Optimization and Equilibrium in Energy Economics, IPAM-UCLA, Winter 2016

Computing Equilibria: *Stochastic Environment*

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1962-63

UC-Berkeley-G. Debreu — Theory of Value

Arrow-Debreu model

pure-exchange economy: goods $\in \mathbb{R}^L$, prices $p = (p_1, \ldots, p_L)$, free disposal

agents: $i \in I$, $|I|$ finite ---- initial holdings: $(e_i, i \in I)$

demand functions: $x_i(p) \in \arg\max \left\{ u_i(x) \,\middle|\, \langle p, x \rangle \leq \langle p, e_i \rangle,\ x \in C_i \right\}$

utility fcn: u_i : dom $u_i = C_i \to \mathbb{R}$, usc, concave $\not\Rightarrow C_i$ closed (but convex)

excess supply: $s(p) = \sum_{i \in I} \big(e_i - x_i(p)\big)$, market clearing: $s(p) \geq 0$

& "simple" firms: exclusive concerned with shareholders

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$$\bar{p} \geq 0 \text{ equilibrium } \Leftrightarrow s(\bar{p}) \geq 0$$

& "simple" firms: exclusive concerned with shareholders

Chap.7 — Stochastic version

- (5 pages) -

$$\max_{x^0, x^1_\bullet \in \mathcal{M}} E\left\{u_i(x^0, x^1_{\xi^1}, x^2_{\xi^1,\xi^2}, \ldots)\right\}$$ *i*-agent

such that $\left\langle p^t_\xi, \sum_{\tau \le t}\left(e^\tau_{\xi^1,\ldots\xi^\tau} - x^\tau_{\xi^1,\ldots\xi^\tau}\right)\right\rangle \ge 0,\ \forall \xi = (\xi^1, \xi^2, \ldots),\ \ \mathrm{t} = 0, 1, \ldots$

$$\sum_{i \in I}\left(\sum_{\tau \le t}\left(e^\tau_{\xi^1,\ldots\xi^\tau} - x^\tau_{\xi^1,\ldots\xi^\tau}\right)\right) \ge 0, \forall \xi, t$$ Clearing the market

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$$\text{such that } \left\langle p^t_{\xi}, \sum\nolimits_{\tau \le t} \left(e^{\tau}_{\xi^1,\dots\xi^{\tau}} - x^{\tau}_{\xi^1,\dots\xi^{\tau}}\right)\right\rangle \ge 0,\ \forall \xi = (\xi^1, \xi^2, \dots),\ \ \mathrm{t} = 0,1,\dots$$

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Key Assumption (via K. Arrow): all contingencies available at time 0

$\Rightarrow$ complete market, i.e., all ξ's can be dealt with separately

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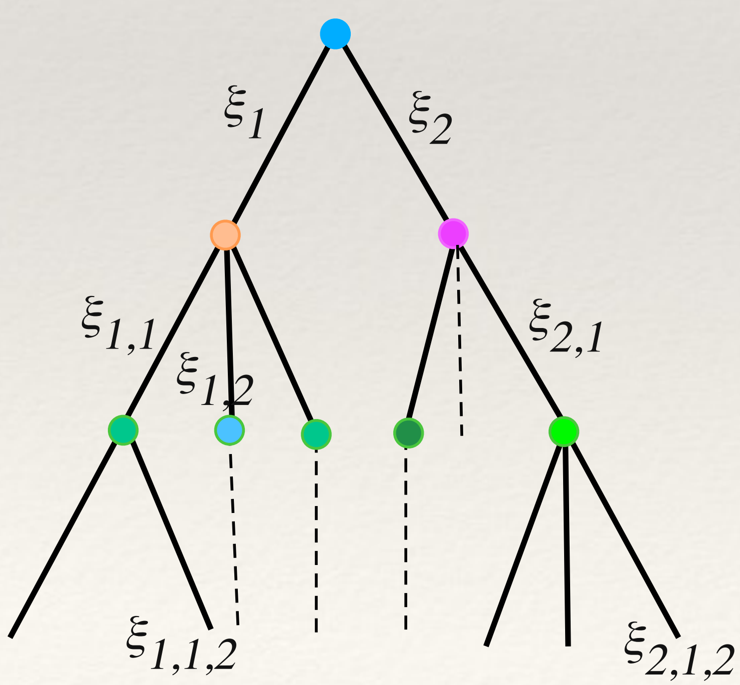

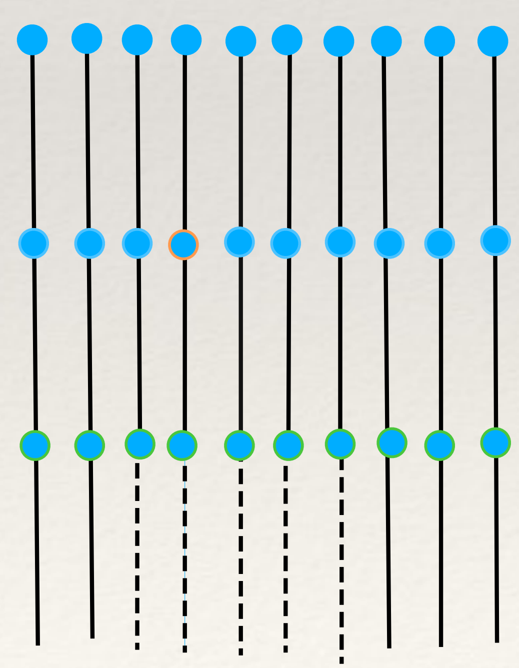

$\exists \left(p_\xi = (p^0_\xi, p^1_\xi, \ldots)\right)$

$(\forall\ \xi)$ equilibrium prices.

2000

Banco Central de Chile

+ Alejandro Jofré at CMM (U. de Chile)

CMM
CENTRO DE
MODELAMIENTO
MATEMATICO
UMI U. DE CHILE-CNRS

Re?-formulating the stochastic model

agent $i \in \mathcal{I}$ solves two (multi)-stage stochastic program ...
all contingencies not included at time 0 (= stage 1)

$\Longrightarrow$ *Incomplete Market* (Radner '72, Arrow, Debreu, Hart, ...)

$$\max u_i^0(x^0) + \mathbb{E}\{u_i^1(x_\xi^1)\} :: \langle p^0, (e_i^0 - x^0 - T_i^0 z)\rangle \geq 0, \quad x^0 \in C^0$$
$$\langle p_\xi^1, (e_{\xi,i}^1 + T_{\xi,i}^1 z - x_\xi^1)\rangle \geq 0, \; x_\xi^1 \in C_\xi^1, \quad \forall \xi \; a.s. \, (\in \Xi)$$

Price system: $p^\diamond = \big(p^0, (p_\xi^1, \xi \in \Xi)\big)$

Demand: $\Big(\bar{x}_i^0(p^\diamond), \big(\bar{x}_{\xi,i}^1(p^\diamond)\big), i \in \mathcal{I}\Big)$

Excess supply: $s^0(p^\diamond) = \sum_{i\in\mathcal{I}}(e_i^0 - \bar{x}_i^0 - T_i^0 \bar{z}_i)$,

$s^1(p^\diamond) = \sum_{i\in\mathcal{I}}(e_{\xi,i}^1 + T_{\xi,i}^1 \bar{z}_i - \bar{x}_{\xi,i}^0), \quad \forall \xi \in \Xi$

Equilibrium: $s^0(\bar{p}^\diamond) \geq 0, \quad s^1(\bar{p}^\diamond) \geq 0 \;\forall \xi \in \Xi,$ find $\bar{p}^\diamond$!

Solution Procedures

?obvious strategies?

- I. iterate on $p^\diamond$, calculate $s(p^\diamond)$,
  - i.e., for each i-agent solve the stochastic program
  - adjust $p^\diamond$, e.g., depending on sign of $s_l(p^\diamond)$ - l-good
- II. reduction to a Variational Inequality
  - or finding a fixed point of an inclusion (set-valued)
  - opt. conditions for each i-agent & market clearing
- III. "complete" the market (generate contingencies prices)
  - $p^\diamond = \big(p^0, \, (p^1_\xi, \xi \in \Xi)\big)$ to be computed for each ξ

Deterministic 1-stage version

1970+ pure exchange +++ finding a fixed point

Walras' law: $\bar{p} \perp s(\bar{p}) \sim \bar{p}_l s_l(\bar{p}) = 0,\ l = 1,\dots,L,\qquad s(p) = s(\alpha p)$ for $\alpha > 0$

scaling: $\bar{p} \in \Delta = \left\{ p \in \mathbb{R}^L_+ \,\middle|\, \sum_l p_l = 1 \right\}$ since $\forall\, \alpha > 0 : \langle \alpha p, x \rangle \le \langle \alpha p, e_i \rangle$

find $\bar{p}\ (\in \Delta)$ such that $\quad 0 \le \bar{p} \perp s(\bar{p}) \ge 0$

one possible way
(**not ideal**, indeterminacy)

0. (very) special instances: via convex programming (*monotonicity*)

I. 1. tâtonnement: $\dot{p} = -s(p),\ p(0) = p^0$, when s differentiable, $e_i \in \operatorname{int} X_i$

Smale's variant: $\nabla s(p)\dot{p} = \lambda s(p)$, $\operatorname{sgn}(\lambda) = (-1)^L \operatorname{sgn} \det\big(\nabla s(p)\big)$

2. finding $\bar{p}$ as a fixed point of $s(p) - p$ in Δ

II simplicial methods: '73 Scarf-Hansen, piece-wise linear homotopy

3. solving a corresponding variational inequality (smoothing, PATH, ...)

