Uniform Kazhdan Constant for some families of linear groups

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Almost invariant vectors

Definition. Let Γ be a discrete group, $S \subset \Gamma$, $\epsilon > 0$, and $(\rho, \mathcal{H})$ be a unitary representation of Γ.
A unit vector $v \in \mathcal{H}$ is called (S, ϵ)-invariant, if for all $s \in S$, $\|\rho(s)v - v\| \leq \epsilon$.

Kazhdan constant

Definition. A discrete group Γ is said to have Kazhdan property (T), if there exist a finite generating set S and $\epsilon > 0$, such that every unitary representation with (S, ϵ)-invariant vector, contains a non-zero Γ-invariant vector.
In that case $(|S|, \epsilon)$ is called a Kazhdan constant for Γ.

Examples

- $SL_n(\mathbb{Z})$ in $SL_n(\mathbb{R})$ for $n \geq 3$ [Kazhdan]

- $SL_n(\mathbb{Z}[x_1, ..., x_k])$ for $n > k + 2$ [Shalom].

- $SL_2(\mathbb{Z})$ does not have property (T).

Applications

1) Expander graphs.

2) Bound on the mixing time of a random walk.

3) Solution of the Banach-Ruziewicz theorem about uniqueness of rotation-invariant means on the unit sphere (Margulis and Sullivan).

Elementary Linear Group

R - a f.g associative ring with unit generated by $a_1 = 1, a_2, ..., a_l$.

$e_{ij}(r)$ - An elementary matrix. This is $n \times n$ matrix with 1 along the diagonal, $r \in R$ in the (i, j) position, and zero elsewhere.

$EL_d(R)$ - the elementary group.

$EL_d(\mathbb{Z}[x_1, ..., x_k] = SL_d(\mathbb{Z}[x_1, ..., x_k])$ for $d \geq 3$ (Suslin).

Generating Set for $EL_d(R)$

$S_d(R)$ – is the set of $2(d^2-d)$ elementary matrices with ±1 off the diagonal and the set of $4l(d-1)$ elementary matrices $e_{ij}(\pm a_m)$ with $|i-j|=1$ and $1\leq m\leq l$.

- For $d\geq 3$, $EL_d(R)$ is generated by the set $S_d(R)$.

Unimodularity

Definition. A sequence $\{a_1, ..., a_n\}$ in a ring R is said to be left unimodular if

$$Ra_1 + ... + Ra_n = R.$$

In case $n \geq 2$, such a sequence is said to be reducible if there exist $r_1, ..., r_{n-1} \in R$ so that

$$R(a_1 + r_1 a_n) + ... + R(a_{n-1} + r_{n-1} a_n) = R.$$

Stable Range

Definition. A ring R is said to have left stable range $\leq n$, if every left unimodular sequence of length $> n$ is reducible. The smallest such n is said to be the left stable range of R.

Examples:
1) $sr(\mathbb{F}) = 1$.
2) $sr(\mathbb{Z}) = 2$.

Previous work

Theorem (Shalom [2006]) Let R be a f.g associative ring with a unit and with stable range r. Then for all $n > max\{2, r\}$, the group $EL_n(R)$ has Kazhdan property (T).

Shalom [1999] (with a complement by Kassabov [2005]) showed that for $n \geq 3$, $SL_n(\mathbb{Z})$ has Kazhdan constant $(2(n^2 - n), \epsilon_n)$ with $(42\sqrt{n} + 860)^{-1} \leq \epsilon_n < 2n^{-\frac{1}{2}}$

- They have taken the elementary matrices as the set of generators.

Previous work

Theorem (Kassabov [2005]) There exist $k \in \mathbb{N}$ and $\epsilon > 0$ s.t for every finite field $\mathbb{F}$, and for every $n \geq 3$, $SL_n(\mathbb{F})$ has Kazhdan constant (k, ϵ).

Main results

Theorem 1.1 [H]. There exist $k \in \mathbb{N}$ and $0 < \epsilon$ s.t. for every $n \geq 3$, $SL_n(\mathbb{Z})$ has Kazhdan constant (k, ϵ).

Main Theorem [H]. Let R be an associative ring generated by l elements with stable range r, and assume that the group $EL_d(R)$ for some $d \geq r+1$, has Kazhdan constant (k_0, ϵ_0). Then there exist $\epsilon = \epsilon(\epsilon_0, l)$ and $k = k(k_0, l)$, s.t. for every $n \geq d$, $EL_n(R)$ has Kazhdan constant (k, ϵ).

Center of mass

Lemma: Let $(\rho, \mathcal{H})$ be a unitary representation of a group Γ. Suppose that for some unit vector $v \in \mathcal{H}$, one has for all $g \in \Gamma$: $\|\rho(g)v - v\| < \sqrt{2}$.
Then there exists a non-zero Γ-invariant vector in $\mathcal{H}$.

The proof is based on the center of the mass property of Hilbert spaces.

Bounded elementary generation property

Definition: The group $\Gamma = EL_d(R)$ is said to have the bounded elementary generation property (BG) if there is a number $N = BE_d(R)$ such that every element of Γ can be written as a product of at most N elementary matrices.

For $n \geq 3$:

- Carter and Keller[1983] - $SL_n(\mathcal{O})$- Yes.

- van der Kallen [1980] - $SL_n(\mathbb{C}[x])$ - No.

- Open problem: $SL_n(\mathbb{Z}[x])$?

BG and Kazhdan constant I

Lemma [Burger 91, Shalom 99, Kassabov 05]:

Let $(\rho, \mathcal{H})$ be a unitary representation of the group $EL_d(R)$. Let $v \in \mathcal{H}$ be a unit vector s.t $\|\rho(s)v - v\| < \epsilon$ for all $s \in S_d(R)$.
Then $\|\rho(g)v - v\| \leq 2M(l)\epsilon$ for every elementary matrix g.

