What is the Riemann Hypothesis for Zeta Functions of Irregular Graphs?

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Joint work with H. M. Stark, M. D. Horton, etc.

What is an expander graph X?

X finite connected (irregular) graph, not a cycle, no degree 1 vertices

4 Ideas

1) spectral property of some matrix
   Adjacency matrix, Laplacian, edge matrix $W$ for $X$
   Lubotzky: compare with spectrum analogous operator
   on universal cover of $X$ (or some graph $X$ covers)

2) $X$ behaves like a random graph.

3) Information is passed quickly in the gossip network
   based on $X$;
   $|\partial S|/|S| \geq c$ if $S$ is a set of vertices and $|S| \leq |X|/2$.
   See Fan Chung's papers on Cheeger constant $c$ on her
   website. Note that it is closely related to the size of
   the smallest non-0 eigenvalue of the Laplacian

4) Random walker gets lost FAST.
Primes in Graphs

are equivalence classes \([C]\) of closed backtrackless
tailless primitive paths \(C\)

**DEFINITIONS**

- **backtrack**
- **equivalence class**: change starting point
- **tail**

Here \(\alpha\) is the start of the path

**non-primitive**: go around path more than once

EXAMPLES of Primes in a Graph

\[[C] = [e_1e_2e_3]\]
\[[D] = [e_4e_5e_3]\]
\[[E] = [e_1e_2e_3e_4e_5e_3]\]

\(\nu(C) = 3, \ \nu(D) = 4, \ \nu(E) = 6\)

\(E = CD\)

another prime \([C\cdot D]\), \(n=2,3,4,\ldots\)
ininitely many primes
Ihara Zeta Function

\[ \zeta(u,X) = \prod_{[C] \text{ prime}} \left(1-u^{\nu(C)}\right)^{-1} \]

\[ \zeta(u,X)^{-1} = (1-u^2)^{r-1} \det(I-Au+Qu^2) \]

for \( u \) complex, \(|u| \) small

\( A \) = adjacency matrix, \( Q + I \) = diagonal matrix of degrees, \( r \) = rank fundamental group

For \( q+1 \) - regular graph, \( u = q^{-s} \) makes Ihara zeta more like Riemann zeta.

\( f(s) = \zeta(q^{-s}) \) has a functional equation relating \( f(s) \) and \( f(1-s) \).

Riemann Hypothesis (RH) says \( \zeta(q^{-s}) \) has no poles with \( 0 < \text{Re } s < 1 \) unless \( \text{Re } s = \frac{1}{2} \).

RH means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the spectrum for the universal covering tree which is the interval \((-2\sqrt{q}, 2\sqrt{q})\) [see Lubotzky, Phillips & Sarnak, Combinatorica, 8 (1988)].

and thus a good expander
Possible Locations of Poles $u$ of $\zeta(u)$ for $q+1$ Regular Graph

1/q is always the closest pole to the origin in absolute value Circle of radius $1/\sqrt{q}$ from part of spectrum of adjacency matrix satisfying the Ramanujan inequality
Real poles ($\neq \pm q^{-1/2}, \pm 1$) correspond to non-Ramanujan eigenvalues of $A$.

Alon conjecture for regular graphs says RH true for “most” regular graphs but can be false. See Joel Friedman’s website (www.math.ubc.ca/~jf) for a paper proving a random regular graph is almost Ramanujan.

What is the meaning of the RH & Ramanujan for irregular graphs?

For irregular graph, natural change of variables is $u=R^s$, where $R =$ radius of convergence of Dirichlet series for Ihara zeta.

Note: $R$ is closest pole of zeta to 0.

Then the critical strip is $0 \leq \Re s \leq 1$ and we get

Graph theory RH: $\zeta(u)$ is pole free in $R < |u| < \sqrt{R}$.

$\rho = \max |\lambda|$, $\lambda$ in spectrum $A=$adjacency matrix of graph $X$

$\rho' = \max |\lambda| \neq \rho$, $\lambda$ in spectrum $A=$adjacency matrix of graph $X$

$\sigma =$ spectral radius Adjacency operator on universal cover of $X$

Lubotzky says a connected irregular graph is Ramanujan if $\rho' \leq \sigma$
Some Facts About the Constants
Let $d=\text{average degree}$, Hoory, J. Comb. Theory, 93 (2005) shows

$$2(d-1)^{1/2} \leq \sigma$$

$R$ is closest pole of zeta to 0

$\rho = \text{maximum } |\lambda|, \lambda \text{ in spectrum } A=\text{adjacency matrix of } X$

$\rho' = \text{maximum } |\lambda| \neq \rho, \lambda \text{ in spectrum } A=\text{adjacency matrix of } X$

$\sigma = \text{spectral radius } A=\text{Adjacency operator on universal cover of } X$

One can show that $\rho \geq d$

Examples all have $\rho \geq 1 + (1/R) \geq d$

Can only show $\rho \geq (p/q)+(1/R)$,

where $q+1=\text{max degree}$, $p+1=\text{min degree}$

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Labeling Edges of Graphs

$X = \text{finite connected (not-necessarily regular graph)}$

Orient the $m$ edges.

Label them as follows.

Here the inverse edge has opposite orientation.

