Recoverability for optimized quantum f-divergences

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Data-processing inequality

Relative entropy:

$$D(\rho\|\sigma) = \operatorname{tr}(\rho[\log\rho - \log\sigma])\ ,$$

where ρ and σ are states

Data Processing inequality (DPI) [Lindblad '75, Uhlmann '77]:

$$D(\rho\|\sigma) \geq D(\Phi(\rho)\|\Phi(\sigma))\ ,$$

where Φ is a quantum channel (CPTP map)

Refinement of data-processing inequality

Junge, Renner, Sutter, W. and Winter, '15

$$
\begin{aligned}
D(\rho\|\sigma) - D(\Phi(\rho)\|\Phi(\sigma)) &\geq -\int_{\mathbb{R}} \log F(\rho, (R_\sigma^{\frac{t}{2}} \circ \Phi)(\rho))\, d\beta_0(t) \\
&\geq -\log F(\rho, (R_\sigma^u \circ \Phi)(\rho)) \\
&\geq \frac{1}{4} \left\|\rho - R_\sigma^u(\Phi(\rho))\right\|_1^2 ,
\end{aligned}
$$

$$
\begin{aligned}
F(\omega, \tau) &:= \|\sqrt{\omega}\sqrt{\tau}\|_1^2, \\
R_\sigma(x) &:= \sigma^{1/2}\Phi^\dagger(\Phi(\sigma)^{-1/2} x \Phi(\sigma)^{-1/2})\sigma^{1/2} , \\
R_\sigma^t(x) &:= \sigma^{it} R_\sigma(\Phi(\sigma)^{-it} x \Phi(\sigma)^{it})\sigma^{-it} , \\
R_\sigma^u &:= \int_{\mathbb{R}} R_\sigma^{\frac{t}{2}}\, d\beta_0(t) .
\end{aligned}
$$

Refinement of Data-Processing inequality (ctd.)

Carlen-Vershynina, '17

$$D(\rho\|\sigma) - D(\Phi(\rho)\|\Phi(\sigma)) \geq \left(\frac{\pi}{8}\right)^4 \|\Delta(\sigma,\rho)\|^{-2} \|\sigma - R_\rho(\Phi(\sigma))\|_1^4 ,$$

where

$$\Delta(\sigma,\rho)(X) \coloneqq \sigma X \rho^{-1}$$

is the relative modular operator.

Quantum f-divergence and Rényi relative entropy

Quantum f-divergence (Petz '85): for a function $f:(0,\infty)\to\mathbb{R}$,

$$Q_f(\rho\|\sigma) \coloneqq \operatorname{tr}\left(\rho^{1/2} f(\Delta(\sigma,\rho))\rho^{1/2}\right) = \langle \rho^{1/2} | f(\Delta(\sigma,\rho)) | \rho^{1/2}\rangle$$

Special cases:

- $Q_{-\log x}(\rho\|\sigma) = D(\rho\|\sigma)$.
- $Q_s(\rho\|\sigma) \coloneqq Q_{x^s}(\rho\|\sigma) = \operatorname{tr}(\rho^{1-s}\sigma^s)$ for $s \in [-1,0)\cup(0,1]$, .
- Petz-Rényi relative entropy:

$$D_\alpha(\rho\|\sigma) \coloneqq \frac{1}{\alpha-1}\log Q_{1-\alpha}(\rho\|\sigma).$$

Data processing for f-divergence and Rényi relative entropy

Data Processing inequality (Petz '85): for operator monotone decreasing f,

$$Q_f(\rho\|\sigma) \geq Q_f(\Phi(\rho)\|\Phi(\sigma)) .$$

Carlen-Vershynina, '18

$$|Q_s(\rho\|\sigma) - Q_s(\Phi(\rho)\|\Phi(\sigma))| \geq K(s, \|\Delta_{\sigma,\rho}\|)\, \|\sigma - R_\rho(\Phi(\sigma))\|_1^{4+2|s|} ,$$

where K is an explicit constant.

Sandwiched relative entropy and optimized f-divergence

Sandwiched Rényi Relative entropy: for $\alpha \in [1/2, 1) \cup (1, \infty]$ and $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{\alpha}{\alpha - 1} \log \widetilde{Q}_\alpha(\rho\|\sigma),$$

$$\widetilde{Q}_\alpha(\rho\|\sigma) := (\text{tr}|\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}|^\alpha)^{1/\alpha}$$

- Extension to general von Neumann algebras via L_p-norms Berta et al., Jenčová.

