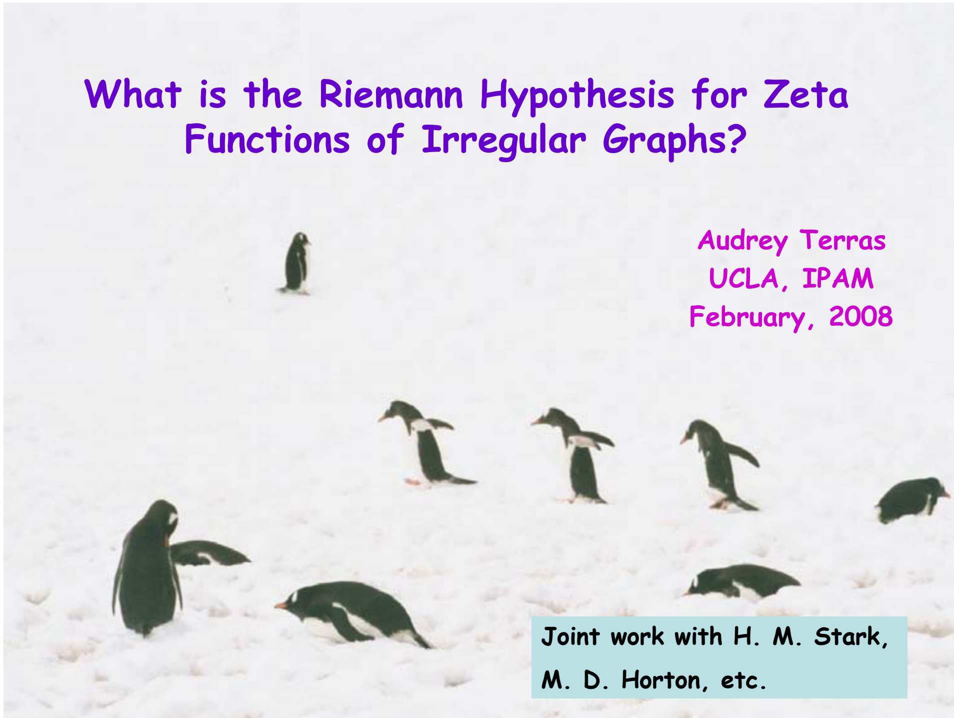


What is the Riemann Hypothesis for Zeta Functions of Irregular Graphs?

Audrey Terras
UCLA, IPAM
February, 2008

Joint work with H. M. Stark, M. D. Horton, etc.

What is an expander graph X?

X finite connected (irregular) graph, not a cycle, no degree 1 vertices

4 Ideas

1) spectral property of some matrix
   Adjacency matrix, Laplacian, edge matrix W for X
   Lubotzky: compare with spectrum analogous operator on universal cover of X (or some graph X covers)

2) X behaves like a random graph.

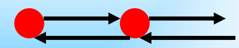
3) Information is passed quickly in the gossip network based on X;
   $|\partial S|/|S| \geq c$ if S is a set of vertices and $|S| \leq |X|/2$.
   See Fan Chung's papers on Cheeger constant c on her website. Note that it is closely related to the size of the smallest non-0 eigenvalue of the Laplacian

4) Random walker gets lost FAST.

