Quasi-relative entropy: the closest separable state and reversed Pinsker inequality

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Entropy Inequalities, Quantum Information and Quantum Physics

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Outline of the talk

- Quasi-relative entropy (f-divergence)
- Pinsker and reversed Pinsker inequalities
- Reversed Pinsker for Tsallis entropy
- Reversed Pinsker for quasi-relative entropy
- The 'closest' separable state
- Future Research

Quantum Relative Entropy

The quantum relative entropy between two states ρ and σ is as follows

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Additivity of relative entropy

$$S(\rho_1 \otimes \rho_2||\sigma_1 \otimes \sigma_2) = S(\rho_1||\sigma_1) + S(\rho_2||\sigma_2)$$

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Monotonicity of Quantum Relative Entropy (Data Processing Inequality)

$$S(\rho\|\sigma) \geq S(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$

Completely Positive Trace Preserving (CPTP) map

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In particular,

$$S(\rho^{AB}\|\sigma^{AB}) \geq S(\rho^{A}\|\sigma^{A})$$

Equivalent statements

Strong Sub-additivity of quantum entropy

For a tri-partite state ρ^{ABC}

$$S(\rho^{ABC}) + S(\rho^{B}) \leq S(\rho^{AB}) + S(\rho^{BC})$$

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For $\rho = \sum_j p_j \rho_j$ and $\sigma = \sum_j p_j \sigma_j$

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all "equivalent"

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e.g. Monotonicity under partial traces holds for all states iff Strong sub-additivity relation holds for all states

Equivalent statements

Stinespring factorization theorem

$$\mathcal{N}(\rho) = \mathrm{Tr}_E(U^*(\rho \otimes 1_E)U)$$

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see "Strong Subadditivity of Quantum Entropy" on Wikipedia

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$$\|\rho - \sigma\|_1 = \max_{-I \leq \Lambda \leq I} \mathrm{Tr}\{\Lambda(\rho - \sigma)\}$$

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Triangle Inequality $$\|\rho-\sigma\|_1 \leq \|\rho-\tau\|_1 + \|\tau-\sigma\|_1$$

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In particular,

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Quasi-relative entropy

Let $f:(0,\infty)\to\mathbb{R}$ be operator monotone decreasing, and $f(1)=0$, *quasi-relative entropy*, or f*-divergence* is

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$$S_f(\rho \| \sigma) = \sum_{j,k} \lambda_j f\left(\frac{\mu_k}{\lambda_j}\right) |\langle\phi_k||\psi_j\rangle|^2$$

For $f(x) = -\log x$, the quasi-relative entropy becomes the relative entropy

$$S_{-\log}(\rho \| \sigma) = S(\rho \| \sigma) = \mathrm{Tr}(\rho \log \rho - \rho \log \sigma)$$

Quasi-relative entropy

For $q \in (0,2)$, the function $f_q(x) = \frac{1}{1-q}(1 - x^{1-q})$ gives *Tsallis q-entropy*

$$S_q(\rho\|\sigma) = \frac{1}{1-q}\left(1 - \mathrm{Tr}(\rho^q \sigma^{1-q})\right)$$

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$$S_q(\rho \| \sigma) = \frac{1}{1-q} \left(1 - \mathrm{Tr}(\rho^q \sigma^{1-q})\right)$$

is used in entanglement and thermodynamics, nonextensive statistics, optical lattice theory, particle charging, statistical mechanics, and others - see "*Quantum Entropies*" on Scholarpedia

Quasi-relative entropy

For $\alpha \in (0,1)$, the function $f_\alpha(x) = 1 - x^{1-\alpha}$ gives

$$S_\alpha(\rho\|\sigma) = 1 - \mathrm{Tr}(\rho^\alpha \sigma^{1-\alpha})$$

which defines *Renyi relative entropy*

$$S_\alpha^R(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{Tr}(\rho^\alpha \sigma^{1-\alpha}) = \frac{1}{\alpha - 1} \log(1 - S_\alpha(\rho\|\sigma)))$$

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is used in hypothesis testing (Csiszar '95); entanglement-assisted LOCC conversion; strong converse problem in quantum hypothesis testing (Mosonyi, Ogawa, '15); strong converse problem for the classical capacity of a quantum channel (Wilde et. al., '14)