Optimized f-divergence [W'18]: for f operator monotone decreasing,

$$\widetilde{Q}_f(\rho\|\sigma) := \sup_{\omega \text{ state}} \text{tr}(\rho^{1/2} f(\Delta(\sigma, \omega))\rho^{1/2}) \ ,$$

Sandwiched relative entropy and optimized f-divergence

Special cases:

- $\widetilde{Q}_{-\log x}(\rho\|\sigma) = D(\rho\|\sigma)$.
- $\widetilde{Q}_{x^s}(\rho\|\sigma) = \operatorname{tr}(|\sigma^{s/2}\rho\sigma^{s/2}|^{\frac{1}{1+s}})^{1+s}$ for $s \in [-1,0) \cup (0,1]$.
- $\widetilde{D}_\alpha(\rho\|\sigma) := \alpha' \log \widetilde{Q}_{x^{-\frac{1}{\alpha'}}}(\rho\|\sigma) = \alpha' \log \widetilde{Q}_\alpha(\rho\|\sigma)$.

Data Processing inequality: for operator monotone decreasing f

$$\widetilde{Q}_f(\rho\|\sigma) \geq \widetilde{Q}_f(\Phi(\rho)\|\Phi(\sigma)) \,.$$

Simple proof of data processing from [W'18] (I)

- Consider partial trace $\Phi = \text{tr}_A : L(H_{AB}) \to L(H_B)$.
- Define the isometry $V_\rho : L(H_B) \to L(H_{AB})$ as

$$V_\rho(X) = (I_A \otimes X\rho_B^{-\frac{1}{2}})\rho^{\frac{1}{2}}.$$

- Note that $V_\rho(\rho_B^{\frac{1}{2}}) = \rho_{AB}^{\frac{1}{2}}$

Key Observation

$$V_\rho^* \Delta(\sigma, R_\rho(\omega_B)) V_\rho = \Delta(\sigma_B, \omega_B)$$

Simple proof of data processing from [W'18] (II)

Since f is operator monotone deceasing,

$$\begin{aligned}
\widetilde{Q}_f(\rho_B\|\sigma_B) &= \sup_{\omega_B} \langle \rho_B^{1/2} | f(\Delta(\sigma_B, \omega_B)) | \rho_B^{1/2} \rangle \\
&= \sup_{\omega_B} \langle \rho_B^{1/2} | f(V_\rho^* \Delta(\sigma, R_\rho(\omega_B)) V_\rho) | \rho_B^{1/2} \rangle \\
&\leq \sup_{\omega_B} \langle \rho_B^{1/2} | V_\rho^* f(\Delta(\sigma, R_\rho(\omega_B))) V_\rho | \rho_B^{1/2} \rangle \\
&= \sup_{\omega_B} \langle \rho^{1/2} | f(\Delta(\sigma, R_\rho(\omega_B))) | \rho^{1/2} \rangle \\
&\leq \sup_{\omega} \langle \rho^{1/2} | f(\Delta(\sigma, \omega)) | \rho^{1/2} \rangle \\
&= \widetilde{Q}_f(\rho\|\sigma)
\end{aligned}$$

This is what we are trying to refine...

First result [Gao-W., '20]

Recovery of σ:

$$D(\rho\|\sigma) - D(\Phi(\rho)\|\Phi(\sigma)) \geq \left(\frac{\pi}{8\cosh(\pi t)}\right)^4 Q_{x^2}(\rho\|\sigma)^{-1} \left\|\sigma - R^t_\rho(\Phi(\sigma))\right\|_1^4$$

where $t \in \mathbb{R}$ and $Q_{x^2}(\rho\|\sigma) := \operatorname{tr}(\sigma^2\rho^{-1}) \leq \|\Delta(\sigma,\rho)\|$

- Note that $Q_2(\rho\|\sigma)$ can be finite for faithful bosonic Gaussian states, while $\|\Delta(\sigma,\rho)\|$ is infinite

Another result [Gao-W., '20]

Recovery of ρ:

$$D(\rho\|\sigma) - D(\Phi(\rho)\|\Phi(\sigma)) \geq \left(\frac{K\pi}{2\cosh(\pi t)}\left\|\rho - R_{\sigma}^{t}(\Phi(\rho))\right\|_1\right)^{2+\epsilon}$$

where the constant K is explicit and is a function of $Q_{x^{-1}}(\rho\|\sigma)$ and ϵ, and

$$Q_{x^{-1}}(\rho\|\sigma) := \operatorname{tr}(\rho^2\sigma^{-1}) \leq \left\|\Delta^{-1}\right\|$$

