Ramanujan Complexes and Their Applications

In memory of Beth Samuels

Winnie Li

Pennsylvania State University

Beth Sharon Samuels (1975-2007)

- Grew up in LA
- 1993-1997 BA in math, Columbia University, Cum Laude
- 1999-2005 PhD in math, Yale University

  Thesis advisor: I. I. Piatetski-Shapiro

  Thesis title: "Ramanujan complexes, non-uniform quotients and isospectrality"
- Assistant Prof at UC Berkeley, 2005-2007
- Graduated from a three year program in Bible and Talmud at the Drisha Institute for Jewish Education in New York
- Torah scholar
- auxiliary police officer in Manhattan

1. Spectral bounds for graphs

Let X_j be a family of $(q+1)$-regular graphs with $|X_j| \to \infty$ as $j \to \infty$. Denote by $\lambda^+(X_j)$ the largest eigenvalue $< q+1$, and $\lambda^-(X_j)$ the smallest eigenvalue $> -(q+1)$.

Alon-Boppana:

$$\liminf_{j\to\infty} \lambda^+(X_j) \geq 2\sqrt{q}.$$

Three proofs by Lubotzky-Phillips-Sarnak, Serre, Nilli, resp.

Li: Assume X_j's satisfy

- the length of shortest odd cycle in X_j tends to ∞ as $j \to \infty$.

Then

$$\limsup_{j\to\infty} \lambda^-(X_j) \leq -2\sqrt{q}.$$

2. Ramanujan graphs

A connected $(q+1)$-regular graph X is called Ramanujan if

$$|\lambda^{\pm}(X)| \leq 2\sqrt{q}.$$

So Ramanujan graphs are optimal expanders from spectral viewpoint.

Explicit constructions of infinite families of Ramanujan graphs for $q = p^a$: Margulis (1988), Lubotzky-Phillips-Sarnak (1988), Mestre-Oesterlé (1986), Pizer (1990), Morgenstern (1994)

3. Another interpretation of R-graphs

The universal cover of $(q+1)$-regular graphs is the infinite $(q+1)$-regular tree $\mathcal{T}$. On it there is also the adjacency operator A. The operator spectrum of A is $[-2\sqrt{q},\ 2\sqrt{q}]$.

Thus a graph is Ramanujan if and only if all of its nontrivial eigenvalues fall in the spectrum of its universal cover.

4. Connection with $PGL_2(F)$

When $q = p^r$ is a prime power, the $(q+1)$-regular tree $\mathcal{T}$ can be identified with the symmetric space $\mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O}_F)$, where F is a local field with ring of integers $\mathcal{O}_F$, and the residue field $\mathcal{O}_F/\pi\mathcal{O}_F$ has cardinality q.

Eg. $F = \mathbb{Q}_p$ with $\mathcal{O}_F = \mathbb{Z}_p$ and $\pi = p$; or $F = \mathbb{F}_q((x))$ with $\mathcal{O}_F = \mathbb{F}_q[[x]]$ and $\pi = x$.

$$\begin{array}{rcl} \mathcal{T} & = & \mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O}_F). \\ \text{vertices} & \leftrightarrow & \mathrm{PGL}_2(\mathcal{O}_F)\text{-cosets} \\ \text{vertex adjacency operator } A & \leftrightarrow & \text{Hecke operator on} \end{array}$$

$$\mathrm{PGL}_2(\mathcal{O}_F)\begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix}\mathrm{PGL}_2(\mathcal{O}_F)$$

5. The Bruhat-Tits building $\mathcal{B}$ on $\mathbf{PGL}_n(F)$

- $G = \mathrm{PGL}_n(F)$, $K = \mathrm{PGL}_n(\mathcal{O}_F)$
- The Bruhat-Tits building $\mathcal{B} = G/K$ is an $(n-1)$-dim'l simplicial complex with vertex set G/K. The chambers are the $(n-1)$-simplices; for $0 \le r \le n-1$, the r-dimensional facets of the chambers are the r-simplices of the building $\mathcal{B}$.
  A chamber is also called an n-hyperedge.
- Each vertex gK has a type in $\mathbb{Z}/n\mathbb{Z}$ given by

$$\tau(gK) = \mathrm{ord}_\pi \det g \mod n.$$

The 1-skeleton of $\mathcal{B}$ is an n-partite graph. The vertices in a chamber have different types.

- There are $n-1$ Hecke operators A_l, $l = 1, ..., n-1$, on double coset $K \begin{pmatrix} \left.\begin{matrix} \pi & & \\ & \ddots & \\ & & \pi \end{matrix}\right\} l & \\ & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \end{pmatrix} K$. Its action on $L^2(G/K)$ is given by

$$A_l f(x) = \sum_{y \text{ adj. to } x,\ \tau(y)=\tau(x)+l} f(y).$$

- $\mathcal{B}$ is $(q+1)$-regular, namely, for $1 \le l \le n-1$, each vertex x of $\mathcal{B}$ has exactly

$$q_{n,l} := \frac{(q^n-1)\cdots(q-1)}{(q^l-1)\cdots(q-1)(q^{n-l}-1)\cdots(q-1)}$$

neighbors of type $\tau(x)+l$.