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Unitary invariance

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Positivity

$$S_f(\rho\|\sigma) \geq 0$$

and $S_f(\rho\|\sigma) = 0$ if and only if $\rho = \sigma$

Data Processing Inequality

Petz '85

$$S_f(\rho\|\sigma) \geq S_f(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \qquad \text{for CPTP map } \mathcal{N}$$

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$$S_f(\rho\|\sigma) \geq S_f(\mathcal{N}(\rho)\|\mathcal{N}(\sigma))$$ for CPTP map $\mathcal{N}$

Trace-distance

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Relative entropy vs Trace distance

Pinsker inequality

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Pinsker inequality for quasi-relative entropy

By Hiai and Mosonyi '16

$$\frac{f''(1)}{2}\|\rho - \sigma\|_1^2 \leq S_f(\rho\|\sigma)$$

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$$\geq ?$$

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$f(x) = -\log x$

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Reversed Pinsker inequality

By Audenaert, Eisert '11

$$S(\rho\|\sigma) \leq (\alpha_\sigma + T)\log(1 + T/\alpha_\sigma) - \alpha_\rho \log(1 + T/\alpha_\rho)$$

$T = \frac{1}{2}\|\rho - \sigma\|_1^2$ $\quad$ α_ω is the minimal non-zero eigenvalue of the state ω

Reversed Pinsker for Tsallis relative entropy

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Rastegin '11

$1 < q \leq 2$

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Reversed Pinsker for quasi-relative entropy

Theorem (V. '19)

Let f any operator monotone decreasing function. Let states ρ and σ be either 2-dimensional qubit states or classical states. Assume one of two conditions: 1) ρ is full rank; 2) $a_f = 0$. Then the following holds

$$S_f(\rho\|\sigma) \leq \|\rho - \sigma\|_1 \left[\frac{\lambda_\rho}{\lambda_\rho - \alpha_\sigma} f(\lambda_\rho^{-1}\alpha_\sigma) - a_f \right]$$

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$\lambda_\rho \in (0, 1]$ is the largest eigenvalue of ρ

$\alpha_\sigma \in (0, 1]$ is the smallest eigenvalue of σ

$a_f = -\lim_{y\uparrow\infty} \frac{f(iy)}{iy}$

Reversed Pinsker for relative entropy

Theorem

In any finite dimensions,

$$S(\rho\|\sigma) \leq \|\rho-\sigma\|_1 \lambda_\rho \frac{\log(\alpha_\rho)-\log(\alpha_\sigma)}{\alpha_\rho-\alpha_\sigma} \leq \frac{\lambda_\rho}{\alpha}\|\rho-\sigma\|_1$$

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For qubits, taking $f(x) = -\log(x)$, we have a slightly improved bound

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Before,

$$S(\rho\|\sigma) \leq (\alpha_\sigma+T)\log(1+T/\alpha_\sigma)-\alpha_\rho\log(1+T/\alpha_\rho)$$

$T = \frac{1}{2}\|\rho-\sigma\|_1^2$

In any dimension larger than four there are states for which our bound is better.

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original $S_q(\rho\|\sigma) \leq \dfrac{1}{q-1}\dfrac{\lambda_\rho^q}{\alpha^q}\,\|\rho - \sigma\|_1$

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Reversed Pinsker for Tsallis relative entropy

For $q \in (0,1)$

$$S_q(\rho\|\sigma) \leq \frac{1}{1-q}\|\rho-\sigma\|_1 \lambda_\rho^q \frac{\alpha_\rho^{1-q}-\alpha_\sigma^{1-q}}{\alpha_\rho-\alpha_\sigma} \leq \|\rho-\sigma\|_1 \frac{\lambda_\rho^q}{\alpha^q}$$

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For qubits

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original $S_q(\rho\|\sigma) \leq \dfrac{1}{\underbrace{1-q}_{}}\|\rho-\sigma\|_1 \dfrac{\lambda_\rho^q}{\alpha_\sigma^q}$

≥ 1 for $q \in (0,1)$

Idea of the proof

Every operator monotone decreasing function f has the following integral representation (Donoghue '74)

$$f(x) = -a_f x - b_f + \int_0^\infty \left(\frac{1}{t+x} - \frac{t}{t^2+1} \right) \mathrm{d}\mu_f(t)$$

here $a_f := -\lim_{y\uparrow\infty} \dfrac{f(iy)}{iy} \geq 0$, $b_f := -\mathrm{Re} f(i) \in \mathbb{R}$ and μ is a positive measure on $(0,\infty)$ s.t.

