Ramanujan Complexes and Their Applications

In memory of Beth Samuels

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Beth Sharon Samuels (1975-2007)

• Grew up in LA
• 1993-1997 BA in math, Columbia University, Cum Laude
• 1999-2005 PhD in math, Yale University
  Thesis advisor: I. I. Piatetski-Shapiro
  Thesis title: "Ramanujan complexes, non-uniform quotients and isospectrality"
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1. Spectral bounds for graphs

Let $X_j$ be a family of $(q + 1)$-regular graphs with $|X_j| \to \infty$ as $j \to \infty$. Denote by $\lambda^+(X_j)$ the largest eigenvalue $< q + 1$, and $\lambda^-(X_j)$ the smallest eigenvalue $> -(q + 1)$.

Alon-Boppana:

$$\liminf_{j \to \infty} \lambda^+(X_j) \geq 2\sqrt{q}.$$ 

Three proofs by Lubotzky-Phillips-Sarnak, Serre, Nilli, resp.

Li: Assume $X_j$’s satisfy

- the length of shortest odd cycle in $X_j$ tends to $\infty$ as $j \to \infty$.

Then

$$\limsup_{j \to \infty} \lambda^-(X_j) \leq -2\sqrt{q}.$$
2. Ramanujan graphs

A connected \((q + 1)\)-regular graph \(X\) is called Ramanujan if

\[ |\lambda^{\pm}(X)| \leq 2\sqrt{q}. \]

So Ramanujan graphs are optimal expanders from spectral viewpoint.

3. Another interpretation of R-graphs

The universal cover of \((q+1)\)-regular graphs is the infinite \((q+1)\)-regular tree \(\mathcal{T}\). On it there is also the adjacency operator \(A\). The operator spectrum of \(A\) is \([-2\sqrt{q}, 2\sqrt{q}]\).

Thus a graph is Ramanujan if and only if all of its nontrivial eigenvalues fall in the spectrum of its universal cover.
4. Connection with $PGL_2(F)$

When $q = p^r$ is a prime power, the $(q+1)$-regular tree $\mathcal{T}$ can be identified with the symmetric space $PGL_2(F)/PGL_2(\mathcal{O}_F)$, where $F$ is a local field with ring of integers $\mathcal{O}_F$, and the residue field $\mathcal{O}_F/\pi \mathcal{O}_F$ has cardinality $q$.

Eg. $F = \mathbb{Q}_p$ with $\mathcal{O}_F = \mathbb{Z}_p$ and $\pi = p$; or $F = \mathbb{F}_q((x))$ with $\mathcal{O}_F = \mathbb{F}_q[[x]]$ and $\pi = x$.

\[ \mathcal{T} = \frac{PGL_2(F)}{PGL_2(\mathcal{O}_F)}. \]

vertices $\leftrightarrow$ $PGL_2(\mathcal{O}_F)$-cosets

vertex adjacency operator $A$ $\leftrightarrow$ Hecke operator on

\[ PGL_2(\mathcal{O}_F) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} PGL_2(\mathcal{O}_F) \]
5. The Bruhat-Tits building $\mathcal{B}$ on $\text{PGL}_n(F)$

- $G = \text{PGL}_n(F)$, $K = \text{PGL}_n(\mathcal{O}_F)$
- The Bruhat-Tits building $\mathcal{B} = G/K$ is an $(n - 1)$-dim’l simplicial complex with vertex set $G/K$. The chambers are the $(n - 1)$-simplices; for $0 \leq r \leq n - 1$, the $r$-dimensional facets of the chambers are the $r$-simplices of the building $\mathcal{B}$. A chamber is also called an $n$-hyperedge.
- Each vertex $gK$ has a type in $\mathbb{Z}/n\mathbb{Z}$ given by
  \[
  \tau(gK) = \text{ord}_\pi \det g \mod n.
  \]

The 1-skeleton of $\mathcal{B}$ is an $n$-partite graph. The vertices in a chamber have different types.
• There are \( n - 1 \) Hecke operators \( A_l, l = 1, \ldots, n - 1 \), on double coset \( K \left[ \begin{array}{c}
\pi \\
\vdots \\
\pi \\
1 \\
\ldots \\
1
\end{array} \right] K \). Its action on \( L^2(G/K) \) is given by

\[
A_l f(x) = \sum_{y \text{ adj. to } x, \tau(y) = \tau(x) + l} f(y).
\]
• $\mathcal{B}$ is $(q + 1)$-regular, namely, for $1 \leq l \leq n - 1$, each vertex $x$ of $\mathcal{B}$ has exactly

$$q_{n,l} := \frac{(q^n - 1) \cdots (q - 1)}{(q^l - 1) \cdots (q - 1)(q^{n-l} - 1) \cdots (q - 1)}$$

neighbors of type $\tau(x) + l$.