Primes in Graphs

are equivalence classes [C] of closed backtrackless tailless primitive paths C

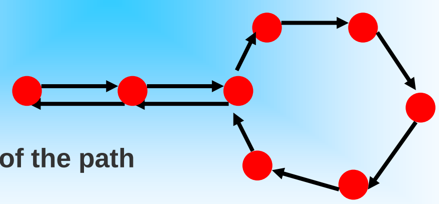
DEFINITIONS backtrack



equivalence class: change starting point

tail

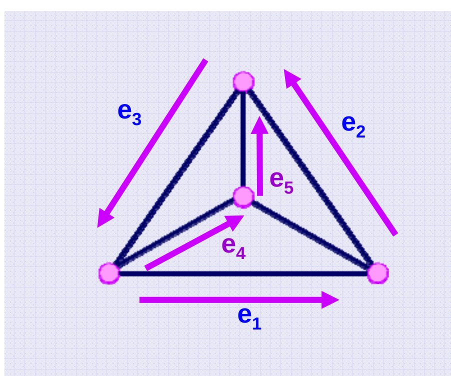
α



Here α is the <u>start</u> of the path

non-primitive: go around path more than once

EXAMPLES of Primes in a Graph



$[C] = [e_1 e_2 e_3]$

$[D]=[e_4 e_5 e_3]$

$[E]=[e_1 e_2 e_3 e_4 e_5 e_3]$

$\nu(C)=3,\ \nu(D)=4,\ \nu(E)=6$

E=CD
another prime $[C^n D]$, n=2,3,4, …
infinitely many primes

Ihara Zeta Function

$$\zeta(u,X)=\prod_{\substack{[C]\\ \text{prime}}}\left(1-u^{\nu(C)}\right)^{-1}$$

for u complex, |u| small

$$\zeta(u,X)^{-1}=(1-u^2)^{r-1}\det(I-Au+Qu^2)$$

A=adjacency matrix, Q +I = diagonal matrix of degrees, r=rank fundamental group

For q+1 – regular graph, $u=q^{-s}$ makes Ihara zeta more like Riemann zeta.

$f(s)=\zeta(q^{-s})$ has a functional equation relating f(s) and f(1-s).

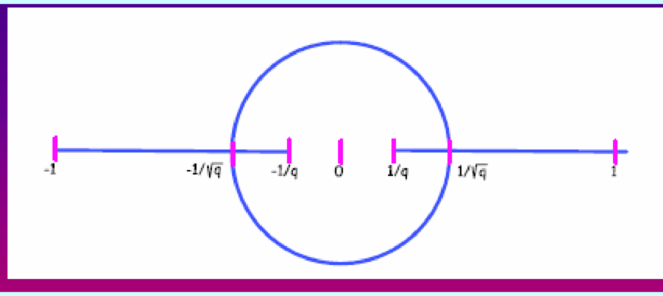
Riemann Hypothesis (RH)

says $\zeta(q^{-s})$ has no poles with $0<\mathrm{Re}\,s<1$ unless $\mathrm{Re}\, s = \frac{1}{2}$.

RH means graph is Ramanujan i.e., non-trivial spectrum of adjacency matrix is contained in the spectrum for the universal covering tree which is the interval $(-2\sqrt{q}, 2\sqrt{q})$ [see Lubotzky, Phillips & Sarnak, *Combinatorica*, 8 (1988)].

and thus a good expander

Possible Locations of Poles u of $\zeta(u)$ for q+1 Regular Graph



$1/q$ is always the closest pole to the origin in absolute value

Circle of radius $1/\sqrt{q}$ from part of spectrum of adjacency matrix satisfying the Ramanujan inequality

Real poles ($\neq \pm q^{-1/2}$, ± 1) correspond to non-Ramanujan eigenvalues of A.

Alon conjecture for regular graphs says RH true for "most" regular graphs but can be false. See Joel Friedman's website (www.math.ubc.ca/~jf) for a paper proving a random regular graph is almost Ramanujan.

What is the meaning of the RH & Ramanujan for irregular graphs?

For irregular graph, natural change of variables is $u=R^s$, where

R = radius of convergence of Dirichlet series for Ihara zeta.

Note: R is closest pole of zeta to 0.

Then the critical strip is $0 \leq \mathrm{Re}\, s \leq 1$ and we get

Graph theory RH: $\zeta(u)$ is pole free in $R < |u| < \sqrt{R}$.