$$\int_0^\infty \frac{1}{t^2+1} \mathrm{d}\mu_f(t) < \infty$$

and $\mu_f(x_1) - \mu_f(x_0) = -\lim_{y\downarrow 0} \frac{1}{\pi} \int_{x_0}^{x_1} \mathrm{Im} f(-x+iy)\, \mathrm{d}x$

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If $f(1) = 0$, then

$$f(x) = a_f(1-x) + \int_0^\infty \left(\frac{1}{t+x} - \frac{1}{t+1} \right) \mathrm{d}\mu_f(t)$$

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$$= 0$$

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using integral representation

$$= \int_0^\infty d\mu_f(t)\ \mathrm{Tr}\{\big((tI + \Delta_{\sigma,\rho})^{-1} - (tI + \Delta_{\rho,\rho})^{-1}\big)\,\rho\}$$

Idea of the proof

$$S_f(\rho\|\sigma) = \mathrm{Tr}\{(f(\Delta_{\sigma,\rho}) - f(\Delta_{\rho,\rho}))\rho\}$$

$$= \int_0^\infty d\mu_f(t)\ \mathrm{Tr}\{\left((tI + \Delta_{\sigma,\rho})^{-1} - (tI + \Delta_{\rho,\rho})^{-1}\right)\rho\}$$

using $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ and $\Delta_{A,B} = L_A R_{B^{-1}}$

$$= \int_0^\infty d\mu_f(t)\ \mathrm{Tr}\{\left((tI + \Delta_{\sigma,\rho})^{-1}(L_\rho - L_\sigma)(tI + \Delta_{\rho,\rho})^{-1}\right)(I)\}$$

Idea of the proof

$$S_f(\rho\|\sigma) = \mathrm{Tr}\{(f(\Delta_{\sigma,\rho}) - f(\Delta_{\rho,\rho}))\rho\}$$

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$$= \int_0^\infty d\mu_f(t)\ (t+1)^{-1}\mathrm{Tr}\{D_t(\rho - \sigma)\}$$

where $D_t = \sum_{jk} \left(t + \frac{\mu_k}{\lambda_j}\right)^{-1} \langle\psi_j||\phi_k\rangle|\psi_j\rangle\langle\phi_k|$, using spectral decomposition of ρ, σ

$$\rho = \sum_j \lambda_j|\psi_j\rangle\langle\psi_j|, \quad \sigma = \sum_k \mu_k|\phi_k\rangle\langle\phi_k|$$

Idea of the proof

Lemma For orthogonal bases $\{|\psi_j\rangle\}$ and $\{|\phi_k\rangle\}$, let

$$D = \sum_{kj} C_{kj} \langle\psi_j||\phi_k\rangle|\psi_j\rangle\langle\phi_k|$$

such that $0 \leq C_{kj} \leq C$ for all k, j and some C. Consider two cases:

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$\square$

Future research

If ρ and σ are d-dimensional states,

$$S_f(\rho\|\sigma) \leq \|\rho - \sigma\|_1 \sqrt{d} \left[\frac{\lambda_\rho}{\lambda_\rho - \alpha_\sigma} f(\lambda_\rho^{-1}\alpha_\sigma) - a_f \right]$$

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How to get rid of the dimension?

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Bipartite state ρ_{AB} can be separable or entangled

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How much entanglement does a state have?

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[Vedral, Plenio '98, +Rippin, Knight '97] A distance should satisfy

- $D(\rho, \sigma) = 0$ if and only if $\rho = \sigma$
- D is invariant under unitary operations: $D(U\rho U^*, U\sigma U^*) = D(\rho, \sigma)$
- Data Processing Inequality: $D(\mathcal{N}(\rho), \mathcal{N}(\sigma)) \leq D(\rho, \sigma)$ for any CPTP map $\mathcal{N}$