Each $(n-2)$-dimensional simplex is contained in $q+1$ chambers.

• Topologically $\mathcal{B}$ is simply connected, so it is the universal cover of its finite quotients, which are $(n - 1)$-dim’l complexes or $n$-hypergraphs.
6. The spectrum of $A_l$ on $\mathcal{B}$

$\sigma_l(z_1, \ldots, z_n) = l$th elementary symmetric polynomial in the variables $z_1, \ldots, z_n$.

$$\Omega_{n,l} = \left\{ \sigma_l(z_1, \ldots, z_n) : z_j \in \mathbb{C}, \ |z_j| = 1 \text{ for } 1 \leq j \leq n, \right. \left. z_1 \cdots z_n = 1 \right\}$$

$\Omega_{n,l}$ is invariant under multiplication by $n$th roots of unity.

MacDonald: The spectrum of $A_l$ on $\mathcal{B}$ is $q^{l(n-l)/2} \Omega_{n,l}$. 
7. Spectral theory of regular complexes

Let \( \{X_j\} \) be a family of finite quotients of \( \mathcal{B} \) with \( |X_j| \to \infty \) as \( j \to \infty \). The trivial eigenvalues of \( A_l \) on \( X_j \) are \( q_{n,l}e^{2\pi ir/n} \).

DeGeorge, Li: Assume each \( X_j \) contains a ball isomorphic to a ball in \( \mathcal{B} \) with radius going to \( \infty \) as \( j \to \infty \). Then, for each \( 1 \leq l \leq n - 1 \), the closure of the collection of eigenvalues of \( A_l(X_j), j \geq 1 \), contains \( q^{l(n-l)/2}\Omega_{n,l} \).

Kang: Denote by \( \lambda_l^+(X) \) the largest nontrivial eigenvalue of \( A_l \) on \( X \) in absolute value. Then

\[
\liminf_{j \to \infty} \lambda_l^+(X_j) \geq q^{l(n-l)/2}\binom{n}{l}.
\]

Note that the lower bound is the radius of the smallest disc containing the spectrum of \( A_l \) on \( \mathcal{B} \).
8. Ramanujan complexes and explicit constructions

A finite \((q + 1)\)-regular quotient \(X\) of \(B\) is called a Ramanujan complex if, for \(1 \leq l \leq n - 1\), all nontrivial eigenvalues of \(A_l(X)\) fall in the region \(q^{l(n-l)/2}\Omega_{n,l}\).

**Theorem.** For \(q\) equal to a prime power and \(n \geq 2\), there exist explicitly constructed infinite families of \((q + 1)\)-regular \((n - 1)\)-dimensional Ramanujan complexes.

The explicit constructions rely on the Ramanujan conjecture over function fields. Choose \(F = \mathbb{F}_q((x))\) and a suitable division algebra \(D\) of degree \(n\) over \(\mathbb{F}_q(x)\). The infinite family comes from quotients \(\Gamma_j \backslash G / K\) by discrete subgroups \(\Gamma_j\) of \(G\) which are suitable congruence subgroups of \(\mathbb{F}_q(x)\)-points of \(D^\times\) mod center.
Three explicit constructions:

(1) Li (2004): Used result of Laumon-Rapoport-Stuhler on Ramanujan conjecture for certain representations of $D^\times$, local Jacquet-Langlands correspondence and trace formula.

Advantage: works for all $n \geq 2$.

Restriction: $D$ should ramify at least at four places.

(2) Lubotzky-Samuels-Vishne (2005): Used result of Lafforgue on Ramanujan conjecture for representations of $PGL_n$ and global Jacquet-Langlands correspondence.

