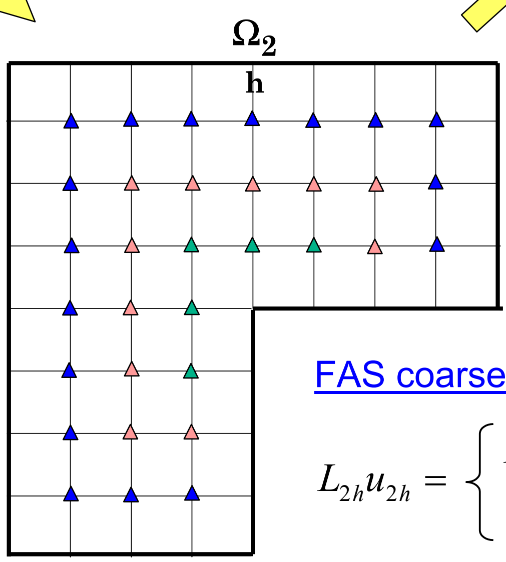
Local Refinement + FAS



Smoothing:

Relaxation on grid Ω_h:
Interface variables are treated as Dirichlet points:





FAS correction:

At all points:

FAS coarse-grid equations:

$$L_{2h}u_{2h} = \begin{cases} I_h^{2h} f_h + \tau_h^{2h} \\ f_h \end{cases}$$

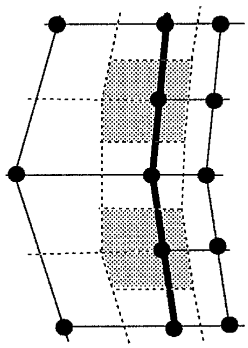
Local Refinement

Modifications

• conservative interpolation

• conservative discretization along interface

Modified interface treatment



Conservative FV discretization along interface

Smoothing by relaxation now includes interface variables

Local Refinement

Requirements on automatic refinement criteria

• Reliably detect local low accuracy

• Terminate automatically (!)

A simple grid refinement criterion

$$L_{2h} u_{2h} = \begin{cases} I_h^{2h} f_h + \tau_h^{2h} \\ f_h \end{cases}$$

τ_h^{2h}: measures to which extent the fine grid solution is different from the coarse grid one.

refine where $h^d \tau_h^{2h} \geq \varepsilon$ (d = dimension)

Example: Euler Equations

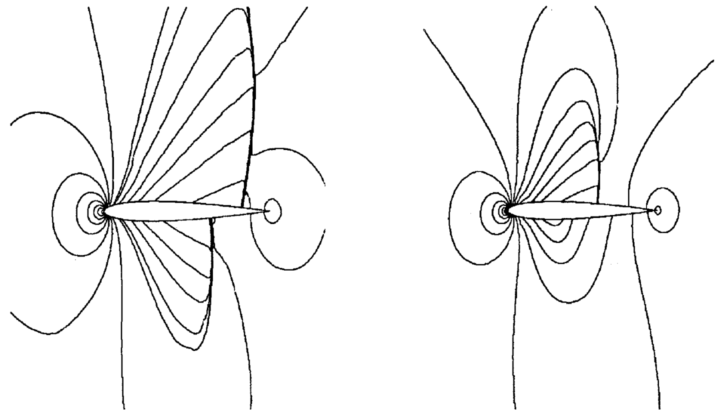
Euler Equations

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0$$

$$\begin{cases} f = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ (E+p)u \end{bmatrix} \quad \text{and} \quad g = \begin{bmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ (E+p)v \end{bmatrix} \\ p = (\gamma - 1)(E - \frac{1}{2}\rho(u^2 + v^2)) \end{cases}$$

$M_\infty = 0.85,\ \alpha = 1.0°$ $\qquad$ $M_\infty = 0.80,\ \alpha = 1.25°$

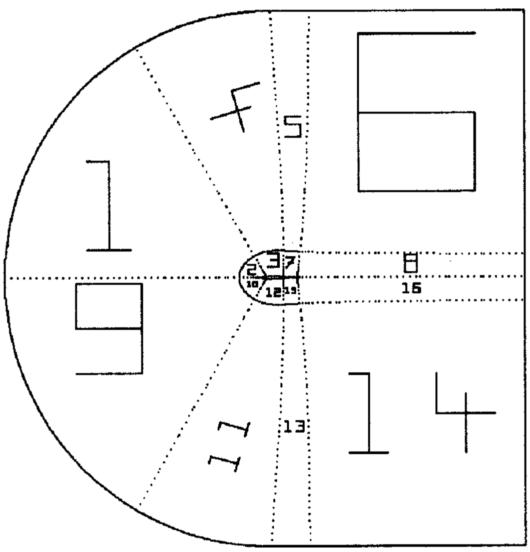
NACA0012 airfoil:

