Biregular expanders and the Ramanujan Conjecture

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Dedicated to the memory of Beth Samuels

joint work with Dan Ciubotaru.

Theorem (Lubotzky, Samuels, Vishne)

Let Γ be a cocompact lattice in $G(F) = PGL_d(F)$.
Then $\Gamma \backslash \mathcal{B}$ is a Ramanujan complex iff every irreducible spherical infinite-dimensional sub-representation of $L^2(\Gamma \backslash G(F))$ is tempered.

Notation

- Graph $X = (V, E)$
- $V = \{v_1, v_2, \ldots, v_m\}$ set of vertices
- E set of edges (subsets of order 2 of V)
- Adjacency matrix of X: $A = (a_{ij})$

Expander Graphs

Definition

An (n, k, c)-expander is a k-regular graph $X_{n,k} = (V, E)$ with $|V| = n$ and

$$\forall\ V' \subseteq V, |V'| \leq \frac{n}{2} : \qquad |\partial(V')| \geq c|V'|$$

$$\partial(V') = \{v \in V \backslash V' \mid (v, v') \in E \text{ for some } v' \in V'\}$$

c - expansion coefficient

Good expanders - Ramanujan graphs

Proposition

$X_{n,k}$ is an (n,k,c)-expander with $2c \geq 1 - \dfrac{|\lambda(X_{n,k})|}{k}$.

- $\lambda(X_{n,k})$ - second largest eigenvalue (in absolute value) of $X_{n,k}$.

Theorem (Alon-Boppana)

$$\liminf_{n\to\infty} |\lambda(X_{n,k})| \geq 2\sqrt{k-1}$$

Definition (Lubotzky, Phillips, Sarnak)

A $(q+1)$-regular graph X is called a Ramanujan graph if $|\lambda(X)| \leq 2\sqrt{q}$.

Ramanujan graphs from groups

- $G = GL_n(\mathbb{Q}_p)$
- Maximal compact subgroups of G: $\{K = GL_n(\mathbb{Z}_p)\}$
- X - Bruhat-Tits building for G
- $n = 2$: X is a $(p+1)$-regular tree
- $n = 3$: X is a $2(p^2 + p + 1)$-regular graph
- G acts simplicially on X
- $\Gamma\backslash X$ finite graph (Γ discrete co-compact subgroup of G)

The Adjacency Operator A

- vertices of $X \leftrightarrow G/K$
- Hecke algebra: $\mathcal{H}(G, K) = \{f : G \to \mathbb{C} \mid f(kgk') = f(g)\}$
  $(\phi * \psi)(g) := \int_G \phi(x)\psi(x^{-1}g)\, dx$
- $\mathcal{H}$ generated by $T_i = \chi(K(1, \ldots, 1, p, \ldots, p)K)$
- $\mathcal{H}$ acts on $L^2(G/K)$ by $(f * \phi)(x) := \int_G \phi(y) f(xy)\, dy$
- T_i acts on $L^2(G/K)$ as a colored adjacency operator.
- T_i's commute.

Eigenvalues of A

- $\lambda \in \mathrm{Spec}(X) \iff T_i(f) = \lambda \cdot f$
- λ (one dimensional) representation of $\mathcal{H}$

$$\begin{array}{c} \text{one dimensional representations of } \mathcal{H} \\ \Big\updownarrow \ \text{1–1 (Satake isomorphism)} \\ \text{irreducible unramified representations of } G \end{array}$$

- $\Gamma\backslash X$ Ramanujan $\Leftrightarrow$ all irreducible unramified (unitary) representations of G appearing in $L^2(G/\Gamma)$ are tempered (RC).

Ramanujan Graphs and Generalizations

- $n = 2$: LPS (Jacquet-Langlands correspondence)
- $n = 3$: B. (Rogawski's classification of representations of $U(3)$ & Arthur's conjectures)
- general n (function filed case): Lubotzky-Samuels-Vishne (Laforgue's work)
- Ramanujan hypergraphs: Li (Laumot-Rapoport-Stuhler)

Biregular Ramanujan Graphs

- $B_{k,l,n}$ is a (k,l)-regular bigraph (biregular, bipartite) on n vertices

Theorem (Li, Feng)

$\liminf_{n\to\infty} |\lambda(B_{k,l,n})| \geq \sqrt{k-1} + \sqrt{l-1}$

Definition (Hashimoto)

A (q_1+1, q_2+1)-bigraph is called *Ramanujan bigraph* if

$$|\lambda^2 - q_1 - q_2| \leq 2\sqrt{q_1 q_2},$$

$\forall\ \lambda \in Spec(X),\ \lambda^2 \neq (1+q_1)(1+q_2)$.