3_b. homotopy continuation method(s): ' $\cdots \Rightarrow$ 11 Dang-Ye

III 4. '01- ... finding a maxinf point of a bifunction (more later)

active Market:

recall: $s(p) = \sum_{i \in I} \left(e_i - x_i(p) \right)$, market clearing: $s(p) \geq 0$

Walrasian: $W(p,g) = \langle g, s(p) \rangle$, $W : \Delta \times \Delta \to \mathbb{R}$ (a bifunction)

= Walrasian auctioneer

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Key observation:

$\bar{p} \in \text{maxinf}\, W$, $W(\bar{p}, \bullet) \geq 0$ on $\Delta \Rightarrow \bar{p}$ is an equilibrium point.

under insatiability, $\bar{p}$ an equilibrium $\Leftrightarrow \bar{p} \in \text{maxinf}\, W$, $W(\bar{p}, \bullet) \geq 0$ on Δ

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Moreover:

p_ε : ε-equilibrium point if $\forall l$ (good), $s_l(p_\varepsilon) \geq -\varepsilon$

$p_\varepsilon \in \varepsilon$-maxinf W, $W(p_\varepsilon,\bullet) \geq -\varepsilon$ on $\Delta \Rightarrow p_\varepsilon$ is an ε-equilibrium point.

with insat., p_ε an ε-equilibrium $\Rightarrow p_\varepsilon \in \varepsilon$-maxinf W, $W(p_\varepsilon,\bullet) \geq -\varepsilon$ on Δ

Stochastic version: hurdles

- I. iterate on $p^\diamond$, adjust based on $s(p^\diamond)$
  - works only under serious (undesirable) restrictions
- II simplicial-methods: desperately slow, ➔ full sol'n
  - full (stochastic) V.I. : humongous VI (stalls)
- III relies heavily on "excellent optimization software"
  - obtains rapidly approximate solutions

flashback

1983

IIASA - Laxenburg, Austria

Hedy Attouch – Jean-Pierre Aubin

Aubin's ineluctable linking

Equilibria: Mechanics
Physics, Economics, Transportation

find Fixed Points
solving V.I.

Coop. & Non-Cooperative Games
Nash equilibrium

Aubin's ineluctable linking

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finding MaxInf
of bifunctions
Ky Fan inequality

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Convergence of saddle points: epi/hypo-convergence

Convergence of MaxInf points: lopsided-convergence
(with Hedy Attouch)

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resurrected 2001: convergence of maxinf/equilibrium points

Convergence of MaxInf points: lopsided-convergence
(with Hedy Attouch)

Lopsided convergence of bifunctions

with Attouch, Jofré, P.Q. Khanh, Royset '83-'16

$\{C, C^\nu \subset \mathbb{R}^n\}, \{D, D^\nu \subset \mathbb{R}^m\} \quad K: C \times D \to \mathbb{R}, \; K^\nu: C^\nu \times D^\nu \to \mathbb{R}$

$K^\nu \to_{\text{lop}} K$ (lopsided convergence) if

(a) $\forall \left(y \in D, (x^\nu \in C^\nu) \to x \in C \right)$,

$\limsup_\nu K^\nu(x^\nu, y^\nu) \leq K(x, y)$ for some $(y^\nu \in D^\nu) \to y \in D$

(b) $\forall\, x \in C, \exists (x^\nu \in C^\nu) \to x$ such that for any $(y^\nu \in D^\nu) \to y$

$\liminf_\nu K^\nu(x^\nu, y^\nu) \leq K(x, y)$ when $y \in D, \; K^\nu(x^\nu, y^\nu) \to \infty \;\; y \notin D$

Lopsided convergence of bifunctions

with Attouch, Jofré, P.Q. Khanh, Royset '83-'16

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ancillary tight

$K^\nu_{C^\nu \times D^\nu} \to_{lop.} K_{C \times D}$ & $C^\nu \to C,\, D^\nu \to D$ compact, as $\varepsilon_\nu \searrow 0$

$\bar{x} \in$ cluster-pts $\{x^\nu \in \varepsilon_\nu\text{-}\operatorname{maxinf} K^\nu_{C^\nu \times D^\nu}\}_{\nu \in \mathbb{N}} \Rightarrow \bar{x} \in \operatorname{maxinf} K_{C \times D}$

Lop-convergence of Walrasians

$W(p,g) = \langle g, s(p) \rangle$ on $\Delta \times \Delta$, p-usc and q-convex

Augmented Walrasian: σ augmenting function

$$\tilde{W}_r(p,g) = \inf_v \left\{ W(p, g-v) + r *_e \sigma^*(v) \right\} \qquad r\sigma(r^{-1}v)$$

$$= \sup_v \left\{ W(p,v) \,\middle|\, \|v-g\|_\square \le r \right\} \qquad \sigma = |\bullet|_\square,\ \iota_{\mathbb{B}} = \sigma^*$$

as $r \to \bar{r} < \infty$, $\tilde{W}_r \to_{\text{lop}} \tilde{W}_{\bar{r}} \simeq W \Rightarrow$ ε-maxinf $W_r \to$ maxinf W as $\varepsilon \searrow 0$

choosing $|\bullet|_\square = |\bullet|_\infty, \mathbb{B} = [-1,1]^L$

or $|\bullet|_\square = |\bullet|_2, \mathbb{B} =$ euclidean unit ball

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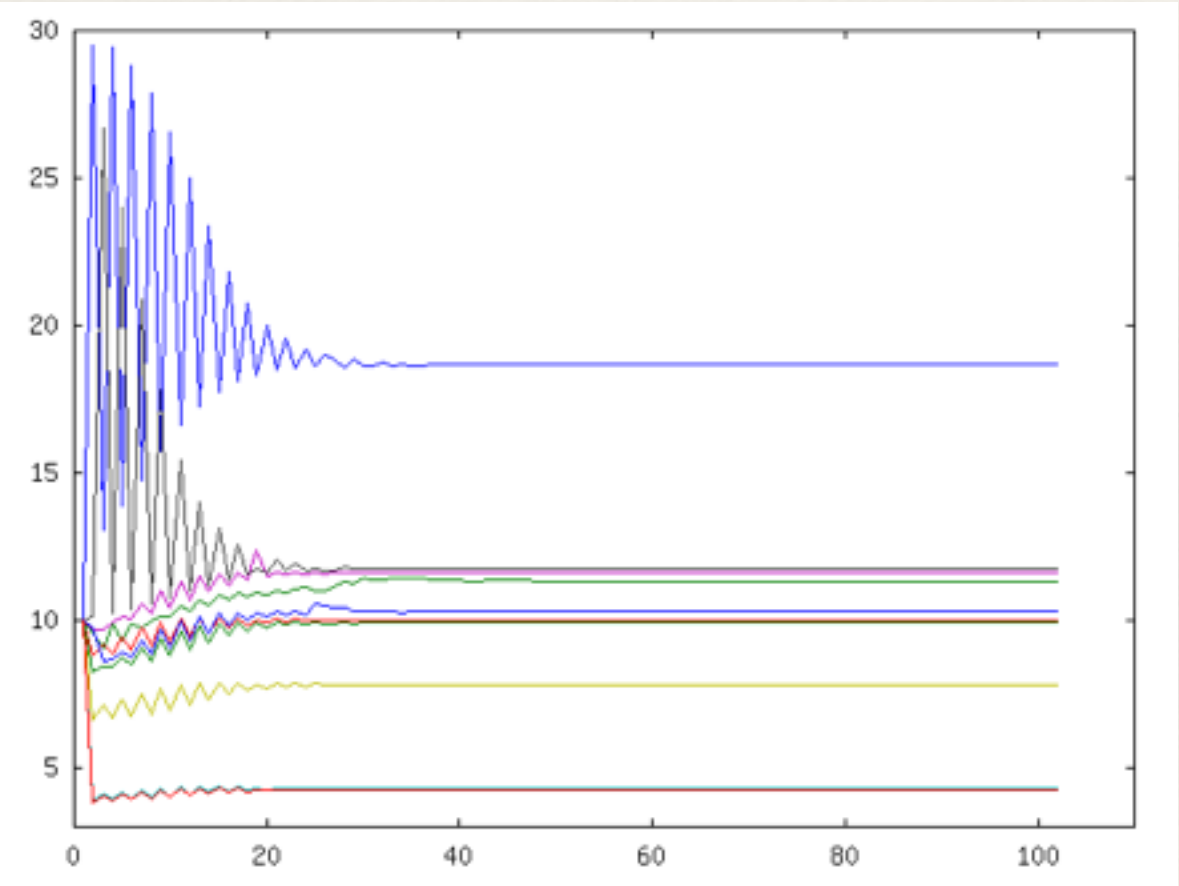
first experiment: 10 agents, 150 goods (two blinks)

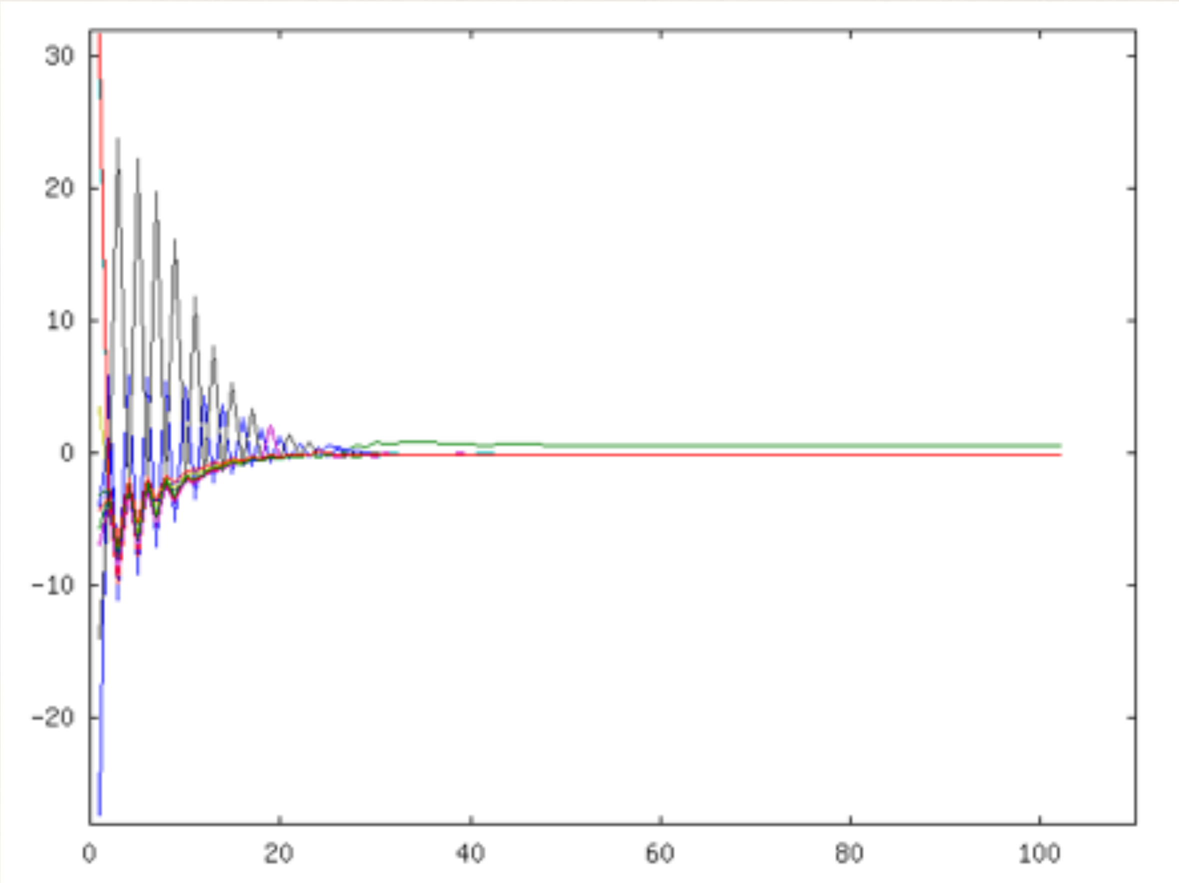
Bagh, Lucero & Wets '03 at UCDavis

Scarf's example

$$u_i(x) = \left(\sum_{l=1}^{L} (a_{il})^{\beta_i^{-1}} (x_l)^{1-\beta_i^{-1}} \right)^{\beta_i(\beta_i-1)^{-1}}$$ CES-utility constant elasticity substitution