Each $(n-2)$-dimensional simplex is contained in $q+1$ chambers.

- Topologically $\mathcal{B}$ is simply connected, so it is the universal cover of its finite quotients, which are $(n-1)$-dim'l complexes or n-hypergraphs.

6. The spectrum of A_l on $\mathcal{B}$

$\sigma_l(z_1, ..., z_n) = l$th elementary symmetric polynomial in the variables $z_1, ..., z_n$.

$$\Omega_{n,l} = \Big\{ \sigma_l(z_1, ..., z_n) : \quad z_j \in \mathbb{C}, \ |z_j| = 1 \text{ for } 1 \le j \le n, \\ z_1 \cdots z_n = 1 \Big\}$$

$\Omega_{n,l}$ is invariant under multiplication by nth roots of unity.

MacDonald: The spectrum of A_l on $\mathcal{B}$ is $q^{l(n-l)/2}\Omega_{n,l}$.

7. Spectral theory of regular complexes

Let $\{X_j\}$ be a family of finite quotients of $\mathcal{B}$ with $|X_j| \to \infty$ as $j \to \infty$. The trivial eigenvalues of A_l on X_j are $q_{n,l}e^{2\pi ir/n}$.

DeGeorge, Li: Assume each X_j contains a ball isomorphic to a ball in $\mathcal{B}$ with radius going to ∞ as $j \to \infty$. Then, for each $1 \leq l \leq n-1$, the closure of the collection of eigenvalues of $A_l(X_j)$, $j \geq 1$, contains $q^{l(n-l)/2}\Omega_{n,l}$.

Kang: Denote by $\lambda_l^+(X)$ the largest nontrivial eigenvalue of A_l on X in absolute value. Then

$$\liminf_{j\to\infty} \lambda_l^+(X_j) \geq q^{l(n-l)/2}\binom{n}{l}.$$

Note that the lower bound is the radius of the smallest disc containing the spectrum of A_l on $\mathcal{B}$.

8. Ramanujan complexes and explicit constructions

A finite $(q+1)$-regular quotient X of $\mathcal{B}$ is called a *Ramanujan complex* if, for $1 \le l \le n-1$, all nontrivial eigenvalues of $A_l(X)$ fall in the region $q^{l(n-l)/2}\Omega_{n,l}$.

Theorem. *For q equal to a prime power and $n \geq 2$, there exist explicitly constructed infinite families of $(q+1)$-regular $(n-1)$-dimensional Ramanujan complexes.*

The explicit constructions rely on the Ramanujan conjecture over function fields. Choose $F = \mathbb{F}_q((x))$ and a suitable division algebra D of degree n over $\mathbb{F}_q(x)$. The infinite family comes from quotients $\Gamma_j \backslash G/K$ by discrete subgroups Γ_j of G which are suitable congruence subgroups of $\mathbb{F}_q(x)$-points of $D^\times$ mod center.

Three explicit constructions:

(1) Li (2004): Used result of Laumon-Rapoport-Stuhler on Ramanujan conjecture for certain representations of $D^{\times}$, local Jacquet-Langlands correspondence and trace formula.
Advantage: works for all $n \geq 2$.
Restriction: D should ramify at least at four places.

(2) Lubotzky-Samuels-Vishne (2005): Used result of Lafforgue on Ramanujan conjecture for representations of PGL_n and global Jacquet-Langlands correspondence.

(3) Sarveniazi (2007): Same theoretical ground and same result as (2), different construction.

Advantage: By taking D ramified only at two places, one obtains (Ramanujan) complexes which are "Cayley graphs" on subgroups of $PGL_n(\mathbb{F}_{q^d})$ containing $PSL_n(\mathbb{F}_{q^d})$ with explicit generators, similar to the Lubotzky-Phillips-Sarnak construction for the case $n = 2$.

Restriction: works where JL correspondence over function fields is established.

9. B. Samuels' published work

Beth Samuels published 4 papers, all joint work with Lubotzky and Vishne, appeared in 2005-06.

- Explicit constructions of Ramanujan complexes of type $\tilde{A}_d$, European J. of Comb. (2005); discussed above.
- Ramanujan complexes of type $\tilde{A}_d$, Israel J. (2005)

  For $n \geq 3$, one has to be careful in choosing the discrete subgroup Γ of G in order that the finite quotient $\Gamma \backslash G/K$ is Ramanujan. A necessary condition is that the corresponding global representations of the group should not contain residual spectrum.