BG and Kazhdan constant II

Theorem [Shalom 1999, Kassabov 2005]: Suppose $d \geq 3$ and R is a f.g associative ring such that $EL_d(R)$ has bounded elementary generation. Then $EL_d(R)$ has property (T) with a lower bound for the Kazhdan constant w.r.t $S_d(R)$.

Example: - $SL_n(\mathbb{Z})$.

A property of Kazhdan group

Lemma: Let $\epsilon > 0$ be a Kazhdan constant of Γ w.r.t S. Let $(\rho, \mathcal{H})$ be a unitary representation of Γ with a unit vector $v \in \mathcal{H}$ which is (δ, S)-invariant (where $\epsilon > \delta > 0$), then v is $(\Gamma, 2\frac{\delta}{\epsilon})$-invariant.

Bounded products of Kazhdan group

Lemma [Shalom 2006, KLN 2006]: If a group Γ is a bounded product of groups with property (T), then Γ is a group with property (T).

Proof of the main Theorem

-Assume that for some $d \geq sr(R)+1$, $EL_d(R)$ has Kazhdan constant (k_0, ϵ_0) w.r.t a generating set F.

- We will show that for any $n \geq d$, the group $EL_{4n}(R)$ has Kazhdan constant $\epsilon(\epsilon_0, l)$ w.r.t $S = F \bigcup S_4(M_n(R))$.

Reminder: $S_4(M_n(R))$ - is the generating set for $EL_4(M_n(R))$ which was defined above.

Proof of the main Theorem

- Let $(\rho, \mathcal{H})$ be a unitary representation of $EL_{4n}(R)$, and suppose $v \in \mathcal{H}$ is a unit vector which is (S, ϵ_1)-invariant (ϵ_1 will be determined latter).
we need to show that v is $(EL_{4n}(R), \sqrt{2})$-invariant.

Proof of the main Theorem

- $EL_4(M_n(R)) \subseteq EL_{4n}(R)$.

- We will show that any element in $EL_{4n}(R)$ is a bounded product of elementary matrices in $EL_4(M_n(R))$ and an element from $EL_d(R)$.

Proof of the main Theorem

Let $g \in EL_{4n}(R)$. We consider g as an element in $GL_4(M_n(R))$.

$$g = \begin{pmatrix} * & * & * & a_1 \\ * & * & * & a_2 \\ * & * & * & a_3 \\ * & * & * & a_4 \end{pmatrix} g^{-1} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}.$$

-$sr(M_n(R)) \leq 1 + [\frac{sr(R)-1}{n}] \leq 2$

$$b_1a_1 + b_2a_2 + b_3a_3 + b_4a_4 = 1$$

$\Rightarrow (a_1, a_2, a_3, a_4)$ is a unimodular sequence.

Proof of the main Theorem

Then by preforming a bounded number of elementary operations we get:

$$g = \begin{pmatrix} * & * & * & a_1 \\ * & * & * & a_2 \\ * & * & * & a_3 \\ * & * & * & a_4 \end{pmatrix} \longrightarrow \begin{pmatrix} A & B & 0 & 0 \\ C & D & 0 & 0 \\ 0 & 0 & Id & 0 \\ 0 & 0 & 0 & Id \end{pmatrix}$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL_2(M_n(R))$$

Proof of the main Theorem

Fact: For $m \geq sr(R) + 2$,

$$EL_{m-1}(R) = EL_m(R) \bigcap GL_{m-1}(R).$$

$$\Rightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in EL_{2n}(R) \subseteq GL_2(M_n(R))$$

Proof of the main Theorem

Lemma [DV] 1:

$$EL_m(R) = ULUL \begin{pmatrix} EL_d(R) & 0 \\ 0 & I_m - d \end{pmatrix}$$

where $m \geq d \geq sr(R) + 1$.

Lemma [DV] 2: U, L are product of 2 commutators.

Therefore

$$EL_m(R) = C_1 C_2 C_3 C_4 C_5 C_6 C_7 C_8 \begin{pmatrix} EL_d(R) & 0 \\ 0 & I_m - d \end{pmatrix}$$

Proof of the main Theorem

Lemma 3:

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & Id & 0 \\ 0 & 0 & 0 & Id \end{pmatrix}$$

where $\begin{pmatrix} * & * \\ * & * \end{pmatrix}$ is a commutator in $GL_2(M_n(R))$, can be written as a product of at most 40 elementary matrices in $EL_4(M_n(R))$.

Proof of the main Theorem

Proposition: Every element in $EL_{4n}(R)$ can be written as a product of at most 340 elementary matrices in $EL_4(M_n(R))$ and an element in $EL_d(R)$.

– Let $g \in EL_{4n}(R)$, g can be written as

$$g = g_1 \cdot \ldots \cdot g_{340} \cdot g_c$$

where the g_i's are elementary matrices of $EL_4(M_n(R))$ and $g_c \in EL_d(R)$.

Proof of the main Theorem

The group $EL_d(R)$ has Kazhdan constant ϵ_0 w.r.t F.
$\rho|_{EL_d(R)}$ is a representation of $EL_d(R)$ with a unit vector v which is (ϵ_1, F) invariant vector, therefore v is $(2 \cdot \frac{\epsilon_1}{\epsilon_0}, EL_d(R))$-invariant.

$$\|\rho(g)v - v\| \leq \sum_{i=1}^{340} \|\rho(g_i)v - v\| + \|\rho(g_c)v - v\| <$$

$$340 \cdot 2M(l+1)\epsilon_1 + 2 \cdot \frac{\epsilon_1}{\epsilon_0} < \sqrt{2}.$$