$i \in I = 5$ agents, $L = 10$ goods (2000 simplicial pivots)



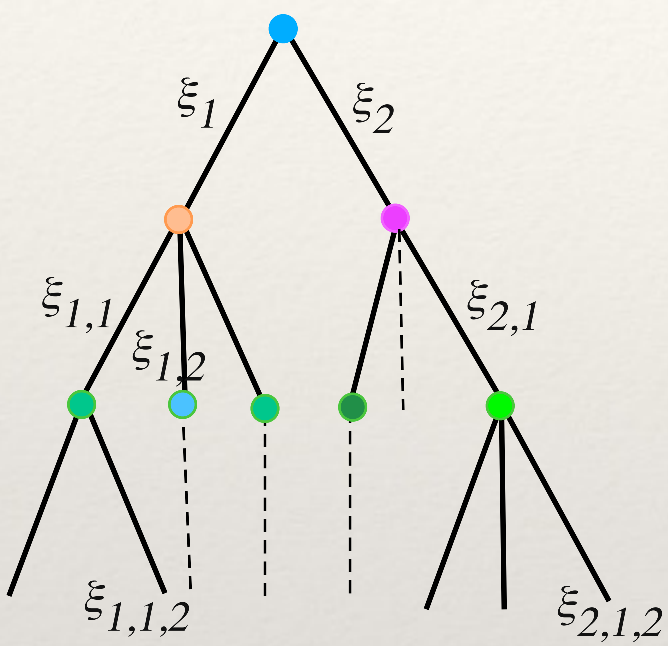


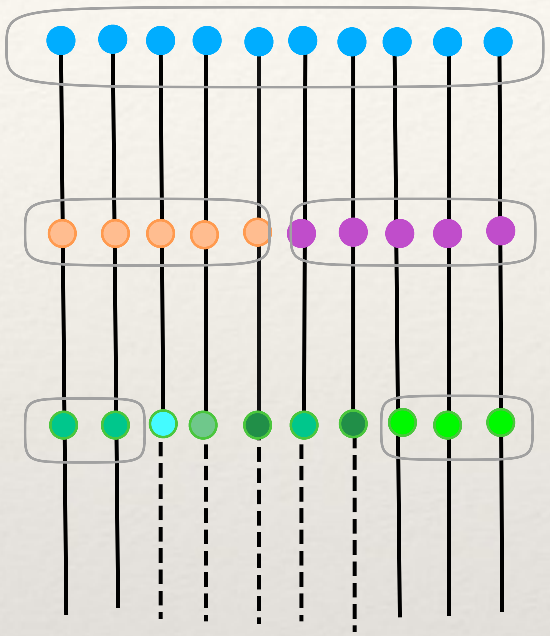
prices and excess supply convergences

Stochastic Environment

Stochastic Environment

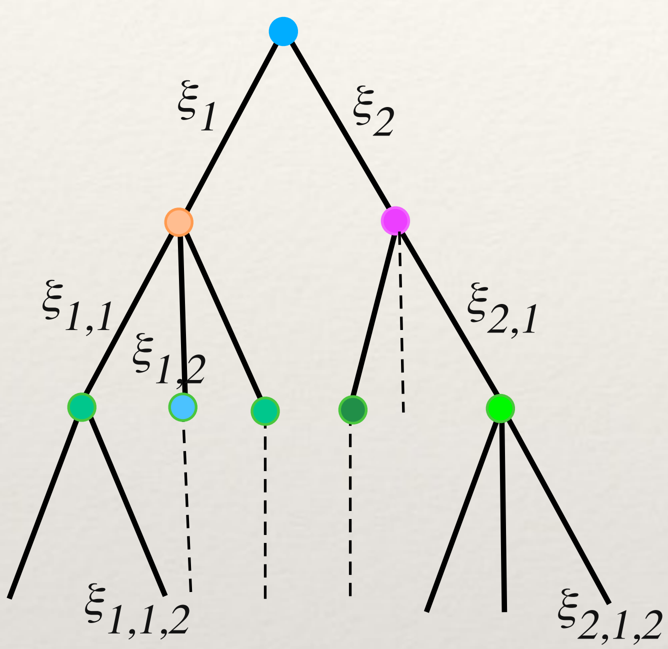
...remember

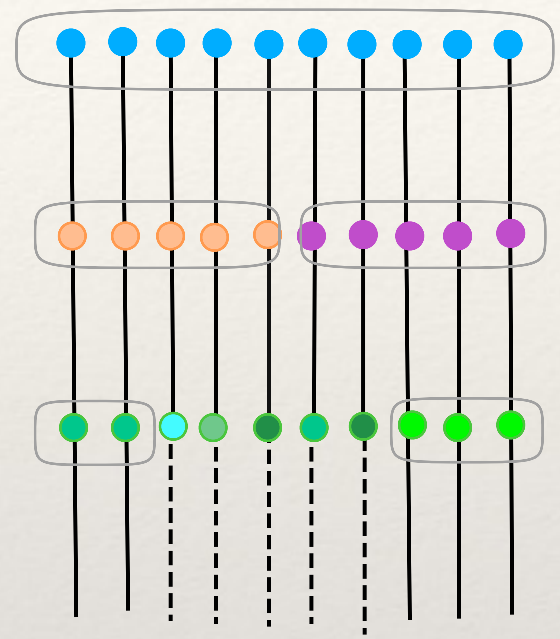




potential decomposition

...remember





potential decomposition

Complete market: all ξ-events can be dealt with separately

would yield $\left(p^0, (p^1_\xi, \xi \in \Xi)\right)$

requires all contingencies available at time 0!

A minimal model

with Julio Deride 2006 … 2016

$$\max_{(x^0, y, x^1)} u_i^0(x^0) + \mathbb{E}\{u_i^1(x_\xi^1)\}$$

i-agent with "home production"

z activities don't affect utility

$$\langle p^0, x^0 + T_i^0 z \rangle \le \langle p^0, e_i^0 \rangle,\ x^0 \in C_i^0,\ y \in Y$$

$$\forall \xi \in \Xi :: \langle p_{\xi,i}^1, x_\xi^1 \rangle \le \langle p^1, e_{\xi,i}^1 + T_{\xi,i}^1 z \rangle,\ x^1 \in C_{\xi,i}^1$$

$$p^\lozenge = \left(p^0, (p_\xi^1, \xi \in \Xi)\right), \quad g^\lozenge = \left(g^0, (g_\xi^1, \xi \in \Xi)\right)$$

excess supply: $s^0(p^\lozenge) = \sum_{i \in I}\left(e_i^0 - \left(x_i^0(p^\lozenge) + T_i^0 z_i(p^\lozenge)\right)\right)$

$$\forall \xi \in \Xi \ :: \ s^1(p^\lozenge) = \sum_{i \in I}\left(\left(e_i^1 + T_{\xi,i}^1 z_i(p^\lozenge)\right) - x_{\xi,i}{}^1(p^\lozenge)\right)$$

equilibrium: $\bar{p}^\lozenge \in \Delta_L \times \Delta_L^{|\Xi|} :: s^0(\bar{p}^\lozenge) \ge 0, s^1(\bar{p}^\lozenge) \ge 0 \ \ (=)$

Walrasian: $W(p^\lozenge, g^\lozenge) = \left\langle (g^\lozenge, \left(s^0(p^\lozenge), s^1(p^\lozenge)\right)\right\rangle$ on $\Delta_L^{1+|\Xi|} \times \Delta_L^{1+|\Xi|}$

$\Rightarrow$ augmented Walrasian, ...

Dealing with future contingencies

$$\max E\left\{ f(\xi; z^0, z^1_\xi) \right\}$$

$$z^0 \in C^0 \subset \mathbb{R}^{n_1},$$

$$z^1_\xi \in C^1_\xi(z^0), \forall \xi.$$

Dealing with future contingencies

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$$z^0_\xi = E\{z^0_\xi\} \; \forall \xi$$

Dealing with future contingencies

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$$z^0_\xi = E\{z^0_\xi\} \ \forall \xi$$
$$w_\xi \perp \ \mathrm{c}^{\mathrm{ste}} \ \text{fcns}$$
$$\Rightarrow E\{w_\xi\} = 0$$

Dealing with future contingencies

$$\max E\left\{ f(\xi; z^0_\xi, z^1_\xi) - \left\langle \bar{w}_\xi, z^0_\xi \right\rangle \right\}$$

$$z^0_\xi \in C^0, \; z^1_\xi \in C^1_\xi(z^0_\xi)$$

$$\forall \xi \in \Xi :$$

$$\max f(\xi; z^0, z^1) - \left\langle \bar{w}_\xi, z^0 \right\rangle$$

$$\max E\left\{ f(\xi; z^0_\xi, z^1_\xi) \right\}$$

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$$w_\xi \perp \text{c}^{\text{ste}} \;\; \text{fcns}$$