(3) Sarveniazi (2007): Same theoretical ground and same result as (2), different construction.
Advantage: By taking $D$ ramified only at two places, one obtains (Ramanujan) complexes which are ”Cayley graphs” on subgroups of $PGL_n(\mathbb{F}_{q^d})$ containing $PSL_n(\mathbb{F}_{q^d})$ with explicit generators, similar to the Lubotzky-Phillips-Sarnak construction for the case $n = 2$.

Restriction: works where JL correspondence over function fields is established.
9. B. Samuels’ published work

Beth Samuels published 4 papers, all joint work with Lubotzky and Vishne, appeared in 2005-06.

- Explicit constructions of Ramanujan complexes of type $\tilde{A}_d$, European J. of Comb. (2005); discussed above.
- Ramanujan complexes of type $\tilde{A}_d$, Israel J. (2005)
  For $n \geq 3$, one has to be careful in choosing the discrete subgroup $\Gamma$ of $G$ in order that the finite quotient $\Gamma \backslash G/K$ is Ramanujan. A necessary condition is that the corresponding global representations of the group should not contain residual spectrum.
For each $n \geq 5$ ($n \neq 6$) and $q > 4n^2 + 1$ a prime power, by properly selecting two different sets of generators, one obtains isospectral, but non-isomorphic, Cayley graphs on $PSL_n(\mathbb{F}_q)$.

Let $F = \mathbb{R}$ or $\mathbb{C}$, $G = PGL_n(F)$, and $K$ its maximal compact subgroup. For $n \geq 3$, given any positive integer $r$, there are $r$ discrete cocompact torsion-free non-commensurable lattices $\Gamma_l$, $1 \leq l \leq r$, in $G$ such that the compact manifolds $\Gamma_l \backslash G/K$ are isospectral. Their construction uses division algebras ramified at the same places but with different invariants.
10. Zeta functions of graphs

• $X$: connected undirected finite graph
• Count backtrackless and tailless cycles.

• *Primitive* cycle: not repeating another cycle more than once.
The Ihara vertex zeta function of $X$ is defined as

$$Z(X; u) = \prod_{[C]} \frac{1}{1 - u^{l(C)}} = \exp \left( \sum_{n \geq 1} \frac{N_n}{n} u^n \right),$$

where $[C]$ runs through all equiv. classes of primitive backtrackless and tailless cycles $C$, $l(C)$ is the length of $C$, and $N_n$ is the number of backtrackless and tailless cycles of length $n$.

Endow two orientations on each edge of a finite graph $X$. Define the neighbors of $u \rightarrow v$ to be the edges $v \rightarrow w$ with $w \neq u$. Associate the edge adjacency matrix $A_e$.

Hashimoto: $N_n = \text{Tr} A_e^n$ so that

$$Z(X, u) = \frac{1}{\det(I - A_e u)}.$$
11. Properties of zeta functions for regular graphs

• Ihara (1968): Let $X$ be a finite $(q + 1)$-regular graph. Then its zeta function $Z(X, u)$ is a rational function of the form

$$Z(X; u) = \frac{(1 - u^2)\chi(X)}{\det(I - Au + qu^2I)} \left( = \frac{1}{\det(I - A_eu)} \right),$$

where $\chi(X) = \#V - \#E$ is the Euler characteristic of $X$ and $A$ is the (vertex) adjacency matrix of $X$.

• $X$ is Ramanujan if and only if $Z(X, u)$ satisfies RH, i.e. the nontrivial poles of $Z(X, u)$ all have absolute value $q^{-1/2}$.
When \( q = p^r \) is a prime power,

\[
\mathcal{T} = \frac{\text{PGL}_2(F)}{\text{PGL}_2(\mathcal{O}_F)}
\]

vertices ↔ \( \text{PGL}_2(\mathcal{O}_F) \)-cosets

vertex adjacency operator \( A \) ↔ Hecke operator on

\[
\text{PGL}_2(\mathcal{O}_F) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \text{PGL}_2(\mathcal{O}_F)
\]

directed edges ↔ \( \mathcal{I} \)-cosets (\( \mathcal{I} = \text{Iwahori subgroup} \))

edge adjacency operator \( A_e \) ↔ Iwahori-Hecke operator on

\[
\mathcal{I} \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} \mathcal{I}
\]