ρ = maximum $|\lambda|$, λ in spectrum A=adjacency matrix of graph X

ρ' = maximum $|\lambda| \neq \rho$, λ in spectrum A=adjacency matrix of graph X

σ = spectral radius Adjacency operator on universal cover of X

Lubotzky says a connected irregular graph is Ramanujan if $\rho' \leq \sigma$

Some Facts About the Constants

Let d=average degree, Hoory, *J. Comb. Theory*, 93 (2005) shows

$$2(d-1)^{1/2} \leq \sigma$$

R is closest pole of zeta to 0

ρ = maximum $|\lambda|$, λ in spectrum A=adjacency matrix of X

ρ'= maximum $|\lambda| \neq \rho$, λ in spectrum A=adjacency matrix of X

σ = spectral radius Adjacency operator on universal cover of X

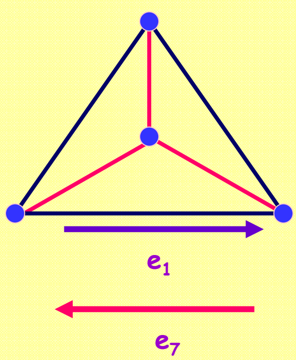
One can show that $\rho \geq d$

Examples all have $\rho \geq 1+(1/R) \geq d$

Can only show $\rho \geq (p/q)+(1/R)$,

where $q+1$=max degree, $p+1$=min degree

Labeling Edges of Graphs



X = finite connected (not-necessarily regular graph)
Orient the m edges.
Label them as follows.
Here the inverse edge has opposite orientation.

$e_1, e_2, \ldots, e_m,$

$e_{m+1}=(e_1)^{-1}, \ldots, e_{2m}=(e_m)^{-1}$

With this labeling, we have the properties of the edge matrix on the next slide.

The Edge Matrix W

Define W to be the $2|E|\times 2|E|$ matrix with i j entry 1 if edge i feeds into edge j, (end vertex of i is start vertex of j) provided that $e_i \neq$ the inverse of e_j, otherwise the i j entry is 0.

Theorem. $\zeta(u,X)^{-1}=\det(I-Wu)$.

Corollary.

The poles of Ihara zeta are the reciprocals of the eigenvalues of W.

The pole R of zeta is:

R=1/Perron-Frobenius eigenvalue of W.

Properties of W

1) $W=\begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$, B and C symmetric

2) Row sums of entries are q_j+1=degree jth vertex
3) Singular Values (square roots eigenvalues of WW^T) are $\{q_1,\ldots,q_n,1,\ldots,1\}$.
4) $(I+W)^{2|E|-1}$ has all positive entries, if $2 \leq r$
   r=rank fundamental group.
   So we can apply Perron-Frobenius theorem to W.

Poles Ihara Zeta

are in region

$$q^{-1} \leq R \leq |u| \leq 1,$$

q+1=maximum degree of vertices of X.

So eigenvalues of W being reciprocals of poles are outside unit circle and inside circle of radius q.

Theorem of Kotani and Sunada

1. If p+1=min vertex degree, and q+1=maximum vertex degree, non-real poles u of zeta satisfy

$$\frac{1}{\sqrt{q}} \le |u| \le \frac{1}{\sqrt{p}}$$

2. Poles symmetric under rotation by $2\pi/\Delta$, where Δ = g.c.d. lengths of primes in graph

Kotani & Sunada, *J. Math. Soc. U. Tokyo*, 7 (2000)

Corollary: Non-real eigenvalues of W are between $\sqrt{p}$ and $\sqrt{q}$.

Work of Joel Friedman, Angel & Hoory

The non-backtracking spectrum of the universal cover of a graph, preprint on Friedman's website

They call W the non-backtracking adjacency matrix

They find 2 dimensional region D which is spectrum of the corresponding operator on the universal cover of the base graph X. One small example is drawn (K4-edge).

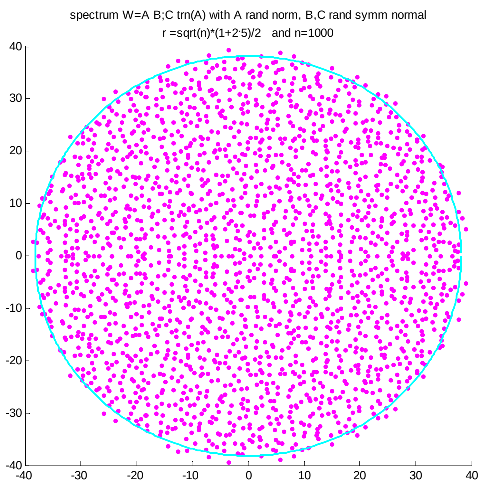
Irregular Graph Analog of Alon Conjecture:

new (i.e., not from X) spectrum of W-matrix for random n-sheeted covering (lift) Y of X should approach region D as n goes to infinity.

$R^{-1/2}$ is the spectral radius of the W-operator on the universal cover of X.