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Dealing with future contingencies

$$\max E\left\{ f(\xi; z_\xi^0, z_\xi^1) - \left\langle \bar{w}_\xi, z_\xi^0 \right\rangle \right\}$$

$$z_\xi^0 \in C^0,\ z_\xi^1 \in C_\xi^1(z_\xi^0)$$

$$\forall \xi \in \Xi :$$

$$\max f(\xi; z^0, z^1) - \left\langle \bar{w}_\xi, z^0 \right\rangle$$

$$\max E\left\{ f(\xi; z_\xi^0, z_\xi^1) \right\}$$

$$z_\xi^0 \in C^0 \subset \mathbb{R}^{n_1},$$

$$z_\xi^1 \in C_\xi^1(z^0),\ \forall \xi.$$

$$z_\xi^0 = E\{z_\xi^0\}\ \forall \xi$$

$$w_\xi \perp \text{ c}^{\text{ste}} \text{ fcns}$$

$$\Rightarrow E\{w_\xi\} = 0$$

finding $\bar{w}_\xi$: Progressive Hedging algorithm

essentially $\mathrm{w}^\nu \to w^{opt}$ updating scheme

Disintegrating the equilibrium problem

with $p^{\Diamond} = \left(p^0, \left\{p_\xi^1\right\}_{\xi\in\Xi}\right)$

prox-term

$$\hat{u}_i^0(x^0) = u_i^0(x^0) - \left\langle \hat{w}_{i,\xi}, (x^0, z)\right\rangle - \frac{\rho}{2}\left\|(x^0, z) - (\tilde{x}^0, \tilde{z})\right\|^2$$

$$\left(\hat{x}_{i,\xi}^0, \hat{z}_{i,\xi}^0, \hat{x}_{i,\xi}^1\right) \in \arg\max\left\{\hat{u}_i^0(x^0) + u_i^1(\xi; x^1)\right\}$$

$$x^0 \in C_i^0,\ x^1 \in C_{\xi,i}^1 \text{ \& budgetary constraints.}$$

solved for each ξ **separately**, for all i

$$\Rightarrow \left(\hat{s}^0(p^{\Diamond}), \hat{s}^1(p^{\Diamond})\right) \,\&\, W(p^{\Diamond}, g^{\Diamond})$$

via augmentation $\rightarrow\ p_{new}^{\Diamond}$

exploits *i*-agent market completion

with $p^\Diamond = \left(p^0, \{p^1_\xi\}_{\xi\in\Xi}\right)$

prox-term

$$\hat{u}^0_i(x^0) = u^0_i(x^0) - \left\langle \hat{w}_{i,\xi}, (x^0, z)\right\rangle - \frac{\rho}{2}\left\|(x^0, z) - (\tilde{x}^0, \tilde{z})\right\|^2$$

$$\left(\hat{x}^0_{i,\xi}, \hat{z}^0_{i,\xi}, \hat{x}^1_{i,\xi}\right) \in \arg\max\left\{\hat{u}^0_i(x^0) + u^1_i(\xi; x^1)\right\}$$

$x^0 \in C^0_i$, $x^1 \in C^1_{\xi,i}$ & budgetary constraints.

solved for each ξ **<u>separately</u>**, for all i

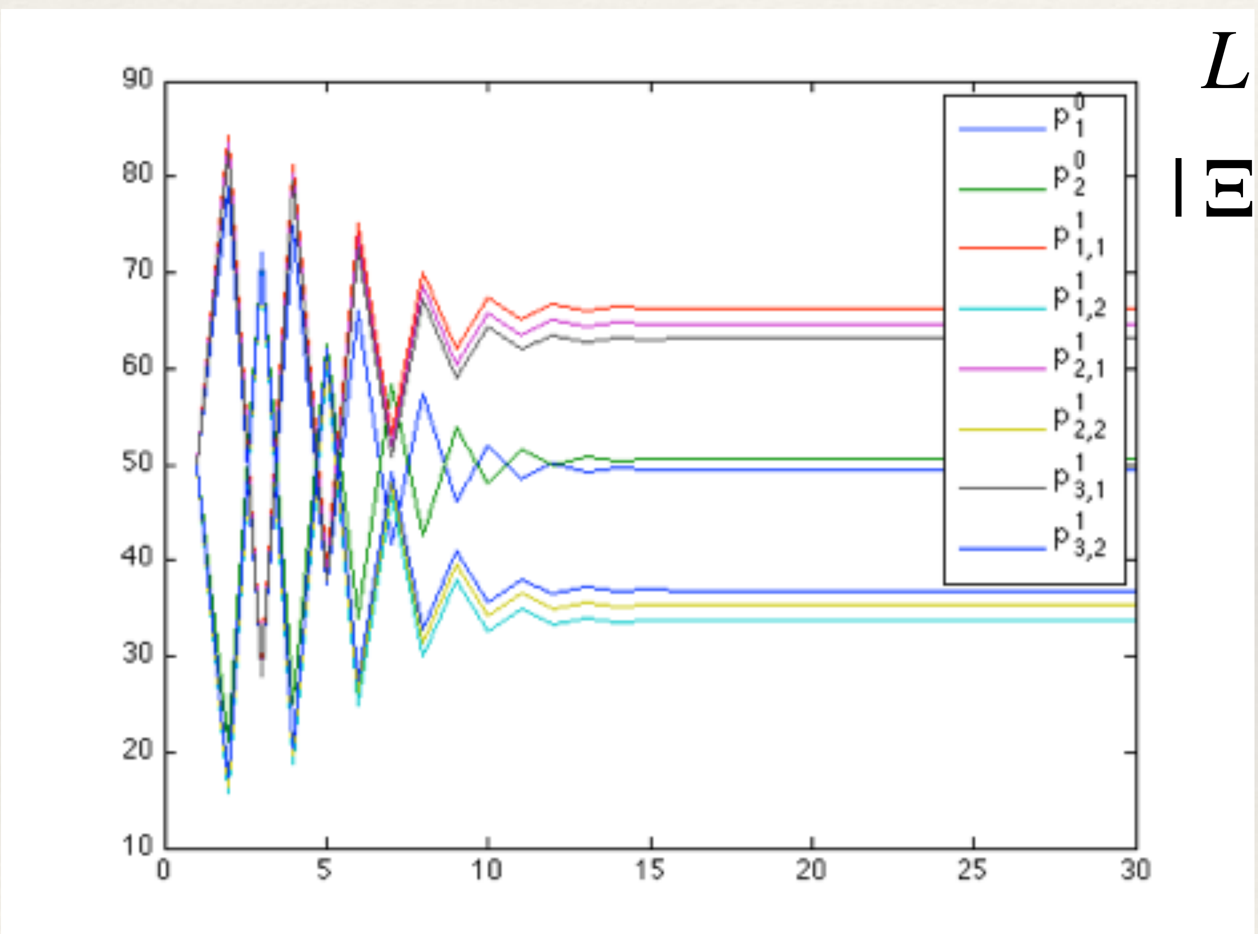
$$\Rightarrow \left(\hat{s}^0(p^\Diamond), \hat{s}^1(p^\Diamond)\right) \;\&\; W(p^\Diamond, g^\Diamond)$$

via augmentation $\rightarrow$ $p^\Diamond_{new}$

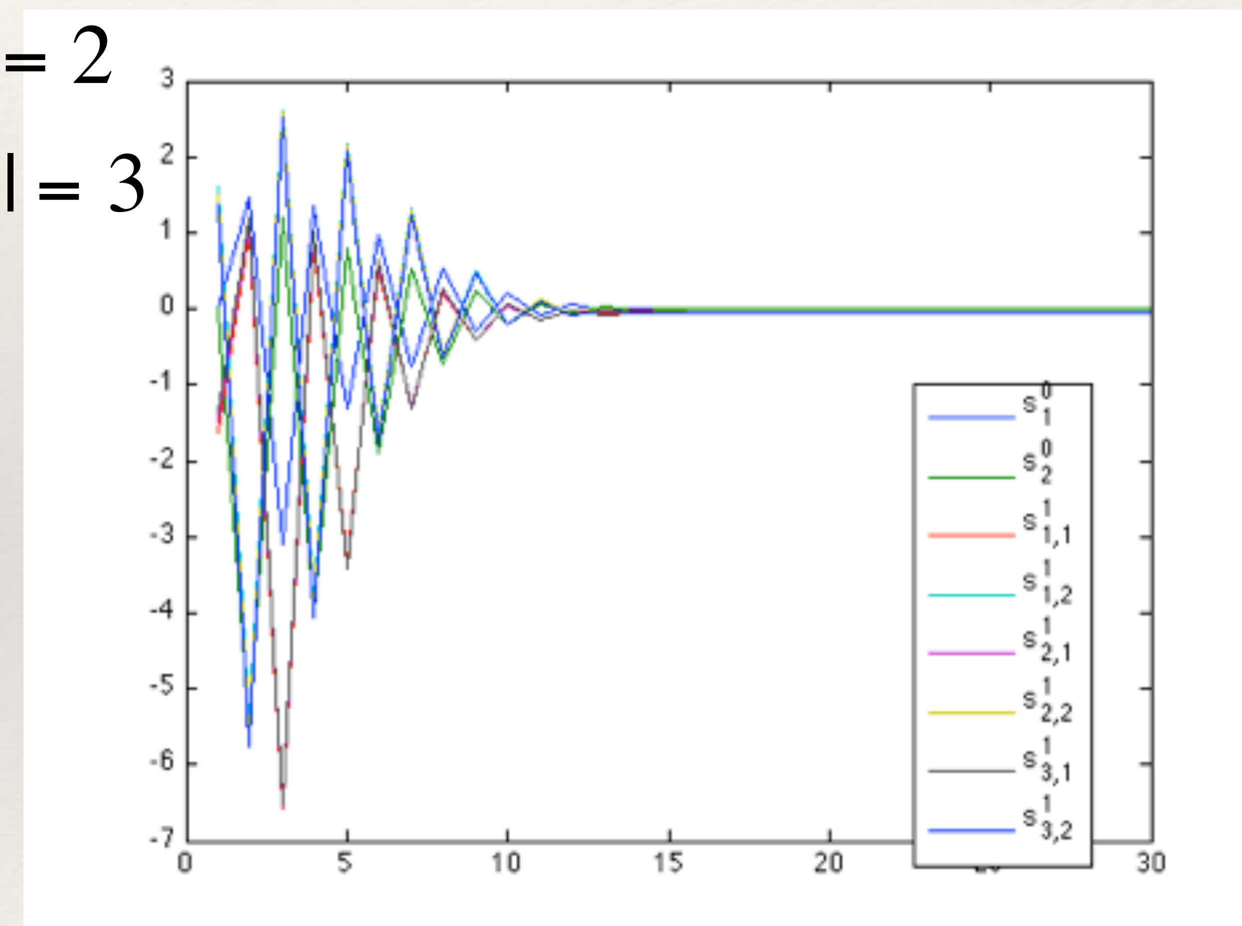
Convergence: exploiting separability

$L = 2$

$|\Xi| = 3$



prices

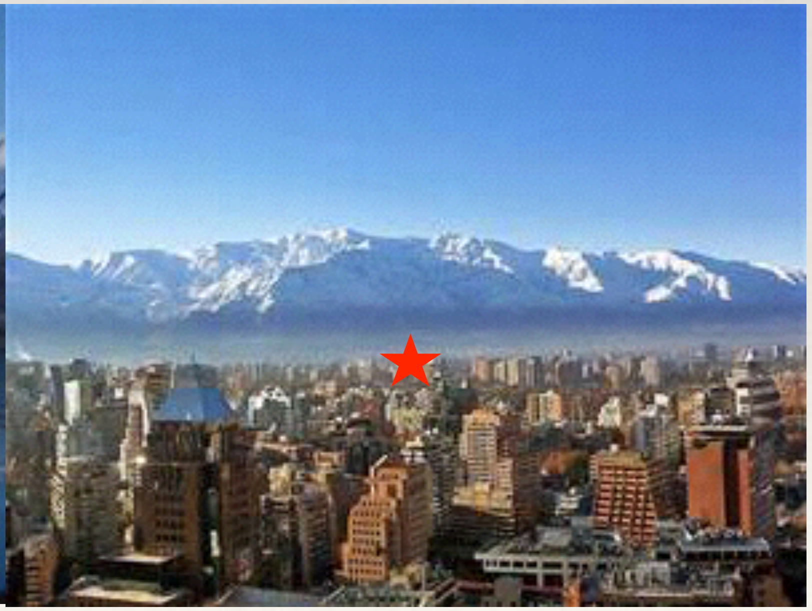


excess supply

2005 – 2016

Chile, Davis, Whidbey Island, Vancouver, Bonn, …

+ R.T. Rockafellar & A. Jofré

"Classical" GEI-model

with monetary market: $k \in K$, contracts (z^+, z^-)