\( X = X_\Gamma = \Gamma \backslash \text{PGL}_2(F)/\text{PGL}_2(\mathcal{O}_F) = \Gamma \backslash \mathcal{T} \) for a torsion free discrete cocompact subgroup \( \Gamma \) of \( \text{PGL}_2(F) \).
12. Our goal

Would like to define a suitable zeta function, which counts tailless cycles up to homotopy in a finite complex arising as a quotient of the building, which possesses the following two properties:

- It is a rational function with closed form expression;
- The complex is Ramanujan if and only if its zeta satisfies RH.

Shall present results for $n = 3$, a joint work with Ming-Hsuan Kang.

This question was previously considered by Deitmar and Deitmar-Hoffman. Partial results.

Fix notation: $G = PGL_3(F)$, $K = PGL_3(O_F)$, and $\mathcal{B} = G/K$ from now on.
13. Parametrizations of the simplices in $B$

- $\sigma = \begin{pmatrix} 1 \\ \pi \\ 1 \end{pmatrix}$. Have a filtration of $K$:

$$K \supset E := K \cap \sigma K \sigma^{-1} \supset B := K \cap \sigma K \sigma^{-1} \cap \sigma^{-1} K \sigma.$$ 

- vertices $\leftrightarrow K$-cosets 
  Each vertex $gK$ has a type $\tau(gK) \in \mathbb{Z}/3\mathbb{Z}$.

- The type of an edge $gK \to g'K$ is $\tau(g'K) - \tau(gK) = 1$ or 2.

- type one edges $\leftrightarrow E$-cosets

- chambers $\leftrightarrow B$-cosets such that $gB$, $g\sigma B$ and $g\sigma^2 B$ represent the same chamber.
14. Operators on $\mathcal{B}$

Hecke operators $A_1$ and $A_2$ are on $K$-double cosets.

The $B$-double cosets define Iwahori-Hecke operators acting on $L^2(G/B)$. The $B$-double cosets of $G$ are represented by the Weyl group $W \rtimes \langle \sigma \rangle$, where $W$ is generated by the three reflections

\[
t_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & \pi^{-1} \\ \pi & 1 \end{pmatrix}, \quad \text{and} \quad t_3 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\]
15. Finite quotients of $\mathcal{B}$

$\Gamma$ : a torsion free discrete subgroup of $G$ with compact quotient.
$X_\Gamma = \Gamma \backslash G/K = \Gamma \backslash \mathcal{B}$

Two assumptions on $\Gamma$:

(I) $\text{ord}_\pi \det \Gamma \subset 3\mathbb{Z}$ so that $\Gamma$ identifies vertices of the same type.

(II) $\Gamma$ is *regular*, that is, the centralizer in $G$ of any nonidentity element in $\Gamma$ is a torus.

Division algebras of degree 3 yield many such $\Gamma$’s.
16. Homotopy cycles and closed galleries in $X_{\Gamma}$

- $\kappa_{\gamma}(gK)$: the homotopy class of the backtrackless paths from $gK$ to $\gamma gK$ in $\mathcal{B}$ and its image in $X_{\Gamma}$, where $\gamma \in \Gamma$.

- Suppose $g^{-1}\gamma g \in K \begin{pmatrix} 1 \\ \pi^m \\ \pi^{m+n} \end{pmatrix} K$. Then all geodesics from $gK$ to $\gamma gK$ use $n$ type one edges and $m$ type two edges and they are homotopic in $\mathcal{B}$. Say $\kappa_{\gamma}(gK)$ has geometric length $n + m$ and algebraic length $n + 2m$. When $m = 0$ or $n = 0$, we say $\kappa_{\gamma}(gK)$ has type one or two, accordingly. No homotopy in this case.
Figure 3: $\kappa_{\gamma}(gK)$
• A cycle \( \kappa_\gamma(gK) \) is called tailless if \( \kappa_\gamma(hK) \) has the same geometric length as \( \kappa_\gamma(gK) \) for all vertices \( hK \) lying on the cycle \( \kappa_\gamma(gK) \). Define primitive and equivalence as before.