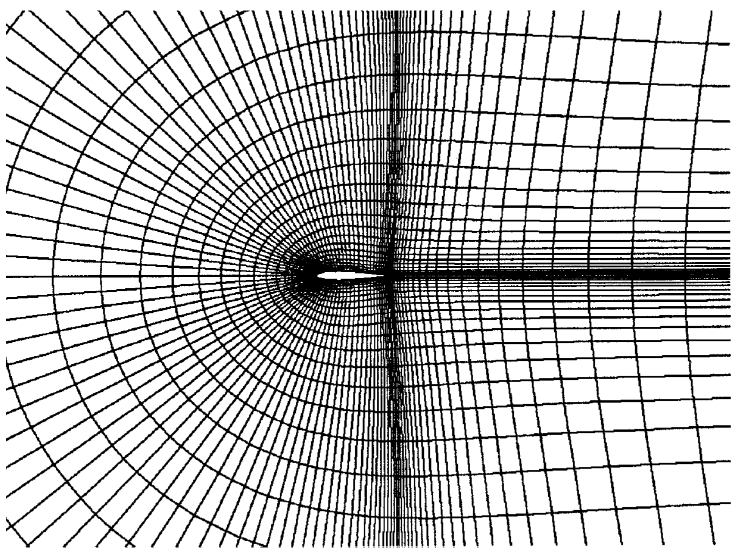


Example: Euler Equations

Global block-structured grid

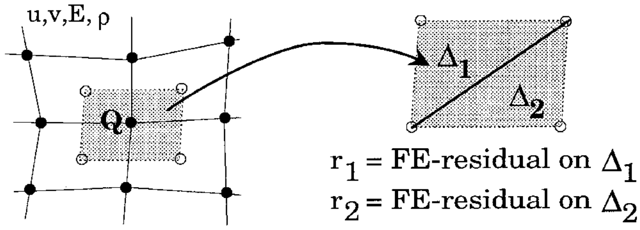
(subdivided for parallel processing)





Example: Euler Equations

Refinement criterion (FE)