$$\max_{(x^0, y, x^1)} u_i\left(x^0, (x^1_\xi, \xi \in \Xi)\right) \quad i\text{-agent integrated utility}$$

$$\left\langle p^0, x^0 + D^0(p^0)y^- \right\rangle + \left\langle q, z^+ \right\rangle \le \left\langle p^0, e^0_i \right\rangle + \left\langle q, y^- \right\rangle, \; x^0 \in C^0_i,$$

$$\forall \xi \in \Xi :: \left\langle p^1_\xi, x^1_\xi + D^1(p^1_\xi)y^- \right\rangle \le \left\langle p^1, e^1_{\xi,i} + D^1(p^1_\xi)y^+ \right\rangle, \; x^1 \in C^1_{\xi,i}$$

Clearing the market: $s^0(p^\Diamond, q) \ge 0, \; s^1(p^\Diamond, q) \ge 0 \;\; \& \; q :: \sum_{i \in I} (y^+_i + y^-_i) = 0$

$$p^\Diamond = \left(p^0, (p^1_\xi, \xi \in \Xi)\right), \quad q \in \mathbb{R}^K_+$$

excess supply: $s^0(p^\Diamond, q) = \sum_{i \in I}\left(e^0_i - \left(x^0_i(p^\Diamond, q) + D^0(p^0)y^-_i(p^\Diamond, q)\right)\right)$

$\forall \xi \in \Xi :: \; s^1(p^\Diamond, q) = \sum_{i \in I}\left(\left(e^1_{\xi,i} + D^1(p^1_\xi)y^+_i(p^\Diamond, q)\right) - \left(x_{\xi,i}{}^1(p^\Diamond, q) + D^1(p^1_\xi)y^-(p^\Diamond, q)\right)\right)$

equilibrium: $\bar{p}^\Diamond \in \Delta_L \times \Delta_L^{|\Xi|} :: s^0(\bar{p}^\Diamond, \bar{q}) \ge 0, \; s^1(\bar{p}^\Diamond, \bar{q}) \ge 0 \;\; (=), \; \bar{y}^+(\bar{p}^\Diamond, \bar{q}) = y^-(\bar{p}^\Diamond, \bar{q})$

Walrasian: $W((p^\Diamond, q), g^\Diamond) = \left\langle g^\Diamond, \left(s^0(p^\Diamond, q), s^1(p^\Diamond, q)\right)\right\rangle$ - $\left[y^+(p^\Diamond, q) - y^-(p^\Diamond, q)\right]^2$ on $\left(\Delta_L^{1+|\Xi|} \times \mathbb{R}^K_+\right) \times \Delta_L^{1+|\Xi|}$

$\Rightarrow$ augmented Walrasian ???

“Classical” model features

- just present & (elusive) future (shoud be ∞-horizon?)
- ★ dawn and *doomsday* framework (no value-utility after time 1)
- ★ universal-endowments: $\in \mathbb{R}^L_{++}$ for all agents, all ξ
- due to the use of differentiable topology methodology
- doesn’t allow for boundary choices (good with no appeal)
- ★ *retention* without value-utility; would remove “doomsday”
- retention would allow to handle money as a generalized good
- plays a vital role in financial markets (& Keynesian behavior)
- *generic* proof of the existence of an equilibrium, not constructive
- ★ *indeterminacy*: no relationship between prices at times 0 & 1

JRW : GEI-model

with monetary market: $k \in K$, contracts (z^+, z^-)

$x = (c, w)$ consumption, retention

good 0 $\approx$ currency, $0 \leq p^\Diamond,\ p_0^\Diamond = (1,1,\ldots,1)$

$\Rightarrow$ goods priced in money, ~~indeterminacy~~

$w^0 \rightarrow A_{\xi,i} w^0$ (retention at time 0 $\rightarrow$ at time 1)

$$p^\Diamond = \left(p^0, (p_\xi^1, \xi \in \Xi)\right), \quad q \in \mathbb{R}_+^K$$

$$\max_{(x^0, y, x^1)} u_i\left(x^0, (x_\xi^1, \xi \in \Xi)\right)$$

$$\left\langle p^0, c^0 + w^0 + D^0(p^0) y^- \right\rangle + \left\langle q, y^+ \right\rangle \leq \left\langle p^0, e_i^0 \right\rangle + \left\langle q, y^- \right\rangle,\ x^0 \in C_i^0,$$

$$\forall \xi \in \Xi :: \left\langle p_\xi^1, c_\xi^1 + w_\xi^1 + D^1(p_\xi^1) y^- \right\rangle \leq \left\langle p^1, e_{\xi,i}^1 + A_{\xi,i} w^0 + D^1(p_\xi^1) y^+ \right\rangle,\ x^1 \in C_{\xi,i}^1$$

Clearing the market: $s^0(p^\Diamond, q) \geq 0,\ s^1(p^\Diamond, q) \geq 0\ \&\ q :: y^+(p^\Diamond, q) + y^-(p^\Diamond, q) = 0$

Existence of equilibrium : good 0 universality attractive
no universal-endowments, boundary cold. not exclude
technique: Variational Analysis (not generic)

2016 Solution Strategy

implementation with Julio Deride, UCD

no arbitrage: (related to "Cass trick" for nominal assets)

$$q = \sum_{\xi\in\Xi} \sigma_\xi \left((D^1_\xi)^T p^1_\xi\right),\ \exists \sigma_\xi > 0, \qquad \text{then set } \tilde p^1_\xi = \sigma_\xi p^1_\xi \ \ \forall \xi$$

$$\max_{(x^0, y, x^1)} u_i\left(x^0, (x^1_\xi, \xi \in \Xi)\right)$$

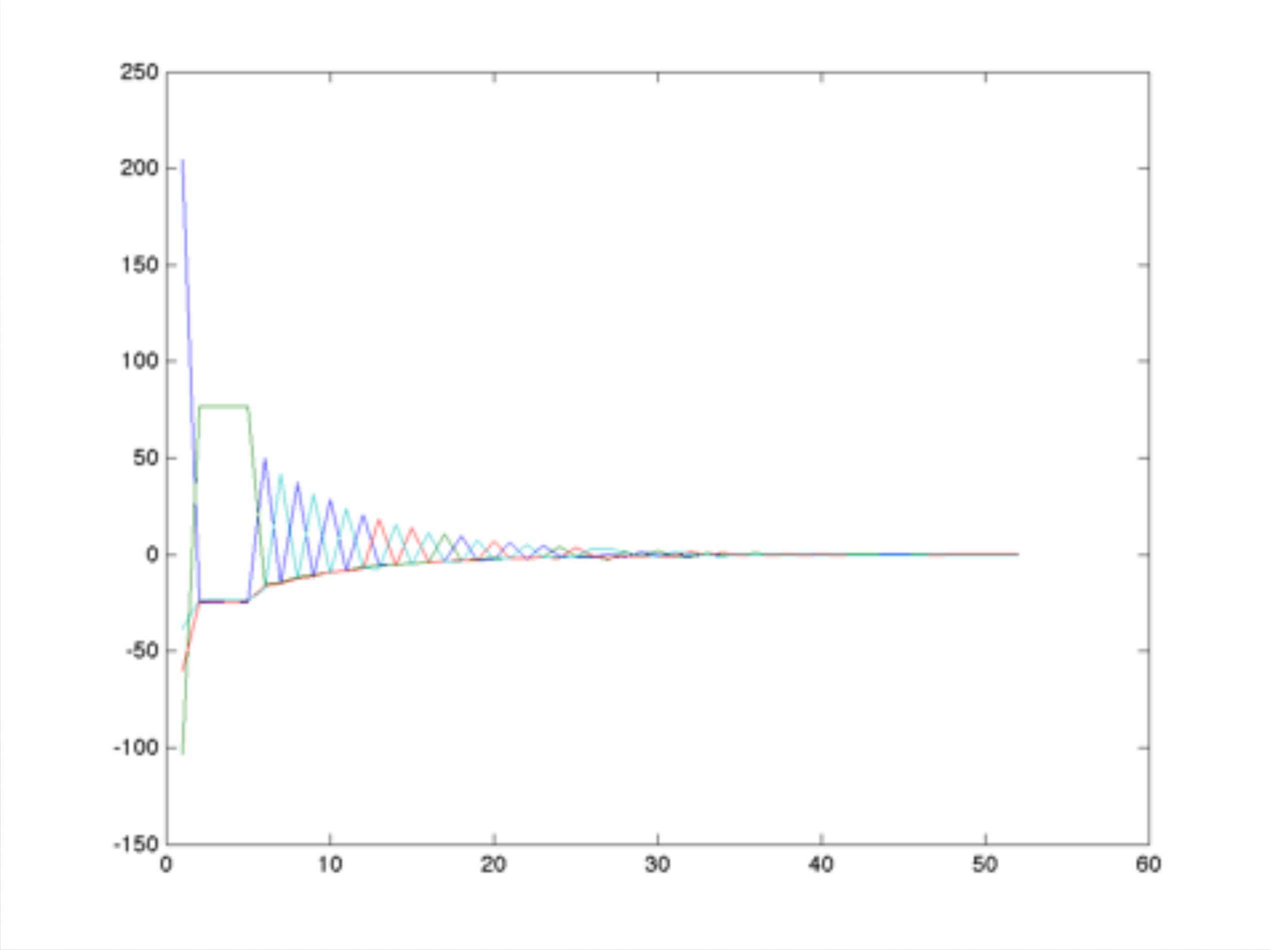
$$\left\langle p^0, c^0 + w^0 + D^0 y^- \right\rangle + \left\langle \sum_{\xi\in\Xi} \tilde p^1_\xi D^1_\xi, y^+ \right\rangle \le \left\langle p^0, e^0_i \right\rangle + \left\langle \sum_{\xi\in\Xi} \tilde p^1_\xi D^1_\xi, y^- \right\rangle,\ c^0 \in C^0_i$$

$$\forall \xi \in \Xi :: \left\langle \tilde p^1_\xi, c^1_\xi + w^1_\xi + D^1(\tilde p^1_\xi) y^- \right\rangle \le \left\langle p^1, e^1_{\xi,i} + A_{\xi,i} w^0 + D^1(\tilde p^1_\xi) y^+ \right\rangle,\ c^1 \in C^1_{\xi,i}$$