Figure 4: a tailless type one cycle
A sequence of edge-adjacent chambers is called a gallery. The least number of reflections needed to go from the chamber \( gB \) to the chamber \( \gamma gB \) is the length of the backtrackless gallery \( \kappa_\gamma(gB) \).
Interested in tailless type one backtrackless closed galleries in $X_\Gamma$. Such a gallery has length $n = 3m$ and is uniquely represented by $\kappa_\gamma(gB)$, resulting from repeating $m$ times the reflection sequence $t_2t_1t_3$.

Figure 6: three type one galleries from a chamber
17. Type one chamber zeta function of $X_\Gamma$

The chamber zeta function of $X_\Gamma$ is defined as

$$Z_B(X_\Gamma, u) = \prod_{[\mathcal{G}]} \frac{1}{1 - u^{l(\mathcal{G})}},$$

where $[\mathcal{G}]$ runs through the equiv. classes of backtrackless primitive tailless type one closed galleries in $X_\Gamma$.

Denote by $L_B$ the Iwahori-Hecke operator supported on $Bt_2\sigma^2B$.

**Theorem.** $Z_B(X_\Gamma, u)$ is a rational function, given by

$$Z_B(X_\Gamma, u) = \frac{1}{\det(I - L_B u)}.$$

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18. Type one edge zeta function of $X_\Gamma$

The type one edge zeta function is defined as

$$Z_E(X_\Gamma, u) = \prod_{[\mathcal{C}]} \frac{1}{1 - u^l A(\mathcal{C})},$$

where $[\mathcal{C}]$ runs through the equiv. classes of backtrackless primitive tailless type one cycles in $X_\Gamma$.

**Theorem.** $Z_E(X_\Gamma, u)$ is a rational function, given by

$$Z_E(X_\Gamma, u) = \frac{1}{\det(I - L_E u)}.$$

Here $L_E$ is an operator on $L^2(G/E)$ given by the $E$-double coset $E(t_2\sigma^2)^2 E$. 
$L_E$ has a combinatorial interpretation: It is the “edge adjacency matrix” on the set of type one edges $\Gamma \backslash G/E$ of $X_\Gamma$ such that the neighbors of a type one edge $v \to v'$ are the $q^2$ type one edges $v' \to v''$ with $v''$ not adjacent to $v$.

Note that the type one vertex cycles traveled in reverse direction are the type two cycles, while the algebraic length is doubled. Define the zeta function of $X_\Gamma$ to be

$$Z(X_\Gamma, u) = Z_E(X_\Gamma, u)Z_E(X_\Gamma, u^2) = \prod_{[\mathcal{C}]} \frac{1}{1 - u'A(\mathcal{C})},$$

where $[\mathcal{C}]$ runs through the equiv. classes of backtrackless primitive tailless type one and type two cycles in $X_\Gamma$. 
19. The Main results for 2-dim’l complexes

Main Theorem (Kang-L.)

(1) $Z(X_\Gamma, u)$ is a rational function given by

$$Z(X_\Gamma, u) = \frac{(1 - u^3)\chi(X)}{\det(I - A_1u + qA_2u^2 - q^3u^3I)\det(I + L_Bu)},$$

where $\chi(X) = \#V - \#E + \#C$ is the Euler characteristic of $X_\Gamma$.

(2) $X_\Gamma$ is a Ramanujan complex if and only if $Z(X_\Gamma, u)$ satisfies RH.
Remarks. (1) Ramanujan complexes are defined in terms of the eigenvalues of $A_1$ and $A_2$, which are equivalent to the nontrivial zeros of $\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I)$ having absolute value $q^{-1}$. In this case, Kang-Li-Wang showed that the nontrivial zeros of $\det(I + L_B u)$ have absolute value $q^{-1/2}$.

(2) The zeta identity can be reformulated in terms of operators:

$$\frac{(1 - u^3)\chi(X)}{\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I)} = \frac{\det(I + L_B u)}{\det(I - L_E u) \det(I - L_E u^2)},$$

compared with the identity for graphs:

$$\frac{(1 - u^2)\chi(X)}{\det(I - Au + qu^2 I)} = \frac{1}{\det(I - Ae u)}.$$