Yet another result [Gao-W., '20]

Recovery of ρ in terms of sandwiched Rényi quasi-relative entropy:

$$|\widetilde{Q}_\alpha(\rho\|\sigma) - \widetilde{Q}_\alpha(\Phi(\rho)\|\Phi(\sigma))| \geq \left(\frac{\tilde{K}\pi}{2\cosh \pi t}\left\|\rho - R^t_\sigma(\Phi(\rho))\right\|_1\right)^{\frac{2+\varepsilon}{1-1/|\alpha'|}}$$

where the constant $\tilde{K}$ is explicit and is a function of α, $\widetilde{Q}_\infty(\rho\|\sigma)$, and ε, and

$$\widetilde{Q}_\infty(\rho\|\sigma) := \inf\{\lambda > 0 \,|\, \rho \leq \lambda\sigma\}$$

- Similar estimates hold for Q_α
- Extended to general von Neumann algebras (including Type III)
- Open Question: Tight recoverability of $\widetilde{Q}_\alpha$ (or $\widetilde{D}_\alpha$) via universal recovery map R^u_σ?

Sketch of proof I (inspired by Carlen, Vershynina)

Take optimal ω_B and denote

$$\Delta \coloneqq \Delta(\sigma, R_\rho(\omega_B)), \qquad \Delta_B \coloneqq \Delta(\sigma_B, \omega_B).$$

Then

$$\begin{aligned}
&\widetilde{Q}_f(\rho\|\sigma) - \widetilde{Q}_f(\rho_B\|\sigma_B) \\
&\geq \int_0^\infty \langle \rho_B^{1/2}|(\Delta_B + \lambda)^{-1}|\rho_B^{1/2}\rangle - \langle \rho^{1/2}|(\Delta + \lambda)^{-1}|\rho^{1/2}\rangle \, d\nu(\lambda) \\
&= \int_0^\infty \langle u_\lambda|(\Delta + \lambda)|u_\lambda\rangle \, d\nu(\lambda)
\end{aligned}$$

where we used integral representation of operator monotone function

$$f(x) = a + bx + \int_0^\infty \left(\frac{\lambda}{\lambda^2 + 1} - \frac{1}{\lambda + x} \right) d\nu(\lambda) \,, \quad b \geq 0,$$

and we set $|u_\lambda\rangle \coloneqq (\Delta + \lambda)^{-1}|\rho^{1/2}\rangle - V_\rho(\Delta_B + \lambda)^{-1}|\rho_B^{1/2}\rangle$.

Sketch of proof II

Now define

$$|u_t\rangle := \Delta^{1/2+it}\frac{\cosh(\pi t)}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}-it}|u_\lambda\rangle d\lambda$$

and use the integral representation

$$\Delta^{-\frac{1}{2}-it} = \frac{\cosh(\pi t)}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}-it}(\lambda+\Delta)^{-1}d\lambda \ .$$

and inequality $\|X^*X - Y^*Y\|_1 \leq 2\|X-Y\|_2$ (for $\|X\|_2 = \|Y\|_2 = 1$), along with some algebraic manipulations to show that

$$\left\|\rho - R_\sigma^{-t}(\rho_B)\right\|_1 \leq 2\,\||u_t\rangle\|_2$$

Sketch of proof III

- Finally, break $\||u_t\rangle\|_2$ into three separate integrals, corresponding to intervals $[0, S]$, $[S, T]$, $[T, \infty)$, apply triangle inequality, and bound these terms to arrive at the following upper bound on $\|\rho - R_\sigma^{-t}(\rho_B)\|_1$:

$$\frac{2\cosh(\pi t)}{\pi}\Bigg(4S^{1/2}\widetilde{Q}_{x^{-1}}(\rho\|\sigma)^{1/2} \\ + (c(S,T)\ln(T/S))^{1/2}(\widetilde{Q}_f(\rho\|\sigma) - \widetilde{Q}_f(\rho_B\|\sigma_B))^{1/2} + 4T^{-1/2}\Bigg)$$

- Now optimize S and T for particular choices of optimized f-divergence to arrive at claimed bounds

Conclusion

- We have improved the bounds of Carlen and Vershynina, because remainder terms involve $Q_{x^{-1}}(\rho\|\sigma)$ or $Q_{x^2}(\rho\|\sigma)$ instead of $\left\|\Delta^{-1}\right\|$
- We have given remainder terms for data processing of sandwiched Rényi (quasi-)relative entropy
- Extended all results to the setting of von Neumann algebras