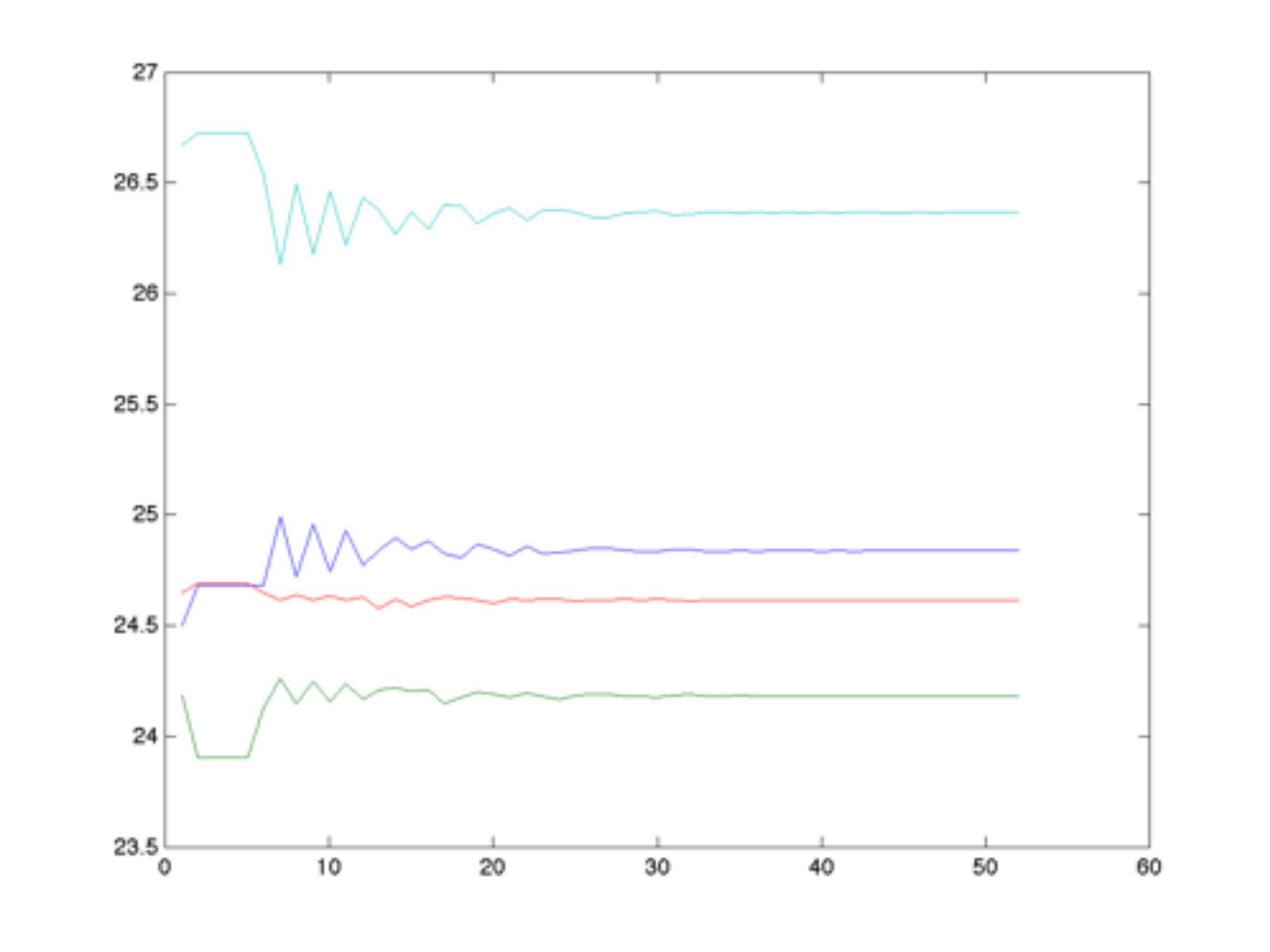
Clearing the market: $s^0(\tilde p^\lozenge) \ge 0,\ s^1(\tilde p^\lozenge) \ge 0$ ~~& q ::~~ $y^+ + y^- = 0$

Walrasian: $W(\tilde p^\lozenge, \tilde g^\lozenge) = \left\langle \tilde g^\lozenge, (s^0(\tilde p^\lozenge), s^1(\tilde p^\lozenge)) \right\rangle - \left[y^+(\tilde p^\lozenge) - y^-(\tilde p^\lozenge) \right]^2$ on $\Delta_L^{1+|\Xi|} \times \Delta_L^{1+|\Xi|}$

Hens-Pilgrim Examples



Convergence of y_k



Convergence of q_k

2005 | | 2008, …

Stanford (Complementarity & Extensions), SierraNevada, Davis, Madison

+ Michael Ferris (& PATH-solver, EMP, …)

★

Disaggregation!

$$\hat{u}_i^{w_{\xi.i}}(x^0,x^1) = u_i^0(\xi,x^0,x^1) - \langle w_{\xi.i},(x^0,y)\rangle - \frac{\rho}{2}\left\|(x^0,y)-(\hat{x}^0,\hat{y})\right\|^2$$

initialization: $\forall \mathrm{i} \in I$

$w_{\xi.i}$ such that $E\{w_{\xi.i}\} = 0$

$\forall \xi \Rightarrow$ equilibrium with $\left\{u_i^{w_{\xi.i}}(\xi;\bullet)\right\}_{i\in I}$

yields: $(\overline{x}^0_{\xi.i},\overline{y}_{\xi.i},\overline{x}^1_{\xi.i}),w^{new}_{\xi.i}, \quad (\overline{p}^0_{\xi.i},\overline{p}^1_{\xi.i})$

agents opt via P.H.

V.I. sol'n via PATH-.

STOP $\forall i \in I : \left(\overline{x}^0_{i,\xi},\overline{y}_{i,\xi}\right) \simeq E\left\{\left(\overline{x}^0_{i,\xi},\overline{y}_{i,\xi}\right)\right\}$

otherwise adjust $\left[w_{i,\xi},\xi\in\Xi\right]_{i\in I}, \ldots$

Disaggregation!

$$\hat{u}_i^{w_{\xi.i}}(x^0,x^1) = u_i^0(\xi,x^0,x^1) - \langle w_{\xi.i},(x^0,y)\rangle - \frac{\rho}{2}\left\|(x^0,y)-(\hat{x}^0,\hat{y})\right\|^2$$

initialization: $\forall\, \mathrm{i} \in I$

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yields: $(\bar{x}^0_{\xi.i},\bar{y}_{\xi.i},\bar{x}^1_{\xi.i}), w^{new}_{\xi.i}, \quad (\bar{p}^0_{\xi.i},\bar{p}^1_{\xi.i})$

agents opt via P.H.

V.I. sol'n via PATH-.

+ prox-term on (p^0,p^1)

STOP $\forall i \in I : \left(\bar{x}^0_{i,\xi},\bar{y}_{i,\xi}\right) \simeq E\left\{\left(\bar{x}^0_{i,\xi},\bar{y}_{i,\xi}\right)\right\}$

otherwise adjust $\left[w_{i,\xi}, \xi \in \Xi\right]_{i\in I}, \ldots$

Disaggregation with PATH Solver

- Economy: (5 agents - 8 goods)
  - Skilled & unskilled workers
  - Businesses: Survival goods & leisure
  - Banker: bonds (riskless), 2 stocks
- 2-stages, solved under # of scenarios (280)
- utilities: CES-functions (gen. Cobb-Douglas)
  - Utility in stage 2 assigned to financial instruments
  - Financial instruments only used for transfer to time 1
- used for calibration (-> deterministic cases)

numerically: `blink' (5000 iterations).

on M. Ferris
semi-slow laptop
using EMP-package
4 min + 2 min for
verification

Disaggregation with PATH Solver

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  - Utility in stage 2 assigned to financial instruments
  - Financial instruments only used for transfer to time 1
- used for calibration (-> deterministic cases)

numerically: `blink' (5000 iterations).

comprehensive V.I.: 800+ variables

Yea! scenario disaggregation, but ...

i-agent: $x_i(p) \in \arg\max \left\{ u_i(x) \,\middle|\, \langle p, x \rangle \le \langle p, e_i \rangle \right\},\ i \in I$

with excess supply $s(p)$: $\quad 0 \le p \perp s(p) \ge 0$

Multi-Optimization with Panoptic Equilibirum Constraint

MOPEC-class ~ maxinf family

Yea! scenario disaggregation, but ...

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Multi-Optimization Problem with Equilibirum Constraint

MOPEC-class ~ maxinf family

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i-agent: $x_i(p) \in \arg\max \left\{ u_i(x) \,\middle|\, \langle p,x \rangle \leq \langle p,e_i \rangle \right\},\ i \in I$

with excess supply $s(p)$: $\quad 0 \leq p \perp s(p) \geq 0$

Multi-Optimization Problem with Equilibirum Constraint

MOPEC-class ~ maxinf family

$$x_i \in \arg\max_{x \in \mathbb{R}^{n_i}} f_i(p,x,x_{-i}),\quad i \in \mathcal{I},\quad x_{\mathcal{I}} = (x_i, i \in \mathcal{I})$$

$$D(p,x_{\mathcal{I}}) \in \partial g(p)\ \left[\text{or } \in N_C(p)\right]$$

with Michael Ferris '11-'?? ... '16?

Examples: Walras, noncooperative games, ... *with Andy Philpott*

stochastic (dynamic): decentralized electricity markets,

joint estimation and optimization, financial equilibrium, ...