So approximate RH for covers of fixed base graph is contained in Alon conjecture

Spectrum of Random Matrix with Properties of W-matrix

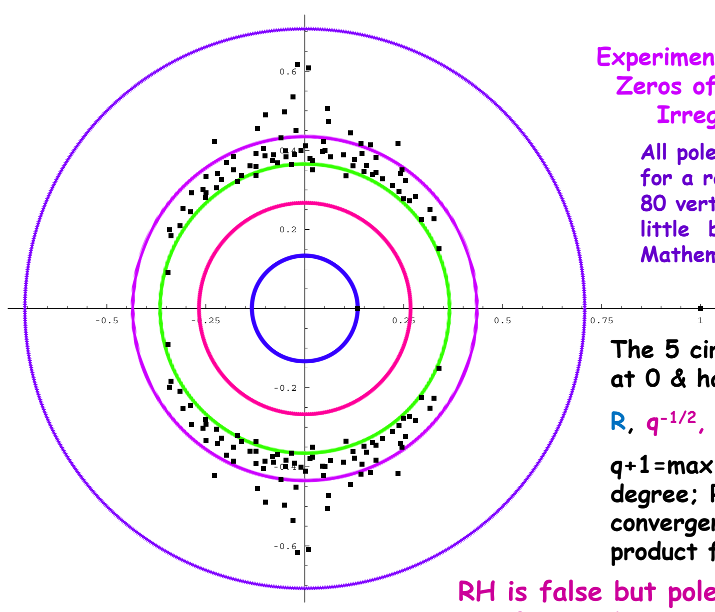


$$W = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$$

B and C symmetric

Girko circle law

We used Matlab command randn(1000) to get A,B,C matrices with random normally distributed entries mean 0 std dev 1



Experiment on Locations of Zeros of Ihara Zeta of Irregular Graphs

All poles but -1 of $\zeta_X(u)$ for a random graph with 80 vertices denoted by little boxes using Mathematica.

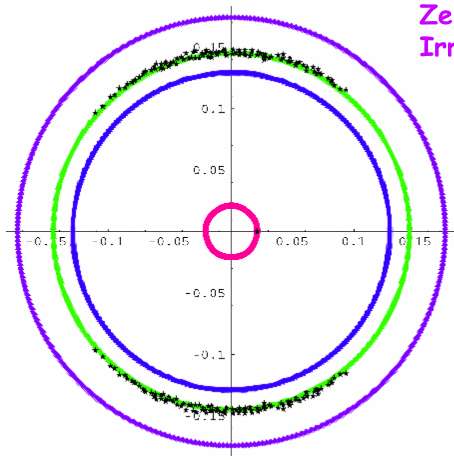
The 5 circles are centered at 0 & have radii

R, $q^{-1/2}$, $R^{1/2}$, $(pq)^{-1/4}$, $p^{-1/2}$

$q+1$=max degree, $p+1$=min degree; R=radius of convergence of Euler product for $\zeta_X(u)$

RH is false but poles are not far inside green circle

RandomGraph[80,1/10] is the Mathematica command we used. It means the probability of an edge between 2 vertices is 1/10.



Experiment on Locations of Zeros of Ihara Zeta of Irregular Graphs

All poles except -1 of $\zeta_X(u)$ for a random graph with 100 vertices are denoted by little boxes, using Mathematica

RandomGraph[100,1/2]

Circles are centered at the origin and have radii

R, $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$

q+1=max degree,

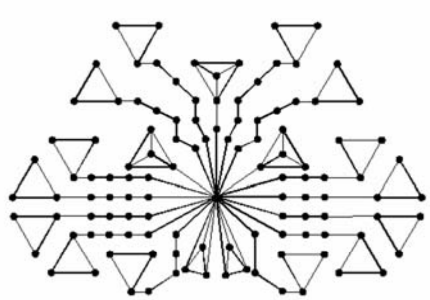
p+1=min degree

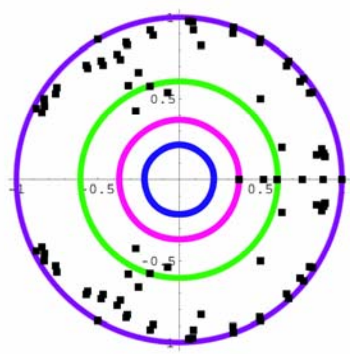
R=radius of convergence

of Euler product for $\zeta_X(u)$

RH is false maybe not as false as in previous example with probability 1/10 of an edge rather than 1/2.

