Some inequalities related to functional calculus

Éric Ricard

Laboratoire de Mathématiques Nicolas Oresme
Université de Caen Normandie, CNRS

February 11, 2021
General problem: How to Obtain a result on $S_p$ from one on $S_q$ via functional calculus?

To understand the Schatten $p$-classes. (more generally n.c. $L_p$ with von Neumann algebras).

To define new bounded maps $S_p \to S_q$. 
- $S_p^n, S_p :$ Schatten $p$-classes $0 < p \leq \infty$ with norm $\| . \|_p$.
  For $p = \infty$, $(M_n, \| . \|_\infty)$.
  $S_p^{n,sa}, S_p^{n+}$

- A Markov map $T : M_n \to M_m$ is a unital completely positive maps that preserves the normalized trace (to deal with n.c. probability spaces).
  Thus $T : S_p^n \to S_p^m$ is contractive.

- $\| . \|$ will denote a unitarily norm on $M_n$.
  Recall that $\| A \| \leq \| B \|$ for all $\| . \|$ iff there is a Markov map $T$ such that $|A| \leq T(|B|)$.

- Most of the results will hold for semi-finite (or type III) von Neumann algebras.
The basic inequality

Ando (88); Birman, Koplienko, Solomjak (75)

If \( f : [0, \infty[ \to [0, \infty[ \) is an operator monotone function, then for \( a, b \in M_n^+ \)

\[
\| f(a) - f(b) \| \leq \| f(|a - b|) \|
\]

The inequality reverses if \( f^{-1} \) is operator monotone.

In particular for \( f(x) = x^{p/q}, q \geq p \), an extension of the Power-Stormer inequality

\[
\| a^{p/q} - b^{p/q} \|_q \leq \| a - b \|_{p/q}^{p/q}.
\]

Based on integral decompositions: \( f(x) = \alpha + \beta x + \int \frac{x}{x+s} d\mu(s) \)

It gives the modulus of continuity of \( x \mapsto x^{p/q} \) from \( S_p^+ \to S_q^+ \).
There are many possible variations

R. (19)

If $f : [0, \infty] \rightarrow [0, \infty]$ is operator monotone, $a, b \in M_n^+$

$$\|(a - b)\exp(f(a) - f(b))\| \leq \|(a - b)\exp(f(|a - b|))\|.$$ 

If $g, h : [0, \infty] \rightarrow [0, \infty]$, $g$ operator convex and $h$ non decreasing

$$\|hf(|a - b|)\| \leq \|h(|b - a|)(f(b) - f(a))\|$$

In the case of power functions

R. (16)

If $p \geq 2$, $a, b \in M_n^+$, then

$$\mathrm{tr}(|a - b|^p) \leq \mathrm{tr}((a - b)(a^{p-1} - b^{p-1}))$$
Original motivation

**Corollary**

If $E : M_n \rightarrow M_n$ is a conditional expectation then for all $x \in M_n^+$ and $p \geq 2$

$$\|x - E(x)\|_p \leq \|x\|_p.$$ 

False if $p < 2$ even in the commutative case.
Ball-Carlen-Lieb (94)

If $1 < p < 2$, $x, y \in M_n$

$$\|x + y\|_p^2 + \|x - y\|_p^2 \geq 2\|x\|_p^2 + 2(p - 1)\|y\|_p^2$$

R.-Xu (16), Cond. version

If $1 < p < 2$, $E : M_n \to M_n$ a cond. expectation, $x \in M_n$

$$\|x\|_p^2 \geq \|E(x)\|_p^2 + (p - 1)\|x - E(x)\|_p^2$$

It reverses if $p > 2$.

Applications to hypercontractivity $L_2 \to L_p$ of Markov semi-groups by iterations.
Ando’s inequality only deals with positive operators and operator monotone functions

**Aleksandrov-Peller (2010)**

If $1 < p < q < \infty$, and $f : \mathbb{R} \to \mathbb{R}$ is $p/q$-Hölder, then for $a, b \in M_n^{sa}$

$$\|f(a) - f(b)\|_q \leq C_{p,q} \|a - b\|_p^{p/q}$$

**Mazur maps $M_{p,q}$ on $M_n$**

Let $0 < p, q < \infty$, if $a$ has polar decomposition $a = u|a|$

$$M_{p,q}(a) = u|a|^{p/q}.$$ 

It is an homeomorphism from $S_p^n \to S_q^n$.