The BDE-example

Brown-DeMarzo-Eaves (Econometrica '96)

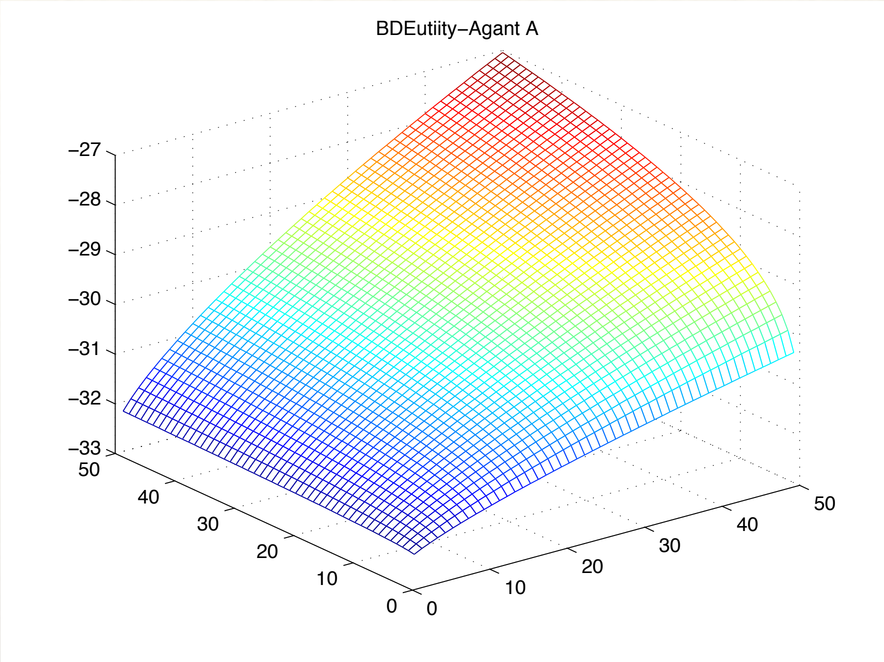
3 agents (2-agent & 3-agent of the same type)

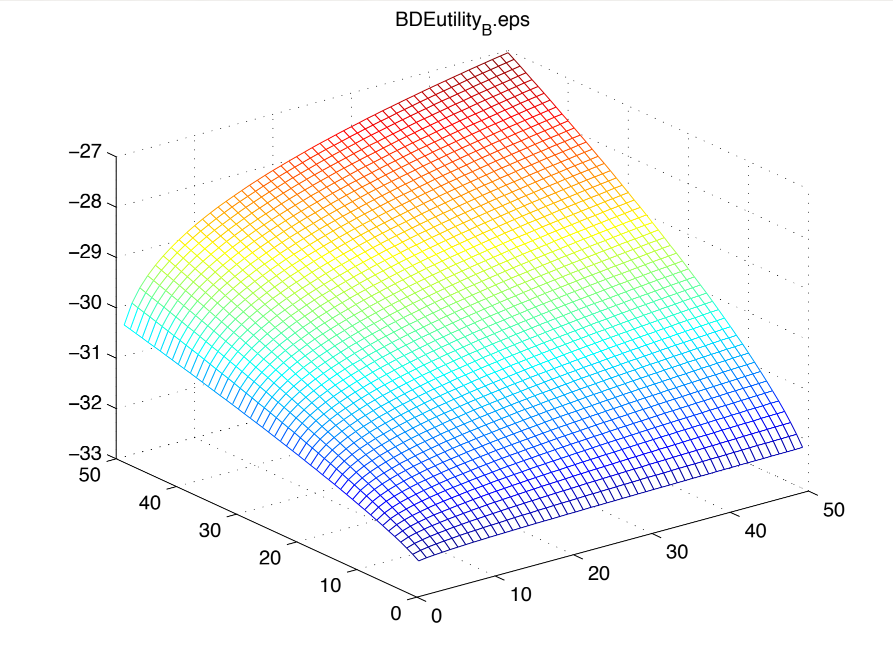
2 goods, $|\Xi| = 3$ (future states), no y-activities

$$u_i^1(\xi; x) = -\left(5.7 - \prod_{l=1}^{2} \left(x_l\right)^{\alpha_{i,l}}\right) = u_1^0(x)$$

$$\alpha_1 = (0.25, 0.75),\ \alpha_{2\&3} = (0.75, 0.25)$$

asset #1: $D_\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, asset #2: $D_\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for all ξ





The BDE-example

3 agents, 2 goods, $|\Xi| = 3$, no y-activities

$$u_i^1(\xi;x) = -\left(5.7 - \prod_{l=1}^{2}(x_l)^{\alpha_{i,l}}\right) = u_1^0(x),\ \alpha_1 = (0.25, 0.75),\ \alpha_{2\&3} = (0.75, 0.25)$$

asset #1: $D_\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$, asset #2: $D_\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$ for all ξ

BDE- solution: $p^0 = (1,\ 0.74)$ (with scaling)

$p_\xi^1 = (1,\ 0.7375;\ 1,\ 0.7174;\ 1,\ 0.6633)$

$q = (??,\ ??)$, $y = (0.94, 0;\ 0.03, 0;\ 0.03, 0)$

change of variables + add unconstrainted agent:

homotopy continuation method (predictor-corrector steps)

The BDE-example

3 agents, 2 goods, $|\Xi| = 3$, no y-activities

$$u_i^1(\xi; x) = -\left(5.7 - \prod_{l=1}^{2}(x_l)^{\alpha_{i,l}}\right) = u_1^0(x),\ \alpha_1 = (0.25, 0.75),\ \alpha_{2\&3} = (0.75, 0.25)$$

asset #1: $D_\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$, asset #2: $D_\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$ for all ξ

BDE- solution: $p^0 = (1,\ 0.74)$ (with scaling)

$p_\xi^1 = (1,\ 0.7375;\ 1,\ 0.7174;\ 1,\ 0.6633)$

$q = (??,\ ??)$, $y = (0.94, 0;\ 0.03, 0;\ 0.03, 0)$ ⟸ all buyers no sellers

change of variables + add unconstrained agent:

homotopy continuation method (predictor-corrector steps)

The BDE-example

3 agents, 2 goods, $|\Xi| = 3$, no y-activities

$$u_i^1(\xi;x) = -\left(5.7 - \prod_{l=1}^{2} (x_l)^{\alpha_{i,l}}\right) = u_1^0(x), \quad \alpha_1 = (0.25, 0.75), \quad \alpha_{2\&3} = (0.75, 0.25)$$

asset #1: $D_\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, asset #2: $D_\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for all ξ

Path Solver solution: $p^0 = (1,\ 0.7338)$ (with scaling & $z\_i \leq 100$)

$p_\xi^1 = (1,\ 0.7158;\ 1,\ 0.7182;\ 1,\ 0.7205)$

$q = (0.9188,\ 0.6600), \quad y = (72.9868, -100; -36.4934, 50;\ -36.4934, 50)$

sol'n time: not noticeable

value transfer for #1-agent: @$t = 0$: -1.0649,

@$t = 1$, scn-1:1.403, scn-2: 1.168, scn-3: 0.933,

2011-2014, ...Hong Kong

+ Xiaojun Chen & Smoothing and ERM

Residual functions → Wardrop Equilibrium

$-F(x) \in N_X(x)$ (deterministic V.I., see Facchinei-Pang)

f residual function if $f \geq 0$ and $f(\bar{x}) = 0 \Rightarrow \bar{x}$ solves V.I.

example: $f(x) =$ dist$(-F(x), N_X(x))$, gap function, ..

$-F(\xi, x, u_\xi) \in N_{X \times U_\xi}(x, u_\xi)$ a.s. (stochastic, see Chen-Wets-Zhang)

f residual function $\Xi \times \mathbb{R}^{n_1+n_2}$ if

(a) prob $\{f(\xi, x, u) \geq 0\} = 1 \;\; \forall (x,u) \in \mathbb{R}^{n_1+n_2}$

(b) $f(\xi, \bar{x}, \bar{u}) = 0 \Rightarrow (\bar{x}, \bar{u})$ solves the ξ-V.I.

Overall goal: minimize $\mathbb{E}\left\{ f(\xi, x, u_\xi) \right\}$

note that (b) does not imply $\inf \mathbb{E}\left\{ f(\xi, x, u_\xi) \right\} = 0$

more about this later

Approximating Stochastic V.I.'s
with same convergence properties

$-F(\xi,x,u_\xi)\in N_{X\times U_\xi}(x,u_\xi)$ P-a.s. to Q-a.s (discretization)

$$K\big(\xi,(x,u_\xi),(y,v_\xi)\big)=\big\langle F\big(\xi,(x,u_\xi)\big),(x,u_\xi)-(y,v_\xi)\big\rangle \text{ on } \big[X\times U_\xi\big]^2$$

gap function: $\varphi\big(\xi,(x,u_\xi)\big)=\sup_{(y,v_\xi)} K\big(\xi,(x,u_\xi),(y,v_\xi)\big)$ on $X\times U_\xi$

$\bar{x}\in\arg\min\mathbb{E}\big\{\varphi\big(\xi,(x,u_\xi)\big)\big\}$ for $x\in X,\ u_\xi\in U_\xi$ P-a.s.

application/computing tractable: substitute $r(\xi,x)$ for u_ξ

$P\to Q$: changes a.s. (size of V.I., fewer or different ξ 's)

