Pro-finite properties and property Tau

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I.e. for two f.g. groups Γ_1 and Γ_2 with the same pro-finite completion then if Γ_1 has P then so does Γ_2.

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- having certain subgroup growth

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Theorem (Lackenby) *Largeness + finitely presented is a pro-finite property (it is even pro-p).*

A group is large if it contains a finite index subgroup which maps onto non-abelian free group.

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It seems that Tau is a pro-finite property since it is "defined" using the pro-finite topology.

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The maximal ϵ such that the existence of an ϵ-almost invariant vector implies the existence of an invariant vector is called Kazhdan constant of G with respect to S and is denoted by $\mathcal{K}(G; S)$.

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Properties T and Tau

A group has Kazhdan property T iff there exist a compact set S such that $\mathcal{K}(G;S) > 0$, i.e., there exists $\epsilon > 0$ such that any representation with (S,ϵ)-almost invariant vector contains the trivial representation.

A group has property Tau iff there exist a finite set S and $\epsilon > 0$ such that any representation which factors through a finite index subgroup with (S,ϵ)-almost invariant vector contains the trivial representation.

Informal definitions for Tau

"Definition" *A group* Γ *has Tau iff its pro-finite completion* $\widehat{\Gamma}$ *has T.*
Since representations of $\widehat{\Gamma}$ *are exactly the same as representation which factors through a finite index subgroup.*

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However this "definition" is also incorrect...

Products

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In most cases such group Γ has (relative) Tau iff the Kazhdan constants $\mathcal{K}(G_i; S_i)$ are uniformly bounded.

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This implies that there exist two groups Γ_1 and Γ_2 which are dense in a same pro-finite group $\mathbb{G}$ such that one has (relative) Tau but the other does not.
However this does not imply that Tau is not pro-finite property, because these groups have different pro-finite completions (in most cases one of the groups has infinite abelian quotient).

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Theorem (K-Nikolov) *Let S_i be a "nice" family of finite simple groups. There exists a finitely generated Γ with pro-finite completion*

$$\prod_i S_i$$

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G is called frame *if also*
(c) The natural surjection $\widehat{G} \to \mathbb{G}$ is an isomorphism.
Lemma *If G is a frame-like subgroup then*

$$\widehat{G} = \mathbb{G} \times C_G$$

i.e., there is a projection $\pi : G \to C_G$.

Gluing frame-like subgroups

Let K_i and L_i are subgroup of G_i such that $G_i = \langle K_i, L_i \rangle$. The pro-finite groups $\mathbb{K} = \prod K_i$ and $\mathbb{L} = \prod L_i$ are a subgroups of $\mathbb{G}$.

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Lemma *If K and L are frame-like subgroup in $\mathbb{K}$ and $\mathbb{L}$ then $G = \langle K, L \rangle$ is a frame-like subgroup in $\mathbb{G}$.*

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Lemma *If K and L are frame-like subgroup in $\mathbb{K}$ and $\mathbb{L}$ then $G = \langle K, L \rangle$ is a frame-like subgroup in $\mathbb{G}$.*

Moreover the congruence kernel C_G is generated by C_K and C_L.

Existence of frames

Theorem (K-Nikolov) *There exits a finitely generated* Γ_0 *which is a frame in* $\prod \mathrm{Alt}(n)$*, in particular* $\widehat{\Gamma_0}$ *is isomorphic to* $\prod \mathrm{Alt}(n)$*.*

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Does it have property Tau? Probably not...

Breaking Tau

Consider the maps:

$$\pi_n : \mathrm{Alt}(n) \star \mathrm{Alt}(n) \to \prod_{n^2 < i \leq (n+1)^2} \mathrm{Alt}(i)$$

where each copy of $\mathrm{Alt}(n)$ is embedded diagonally in $\mathrm{Alt}(n)^{\times n} \subset \mathrm{Alt}(i)$ and the "supports" of the images of the two copies are "offset" by 1 point.

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and such that $\Gamma_1 = \pi(\Gamma_0 \star \Gamma_0)$ does not have property Tau.
The gluing lemma gives that $\widehat{\Gamma_1} = \prod \mathrm{Alt}(n)$.

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Theorem (K) *There exists generating sets S_n for $\mathrm{Alt}(n)$ such that*

- S_n *consists of 100 involutions with no more than 3 fixed points;*
- $\mathcal{K}(\mathrm{Alt}(n); S_n) > \epsilon$ *for some* $\epsilon > 0$*.*

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These two examples show that Tau is NOT a pro-finite property.