From Entropic Curvature Bounds to Logarithmic Sobolev Inequalities

Melchior Wirth
(joint work with Haonan Zhang)

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Bakry-Émery approach

uses Bochner's identity to characterize $\mathrm{Ric}_g \geq K$

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Definition (τ-symmetric quantum Markov semigroup)
$(T_t)_{t\geq 0}$ weak* continuous semigroup of unital, completely positive operators on $\mathcal{M}$ such that

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(T_t) satisfies CLSI(λ) if $(T_t \otimes \mathrm{id}_{\mathcal{N}})$ satisfies MLSI(λ) for all tracial $(\mathcal{N}, \tau_{\mathcal{N}})$

Theorem (Cipriani–Sauvageot)
The generator of a τ-symmetric QMS (with carré du champ) is of the form $\partial^*\partial$ for derivation ∂ with values in normal $\mathcal{M}$-bimodule $\mathcal{H}$.

First-order Differential Calculus

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- (T_t) heat semigroup on complete Riemannian manifold (M, g) $\rightsquigarrow \mathcal{H} = L^2(M; TM)$, $\partial = \nabla$
- (T_t) τ-symmetric QMS on $M_n(\mathbb{C}) \rightsquigarrow$ generator $\mathcal{L}$ has Lindblad form $\mathcal{L} = \sum_{j=1}^d [v_j, [v_j, \cdot\,]] \rightsquigarrow \mathcal{H} = L^2(M_n(\mathbb{C}), \tau)^d$, $\partial A = ([v_j, A])_j$

(Complete) gradient estimate

Λ operator mean $\rightsquigarrow$

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Theorem (W., Zhang '20)
If Λ is the logarithmic mean (and minor technical assumptions), then GE(K,∞) implies MLSI$(2K)$. The same is true for CGE(K,∞) and CLSI$(2K)$.

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- q-Ornstein–Uhlenbeck semigroup on $\Gamma_q(H)$ satisfies $\mathrm{CGE}(1, \infty)$

Stability properties

Proposition (W., Zhang '20)
If (S_t), (T_t) *satisfy* $\mathrm{CGE}(K,\infty)$, *then* $(S_t \otimes T_t)$ *and* $(S_t * T_t)$ *satisfy* $\mathrm{CGE}(\mathrm{K},\infty)$.

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Example
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$T_t\lambda_g = e^{-t\|b(g)\|^2}\lambda_g \rightsquigarrow \mathcal{L} = \sum_j [v_j,[v_j,\cdot\,]]$ with $v_j\delta_g = \langle b(g), e_j\rangle\delta_g$

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Corollary
$(e^{-t(-\Delta)^{1/2}})$ *on* $L^\infty(\mathbb{T}^d)$ *satisfies* $\mathrm{CLSI}(2)$.

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- Is there a finite-dimensional improvement $\mathrm{GE}(K, N)$?