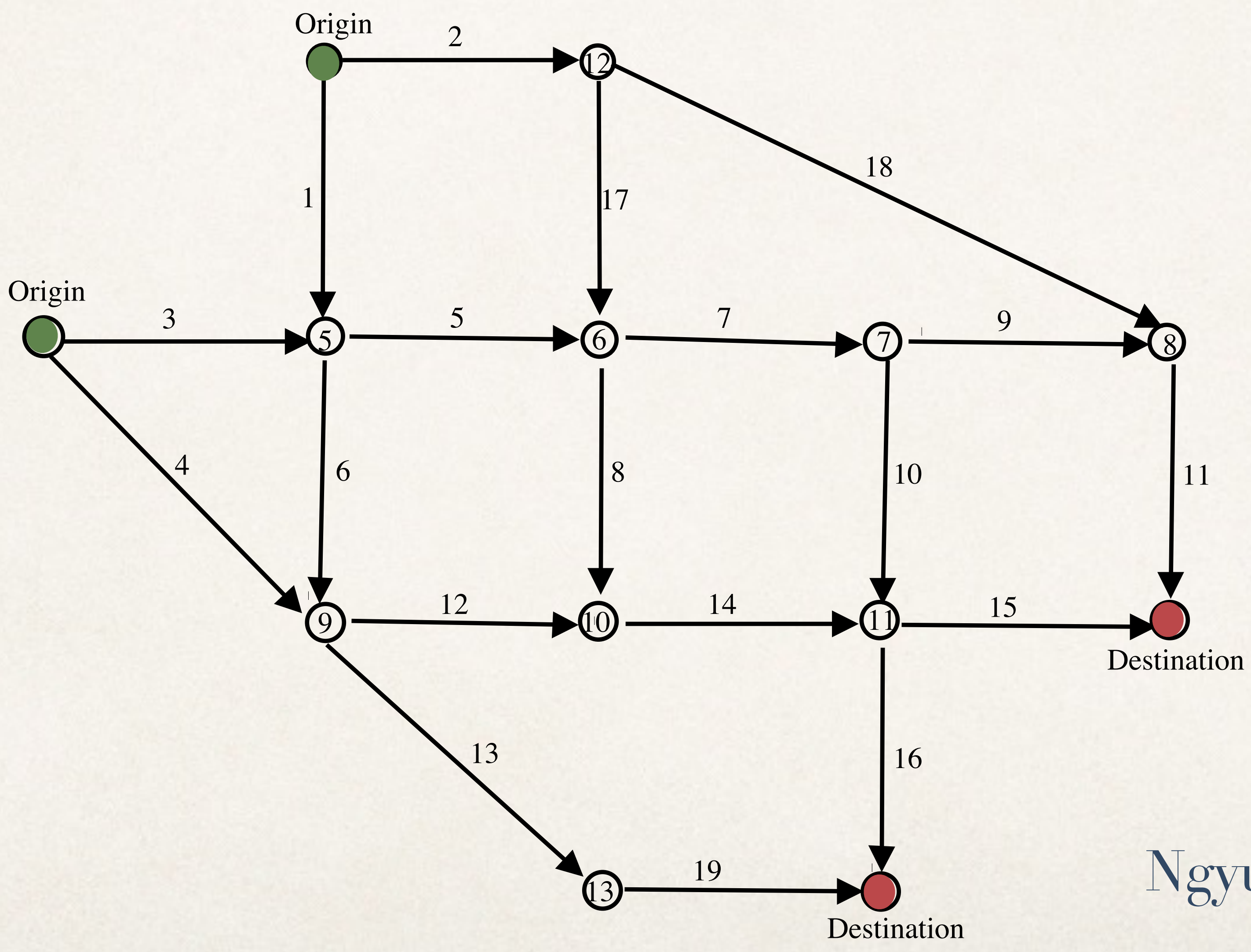
$$K^\nu\big(\xi,(x,u_\xi),(y,v_\xi)\big)=\big\langle F^\nu\big(\xi,(x,u_\xi)\big),(x,u_\xi)-(y,v_\xi)\big\rangle \text{ on } \big[X\times U_\xi^\nu\big]^2$$

gap fcn: $\varphi^\nu\big(\xi,(x,u_\xi)\big)=\sup_{(y,v_\xi)} K^\nu\big(\xi,(x,u_\xi),(y,v_\xi)\big)$ on $X\times U_\xi^\nu$

Network flow: Stochastic version Wardrop Equilibrium

Network with random OD demand: d_ξ

random link capacities c_ξ also affecting travel times $F(\xi, \bullet)$



Ngyuyen-Dupuis net-example

Evaluating
steady x-flow case

network $(\mathcal{N}, \mathcal{A}), |\mathcal{A}| = n$

x "steady" (best) flow, $x - u_\xi$ "daily" flow, $u_\xi \in \mathbb{R}^n$

$x - u_\xi$ must satisfy $A(x - u_\xi) = b_\xi$, $0 \le x - u_\xi \le c_\xi$

$x \in \mathbb{R}^n_+$ and $\forall \xi$, $\exists u_\xi$ satisfying

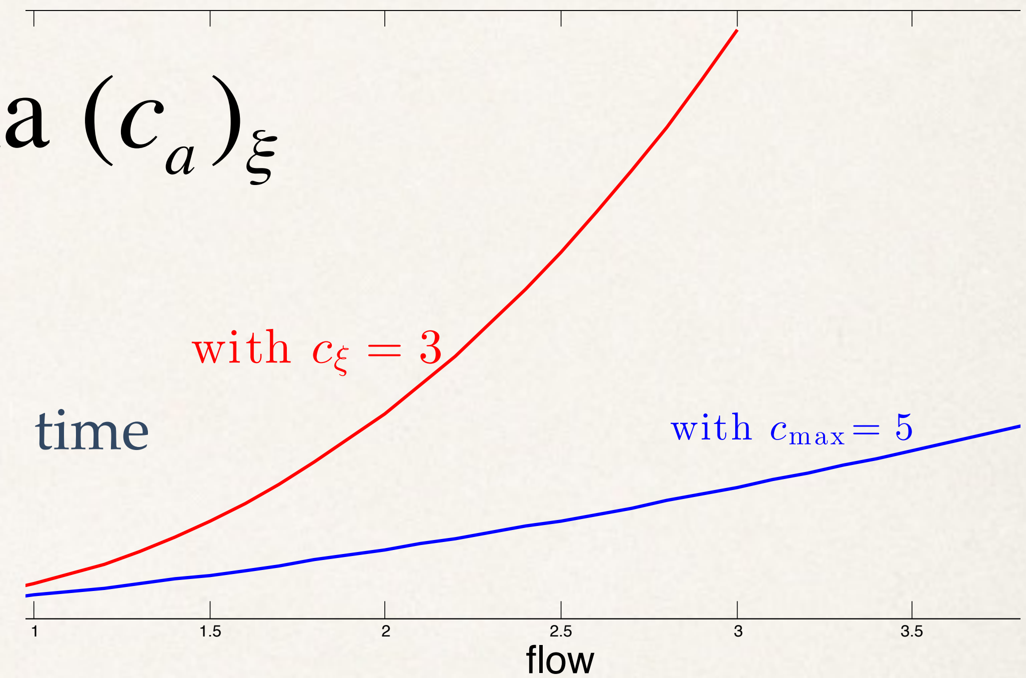
$$D_\xi = \left\{ x \middle| \exists u : Au = Ax - b_\xi,\ x - c_\xi \le u \le x \right\} \quad \forall \xi \text{ (a.s)}$$

$\Rightarrow$ $x \in \mathbb{R}^n_+ \cap D$, $\quad D = \bigcap_{\xi \in \Xi} D_\xi$, Ξ support of ξ.

feasible (steady) flow

finding the “steady” flow

$F_a(\xi, x - u_\xi)$ dependence on ξ via $(c_a)_\xi$



Residual function:

$$\varphi(\xi, x) = \max_{y \in D} \left\langle F(\xi, x - u_\xi - y),\ x - u_\xi - y \right\rangle \ \sim \text{gap fcn}$$

smoothing $\varphi(\xi, x) - \lambda \| x - u_\xi - y \|^2,\ \lambda > 0$

$$\Rightarrow \ \min_{x \in D} \mathbb{E}\left\{ \varphi(\xi, x) - \lambda \| x - u_\xi - y \|^2 \right\}$$

suggested Solution procedure

Algorithm: pick $x \in D$, $\forall \xi: \ x - u_\xi$ Wardrop equilibrium flow

evaluate $\mathbb{E}\{\text{smoothed } \varphi\}$, find better x & repeat

But - no explicit expression for D

$$D = \cap_{\xi \in \Xi} D_\xi \text{ closed, convex (polyhedral?)}$$

- computing $\forall \xi: \ x - u_\xi$ W-equilibrium flow

Remedies:

set $u_\xi = r(\xi, x), \ r \in \mathcal{R}$, e.g., $r(\xi, x) = \text{prj}_{U_\xi(x)}(x)$

$$U_\xi(x) = \left\{ u \,\middle|\, Au = Ax - b_\xi, \ u \in [x - c_\xi, x] \right\}$$

build D sequentially (partially) as needed

suggested Solution procedure

Algorithm: pick $x \in D$, $\forall \xi :\ x - u_\xi$ Wardrop equilibrium flow

evaluate $\mathbb{E}\{\text{smoothed } \varphi\}$, find better x & repeat

But - no explicit expression for D

$D = \cap_{\xi \in \Xi} D_\xi$ closed, convex (polyhedral?)

- computing $\forall \xi :\ x - u_\xi$ W-equilibrium flow

assumed D known with relatively complete recourse

Remedies:

set $u_\xi = r(\xi, x),\ r \in \mathcal{R}$, e.g., $r(\xi, x) = \text{prj}_{U_\xi(x)}(x)$

$$U_\xi(x) = \left\{ u \,\middle|\, Au = Ax - b_\xi,\ u \in [x - c_\xi, x] \right\}$$

build D sequentially (partially) as needed *not required*

infrastructure Design

with T.K. Pong, HK-Poly

Given OD-demand (stochastic)

links potential failures (stochastic)

find network design (links/capacity)

that optimizes network flow

Regularized gap residual function $\alpha > 0$,

$$r(\xi, x, u_\xi) = \max_{v \in C_\xi} \Big\{ \langle (u_\xi - x) - v, F(\xi, u_\xi - x) \rangle - \tfrac{\alpha}{2} \|(u_\xi - x) - v\|^2 \Big\}$$

Leads to the stochastic program

non-convex

$$\begin{array}{ll} \min_{x \in D} & \theta(x) + \lambda \mathbb{E}\{r(\xi, u_\xi) + Q(\xi, x)\} \\ \text{where} & Q(\xi, x) = \inf_y \left\{ \tfrac{1}{2}\langle y, Hy \rangle, \;\middle|\; u_\xi = x + Wy \in C_\xi \right\} \end{array}$$

Solution Procedure: re-designed splitting method

Further readings

- Deride J., Jofré A. & R. Wets, Solving deterministic and stochastic equilibrium problems via the augmented Walrasian, Tech. Report UC-Davis, (2014?)
- Jofré, A. & R. Wets, Variational convergence of bivariate functions: theoretical foundations. Mathematical Programming (2006).
- Jofré, A. & R. Wets, Variational convergence of bivariate functions: motivating examples. SIAM J. on Optimization (2014)
- + literature on Progressive Hedging/Scenario Aggregation Principle.

Further readings

- Jofré, A. & R. Wets, Variational convergence of bivariate functions: theoretical foundations. Mathematical Programming (2006).
- Jofré, A., R.T. Rockafellar R.T & R. Wets. Variational Inequalities and economic equilibrium. Mathematics of Operation Research (2006?)
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- S. P. Dirkse and M. C. Ferris. The PATH solver: A non-monotone stabilization scheme for mixed complementarity problems. Optimization Methods and Software, 5:123-156, 1995.