Graph satisfies Hoory inequality and is thus Ramanujan in Lubotzky's sense.



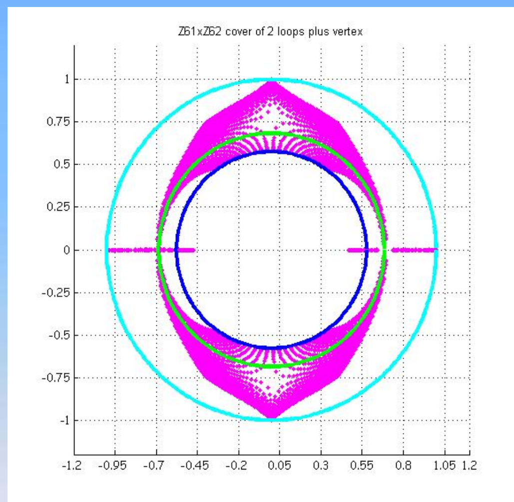


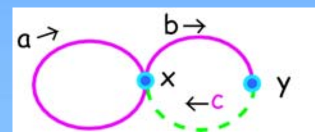
Matthew Horton's Graph has 1/R ≅ e to 7 digits.

Poles of Ihara zeta are boxes on right. Circles have radii $R, q^{-\frac{1}{2}}, R^{\frac{1}{2}}, p^{-\frac{1}{2}}$, if q+1=max deg, p+1=min deg. Here (ρ,1+1/R,d=average degree) ≅(5.26243,3.71828,2.47863)

The RH is false; does not satisfy Hoory inequality. Poles more spread out over plane.

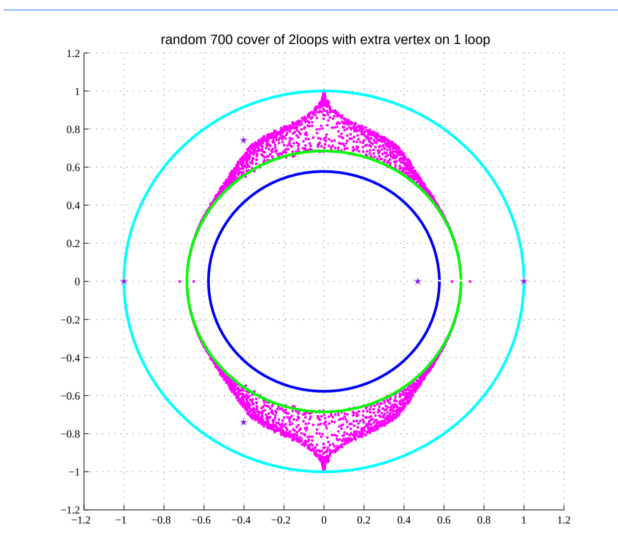
Poles of Ihara Zeta for a $Z_{61}xZ_{62}$-Cover of 2 Loops + Extra Vertex are pink dots

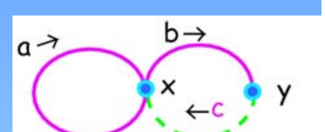




Circles Centers (0,0); Radii: $3^{-1/2}$, $R^{1/2}$,1; R ≅.47

RH very False





Z is random 700 cover of 2 loops plus vertex graph in picture.

The pink dots are at poles of ζ_Z. Circles have radii $q^{-1/2}$, $R^{1/2}$, $p^{-1/2}$, with q=3, p=1, R ≅ .4694. RH approximately True.

References: 3 papers with Harold Stark in *Advances in Math*.

- Paper with Matthew Horton & Harold Stark in Snowbird Proceedings, Contemporary Mathematics, Volume 415 (2006)

  Quantum Graphs and Their Applications, *Contemporary Mathematics*, v. 415, AMS, Providence, RI 2006.

- See my draft of a book:

  www.math.ucsd.edu/~aterras/newbook.pdf

- Draft of new paper: also on my website

  www.math.ucsd.edu/~aterras/cambridge.pdf

- There was a graph zetas special session of this AMS meeting – many interesting papers some on my website.

- For work on directed graphs, see Matthew Horton, Ihara zeta functions of digraphs, *Linear Algebra and its Applications,* 425 (2007) 130–142.