$e_1, e_2, \ldots, e_m$,

$e_{m+1} = (e_1)^{-1}, \ldots, e_{2m} = (e_m)^{-1}$

With this labeling, we have the properties of the edge matrix on the next slide.
The Edge Matrix $W$

Define $W$ to be the $2|E| \times 2|E|$ matrix with $i,j$ entry 1 if edge $i$ feeds into edge $j$, (end vertex of $i$ is start vertex of $j$) provided that $e_i \neq$ the inverse of $e_j$, otherwise the $i,j$ entry is 0.

Theorem. $\zeta(u,X)^{-1} = \det(I-Wu)$.

Corollary. The poles of Ihara zeta are the reciprocals of the eigenvalues of $W$.

The pole $R$ of zeta is:

$R = \frac{1}{\text{Perron-Frobenius eigenvalue of } W}$.

Properties of $W$

1) $W = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$, $B$ and $C$ symmetric

2) Row sums of entries are $q_j + 1$ = degree $j$th vertex

3) Singular Values (square roots eigenvalues of $WW^T$) are \{q_1, \ldots, q_n, 1, \ldots, 1\}.

4) $(I+W)^{2|E|-1}$ has all positive entries, if $2 \leq r \leq \text{rank fundamental group}$.

So we can apply Perron-Frobenius theorem to $W$.

Poles Ihara Zeta

are in region $q^{-1} \leq R \leq |u| \leq 1$,

$q + 1$ = maximum degree of vertices of $X$.

So eigenvalues of $W$ being reciprocals of poles are outside unit circle and inside circle of radius $q$. 

\[ \]
Theorem of Kotani and Sunada

1. If $p+1=\min$ vertex degree, and $q+1=\max$imum vertex degree, non-real poles $u$ of zeta satisfy

\[
\frac{1}{\sqrt{q}} \leq |u| \leq \frac{1}{\sqrt{p}}
\]

2. Poles symmetric under rotation by $2\pi/\Delta$, where

$\Delta = \gcd$ lengths of primes in graph


Corollary: Non-real eigenvalues of $W$ are between $\sqrt{p}$ and $\sqrt{q}$.

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**Work of Joel Friedman, Angel & Hoory**

The non-backtracking spectrum of the universal cover of a graph, preprint on Friedman's website

They call $W$ the non-backtracking adjacency matrix

They find 2 dimensional region $D$ which is spectrum of the corresponding operator on the universal cover of the base graph $X$. One small example is drawn (K4-edge).

Irregular Graph Analog of Alon Conjecture:

new (i.e., not from $X$) spectrum of $W$-matrix for random $n$-sheeted covering (lift) $Y$ of $X$ should approach region $D$ as $n$ goes to infinity.

$R^{-1/2}$ is the spectral radius of the $W$-operator on the universal cover of $X$.

So approximate RH for covers of fixed base graph is contained in Alon conjecture.
**Spectrum of Random Matrix with Properties of W-matrix**

\[
W = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}
\]

- B and C symmetric
- Girko circle law

We used Matlab command `randn(1000)` to get A, B, C matrices with random normally distributed entries mean 0 std dev 1

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**Experiment on Locations of Zeros of Ihara Zeta of Irregular Graphs**

All poles but -1 of \(\zeta_X(u)\) for a random graph with 80 vertices denoted by little boxes using Mathematica.

The 5 circles are centered at 0 & have radii \(R, q^{1/2}, R^{1/2}, (pq)^{-1/4}, p^{1/2}\)

\(q+1=\text{max degree},\ p+1=\text{min degree}; R=\text{radius of convergence of Euler product for } \zeta_X(u)\)

RH is false but poles are not far inside green circle

RandomGraph[80,1/10] is the Mathematica command we used. It means the probability of an edge between 2 vertices is 1/10.
Experiment on Locations of Zeros of Ihara Zeta of Irregular Graphs

All poles except -1 of $\zeta_X(u)$ for a random graph with 100 vertices are denoted by little boxes, using Mathematica

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RandomGraph[100, 1/2]
```

Circles are centered at the origin and have radii $R$, $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$

$q+1=$ max degree,
$p+1=$ min degree

$R=$ radius of convergence of Euler product for $\zeta_X(u)$

RH is false maybe not as false as in previous example with probability 1/10 of an edge rather than 1/2.

Graph satisfies Hoory inequality and is thus Ramanujan in Lubotzky's sense.

Matthew Horton's Graph has $1/R \approx e$ to 7 digits.

Poles of Ihara zeta are boxes on right. Circles have radii $R, q^{1/2}, R^{1/2}, p^{1/2}$, if $q+1=$ max deg, $p+1=$ min deg. Here $(p, 1+1/R, d=average\ degree) \approx (5.26243, 3.71828, 2.47863)$

The RH is false; does not satisfy Hoory inequality. Poles more spread out over plane.
Poles of Ihara Zeta for a $\mathbb{Z}_6 \times \mathbb{Z}_6$-Cover of 2 Loops + Extra Vertex are pink dots

Circles Centers $(0,0)$; Radii: $3^{-1/2}$, $R^{1/2}$, $1$; $R \cong 0.47$
RH very False

$Z$ is random 700 cover of 2 loops plus vertex graph in picture.
The pink dots are at poles of $\zeta_Z$. Circles have radii $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$, with $q=3$, $p=1$, $R \cong 0.4694$. RH approximately True.
References: 3 papers with Harold Stark in *Advances in Math*.


- See my draft of a book:
  
  www.math.ucsd.edu/~aterras/newbook.pdf

- Draft of new paper: also on my website
  
  www.math.ucsd.edu/~aterras/cambridge.pdf

- There was a graph zetas special session of this AMS meeting – many interesting papers some on my website.