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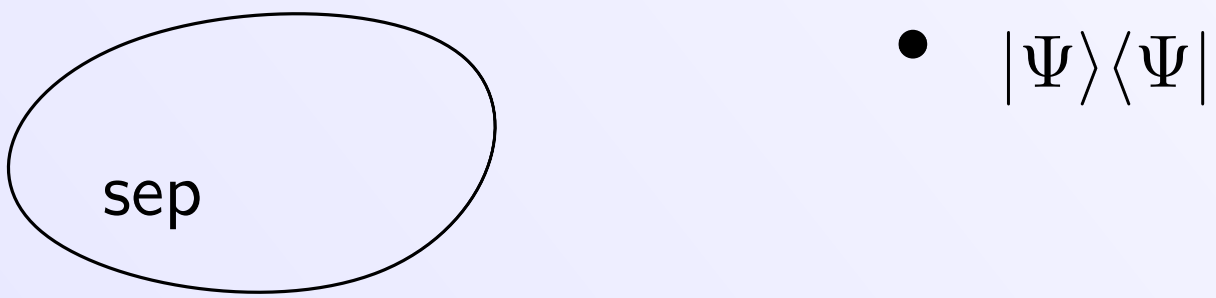

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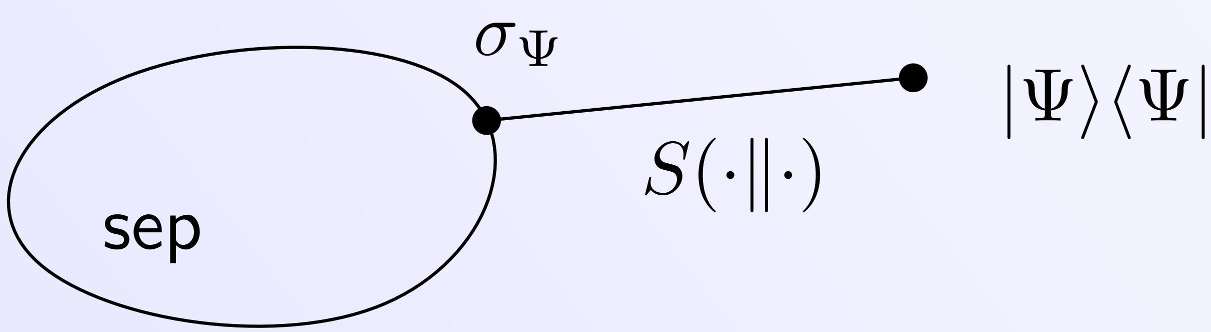

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Theorem (V. '20)

For a maximally entangled state $|\Psi^+\rangle = \sum_j \frac{1}{\sqrt{d}}|jj\rangle$, the quasi-relative entropy of entanglement is reached for a state $\sigma_+ = \sum_j \frac{1}{d}|jj\rangle\langle jj|$, and becomes

$$E_f(|\Psi^+\rangle\langle\Psi^+|) = f(1/d)$$

Closest separable state

Theorem

For a pure state $|\Psi\rangle = \sum_j \sqrt{p_j}|jj\rangle$, the quasi-relative entropy of entanglement for a class of functions, including $-\log(x)$, is reached for a state $\sigma = \sum_j p_j |jj\rangle\langle jj|$, and becomes

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Theorem

For a pure 2-qubit state $|\Psi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$, the quasi-relative entropy of entanglement for a class of functions, including $1 - x^{1-\alpha}$, is reached for a state in the form $\sigma = q|00\rangle\langle 00| + (1-q)|11\rangle\langle 11|$, and becomes

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Entropy of entanglement

$f(x) = -\log x$ gives relative entropy $S_{-\log}(\rho\|\sigma) = \mathrm{Tr}(\rho \log \rho - \rho \log \sigma)$

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Idea of the proof

Let $\rho = |\Psi\rangle\langle\Psi|$ and σ^* be our guess for the closest state. Consider

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This implies that

$$\frac{1}{x}[F(x,\sigma) - F(0,\sigma)] \leq F(1,\sigma) - F(0,\sigma) < 0$$

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Constrain on the function

$$H_f(p) = \int_0^\infty p(t+p)^{-2} d\mu_f(t)$$

is either 1) constant; or 2) monotonically increasing or decreasing

Future research

Is this true?

For a pure state $|\Psi\rangle = \sum_j \sqrt{p_j}|jj\rangle$, the quasi-relative entropy of entanglement for ANY function is reached for a state in the form $\sigma = \sum_j q_j |jj\rangle\langle jj|$, and becomes

$$E_f(|\Psi\rangle\langle\Psi|) = \sum_j p_j f(q_j)$$

Thank you!