Raynaud (02) showed that there are uniformly continuous uniformly in $n$ but without precise estimates (ultrapower techniques). The same holds for $f_{p/q} : x \mapsto |x|^{p/q}$
Aleksandrov-Peller with a $2 \times 2$ trick says that

If $1 < p < q < \infty$, $M_{p,q}, f_{p/q}$ are $p/q$-Hölder as in the commutative case.

The proof works use a Cesaro operator and $C_{p,q} \rightarrow_{p \to 1} \infty$. False with $p = 1$, weak-type.
On the opposite direction, we have using basic algebra

$M_{2,1}$ and $f_2$ are Lipschitz on balls (as in the commutative case).

Question: What happen for other values of $p, q$?
If $0 < p < q < \infty$, $M_{p,q}$, $f_{p/q}$ are $p/q$-Hölder, more precisely with $f = M_{p,q}$ or $f = f_{p/q}$

$$\|f(a) - f(b)\|_q \leq C_{p,q}\|a - b\|^{p/q}_p.$$ 

If $0 < q < p < \infty$, $M_{p,q}$, $f_{p/q}$ are Lipschitz on balls, more precisely

$$\|f(a) - f(b)\|_q \leq C'_{p,q}\|a - b\|_p (\|a\|_p + \|b\|_p)^{p/q - 1}.$$ 

Strange behaviour the constant $C_{1,\theta} \to \theta \to 1 \infty$. 

Much more involved for exponents $< 1$.

Fix $\alpha > 0$, $0 < s < \infty$ and $0 < r \leq \infty$. Let $p$ be so that $\frac{1}{p} = \frac{1}{s} + \frac{1}{r}$ and $q$ so that $\frac{1}{q} = \frac{1 + \alpha}{s} + \frac{1}{r}$. If $d \in M_n^+$, $x \in M_n$:

$$\|xd^{1+\alpha}\|_q \leq C_{\alpha,q}\|d\|_s^\alpha \|dx + xd\|_p.$$
Let \( T : M_n \rightarrow M_n \) be a Markov map for any \( 1 \leq p \leq \infty \) and \( x \in M_n^+ \)

\[
\| T(x) - T(\sqrt{x})^2 \|_{2p} \leq \frac{1}{2} \| T(x^2) - T(x)^2 \|_{\frac{1}{2}p}.
\]

Application to local approximation

**Corollary**

There is \( C > 0 \), such that if \( T \) is Markov, \( y \in M_n \) has polar decomposition \( y = u|y| \) and any \( 0 < \theta \leq 1 \)

\[
\| T(u|y|^{\theta}) - u|y|^{\theta} \|_{\frac{2}{\theta}} \leq C \| T(y) - y \|_{\frac{\theta}{4}} \| y \|_{\frac{3\theta}{4}}.
\]

Another statement if \( \theta > 1 \).
Possible application:

\( x \in L_\infty(\mathbb{R}) \cap L_1(\mathbb{R}), \varepsilon \text{ small} \)

\[
\text{supp}(\hat{x}) \subset [-\varepsilon, \varepsilon] \implies \text{supp}(\hat{x}^2) \subset [-2\varepsilon, 2\varepsilon]
\]

What about the opposite?
Can we say something on \( \text{supp}(\hat{x}) \) assuming that \( \text{supp}(\hat{x}^2) \subset [-\varepsilon, \varepsilon] \)?
If $T : M_n \to M_n$ is Markov,

$N = \{x \mid T(x) = x\}$ is a subalgebra with cond. exp $E$

We set $M_n^0 = \text{Ker } E = N^\perp$.

**Spectral gaps**

We say that $T$ has a $p$-spectral gap if there is $0 < \delta_p < 1$ such that

$$\forall x \in M_n^0, \quad \|T(x)\| \leq (1 - \delta_p)\|x\|_p$$

**Factorizable maps**

$T$ is factorizable if there is a Markov representation $\pi$ and a Markov conditional expectation $E$ such that $T = E\pi$
If $T$ has a 2-spectral gap then it has a $p$-spectral gap for $1 < p < \infty$.

If $T$ is factorizable with a $p$-spectral gap some $1 < p < \infty$ then it has a 2-spectral gap.

1. has also been obtained by Heilman, Mossel (17), Oleszkiewicz using only interpolation. Their quantitative estimate is better for $p > 2$. 