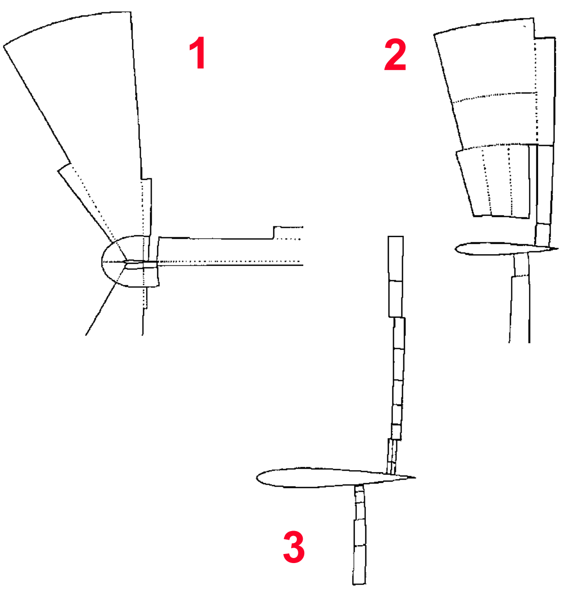
assuming *linear* shape functions

$$r^h(Q) = \sum_{i=1}^{2} \sum_{j=1}^{4} \int_{\Delta_i} | r_{ij} | \, dx$$

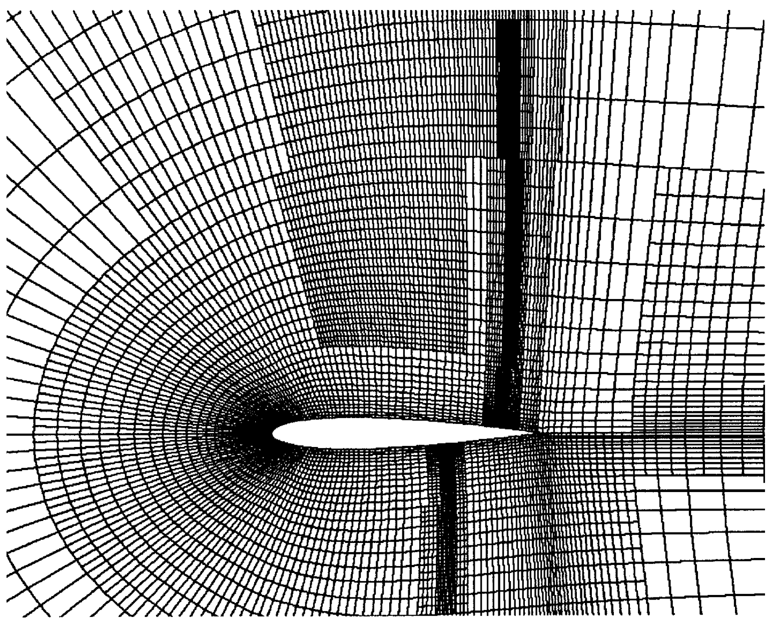
refine near Q if $r^h(Q) \geq \varepsilon$

Example: Euler Equations

Refinement areas

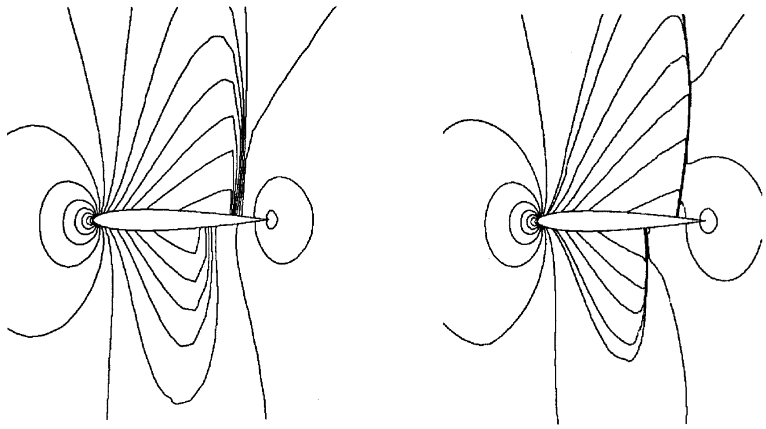


Resulting refined grid (near the profile)



Example: Euler Equations

Without refinement With refinement



	Global fine grid	*Adaptive grid*	*Factor*
# points **Time**	**197 632** **6 115 sec**	**13 866** **590 sec**	**14** **10**

Comparison: global grid vs. adaptive grid
(obtained in parallel on 16 processors)

Literature

Historical Papers

- Discovery of multigrid (theoretical)
  - **Fedorenko, R.P.:** *The speed of convergence of an iterative method*, USSR Comput. Math. and Math. Phys. 4,3 (1964).
  - **Bakhvalov, N.S.:** *On the convergence of a relaxation method with natural constraints on the elliptic operator*, USSR Comput. Math. and Math. Phys. 6,5 (1966).
- Beginning of multigrid
  - **Brandt, A.:** *Multi-level adaptive technique (MLAT) for fast numerical solution to boundary value problems*, Lecture Notes in Physics 18, Springer 1973.
  - **Brandt, A.:** *Multi-level adaptive solutions to boundary value problems*, Math. Comp. 31 (1977).
- Re-discovery of multigrid
  - **Hackbusch, W.:** *On the multigrid method applied to difference equations*, Computing 20 (1978).

Text Books

- Classical
  - **Stüben, K.; Trottenberg, U.:** *Multigrid methods: Fundamental algorithms, model problem analysis and applications*, Lecture Notes in Mathematics 960, Springer (1982).
  - **Brandt, A.:** *Multigrid techniques: 1984 Guide with applications to fluid dynamics*, GMD-Studie No. 85 (1984).
- Theory
  - **Hackbusch, W.:** *Multigrid methods and applications*, Springer Series in Comp. Math. 4, Springer (1985).
  - **McCormick, S. (ed.).:** *Multigrid methods*, Frontiers in Applied Mathematics, Vol. 5, SIAM, Philadelphia (1987).
- Tutorial-level
  - **Briggs, W.:** *A multigrid tutorial*, SIAM, Philadelphia (1987). New edition: 2001.
- Engineers
  - **Wesseling, P.:** *An introduction to multigrid methods*, Pure and Applied Mathematics Series, John Wiley and Sons (1992).
- Engineers and Practitioners
  - **Trottenberg, U.; Oosterlee, C.W.; Schüller, A.:** *Multigrid*, Academic Press, 2001 (with appendices by **Brandt, A., Oswald, P.** and **Stüben, K.**)