Bigraphs from groups

- $G = U_3(\mathbb{Q}_p)$
- $\tilde{X}$ - Bruhat-Tits building of G

  $(q_1 + 1, q_2 + 1) = (p^3 + 1, p + 1)$ biregular, bipartite tree
- $X = \tilde{X}/\Gamma$, Γ discrete, co-compact subgroup of G
- Maximal compact subgroups of: $\{K_1\}, \{K_2\}$.
- $B = K_1 \cap K_2$

Adjacency and the Iwahori-Hecke algebra

- $T_1, T_2 \in End(\mathbb{Z}[E\tilde{X}])$: $T_i(e) := \sum_{\substack{e' \in \tilde{E}_i(e) \\ e' \neq e}} e' \quad (i = 1, 2)$
- T_1, T_2 induce naturally endomorphisms on $M^1(X)$,

$$((T_i)^* f)(e) := \sum_{e' \in E_i(e)} f(e') - f(e) \quad (i = 1, 2).$$

Theorem (Hashimoto)

$$Z_X(u)^{-1} = \det(I - (T_1 T_2)^* u)$$

- Iwahori-Hecke algebra
  $\mathcal{H} = \mathcal{H}(G, B) \cong \mathbb{C}[T_1, T_2]/\langle T_i^2 = (q_i - 1) T_i + q_i,\ i = 1, 2\rangle$

Hashimoto's results

- Finite dim'l irreducible representations φ of $\mathcal{H}$ have dimension 1 or 2 and they are determined by the characteristic polynomial p_φ of $\varphi(T_1 T_2)$.

- Degree two irreducible representations are parameterized by $c \in \mathbb{C}$, $c \neq 0$, $c \neq (q_1+1)(q_2+1)$.

$$p_\varphi(u) = 1 - (c - q_1 - q_2)u + q_1 q_2 u^2.$$

- $$c \in Spec(X)$$
  $$\updownarrow$$
  $\varphi \leftrightarrow \pi$ sherical unitary irreducible representation of G appearing in $L^2(G/\Gamma)$

Iwahori-Hecke Algebra

- (Bernstein-Lusztig presentation) $\mathcal{H} = \mathcal{H}_W \otimes \mathcal{A}$

  $\mathcal{H}_W = \mathbb{C}[T]/\langle T^2 = (z^{2\lambda} - 1)T + z^{2\lambda}\rangle$, $\mathcal{A} = \mathbb{C}[\theta]$

  commutation relation:

  $$\theta T - T\theta^{-1} = (z^{2\lambda} - 1)\theta + (z^{\lambda+\lambda^*} - z^{\lambda-\lambda^*})$$

  (for $SU_3(\mathbb{Q}_p)$, $z = \sqrt{p}$, $\lambda = 3$, $\lambda^* = 1$)

- Iwahori $\leftrightarrow$ Berenstein-Lusztig

  $T = T_1$, $\theta = \frac{1}{\sqrt{q_1 q_2}} T_1 T_2$, $z^{2\lambda} = q_1$ and $z^{2\lambda^*} = q_2$

Tempered representations

- $Tr(\Theta) = \frac{1}{\sqrt{q_1 q_2}} Tr(T_1 T_2) = \frac{c - q_1 - q_2}{\sqrt{q_1 q_2}}$.

$$\text{Ramanujan condition } |c - q_1 - q_2| \leq 2\sqrt{q_1 q_2}$$

$$\updownarrow$$

$$|Tr(\Theta)| \leq 2 \text{ (i.e., } \Theta \text{ is tempered)}$$

- θ acts by a scalar z^{ν}, $\nu \in \mathbb{C}$.
- The eigenvalues of θ are z^{ν} and $z^{-\nu}$.

Ramanujan bigraphs

Theorem (Rogawski)

If G is a compact inner form of $U(3)$ arising from a division algebra with an involution of the second kind, there are no non-tempered representations (the Ramanujan-Petersen conjecture is satisfied).

- Open problem (Arthur): If the group does not satisfy the Ramanujan conjecture, there are very few non-tempered representations. Thus there are very few exceptional eigenvalues. What is their combinatorial meaning?