- Isospectral Cayley graphs of some finite simple groups, Duke Math. J. (2006)

  For each $n \geq 5$ $(n \neq 6)$ and $q > 4n^2 + 1$ a prime power, by properly selecting two different sets of generators, one obtains isospectral, but non-isomorphic, Cayley graphs on $PSL_n(\mathbb{F}_q)$.

- Division algebras and non-commensurable isospectral manifolds, Duke Math. J. (2006)

  Let $F = \mathbb{R}$ or $\mathbb{C}$, $G = PGL_n(F)$, and K its maximal compact subgroup. For $n \geq 3$, given any positive integer r, there are r discrete cocompact torsion-free non-commensurable lattices Γ_l, $1 \leq l \leq r$, in G such that the compact manifolds $\Gamma_l \backslash G/K$ are isospectral. Their construction uses division algebras ramified at the same places but with different invariants.

10. Zeta functions of graphs

- X : connected undirected finite graph
- Count backtrackless and tailless cycles.

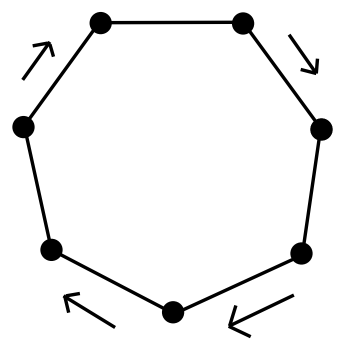

Figure 1: without tail

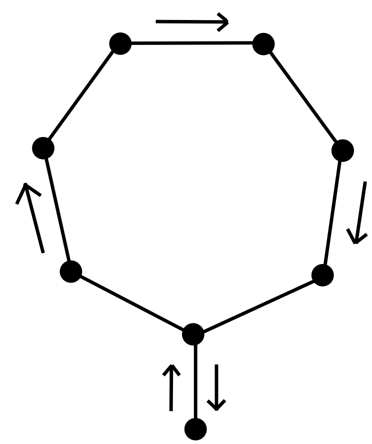

Figure 2: with tail

- *Primitive* cycle: not repeating another cycle more than once.

The Ihara vertex zeta function of X is defined as

$$Z(X;u) = \prod_{[C]} \frac{1}{1 - u^{l(C)}} = \exp\left(\sum_{n \geq 1} \frac{N_n}{n} u^n\right),$$

where $[C]$ runs through all equiv. classes of primitive backtrackless and tailless cycles C, $l(C)$ is the length of C, and N_n is the number of backtrackless and tailless cycles of length n.

Endow two orientations on each edge of a finite graph X. Define the neighbors of $u \to v$ to be the edges $v \to w$ with $w \neq u$. Associate the edge adjacency matrix A_e.

Hashimoto: $N_n = \mathrm{Tr} A_e^n$ so that

$$Z(X, u) = \frac{1}{\det(I - A_e u)}.$$

11. Properties of zeta functions for regular graphs

- Ihara (1968): Let X be a finite $(q+1)$-regular graph. Then its zeta function $Z(X,u)$ is a rational function of the form

$$Z(X;u) = \frac{(1-u^2)^{\chi(X)}}{\det(I - Au + qu^2 I)} \;\Big(= \frac{1}{\det(I - A_e u)}\Big),$$

where $\chi(X) = \#V - \#E$ is the Euler characteristic of X and A is the (vertex) adjacency matrix of X.

- X is Ramanujan if and only if $Z(X,u)$ satisfies RH, i.e. the nontrivial poles of $Z(X,u)$ all have absolute value $q^{-1/2}$.

When $q = p^r$ is a prime power,

$$\begin{array}{rcl} \mathcal{T} & = & \mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O}_F) \\ \text{vertices} & \leftrightarrow & \mathrm{PGL}_2(\mathcal{O}_F)\text{-cosets} \\ \text{vertex adjacency operator } A & \leftrightarrow & \text{Hecke operator on} \\ & & \mathrm{PGL}_2(\mathcal{O}_F)\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\mathrm{PGL}_2(\mathcal{O}_F) \\ \text{directed edges} & \leftrightarrow & \mathcal{I}\text{-cosets } (\mathcal{I} = \text{Iwahori subgroup}) \\ \text{edge adjacency operator } A_e & \leftrightarrow & \text{Iwahori-Hecke operator on} \\ & & \mathcal{I}\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\mathcal{I} \end{array}$$

$X = X_\Gamma = \Gamma\backslash\mathrm{PGL}_2(F)/\mathrm{PGL}_2(\mathcal{O}_F) = \Gamma\backslash\mathcal{T}$ for a torsion free discrete cocompact subgroup Γ of $PGL_2(F)$.

12. Our goal

Would like to define a suitable zeta function, which counts tailless cycles up to homotopy in a finite complex arising as a quotient of the building, which possesses the following two properties:

- It is a rational function with closed form expression;
- The complex is Ramanujan if and only if its zeta satisfies RH.

Shall present results for $n = 3$, a joint work with Ming-Hsuan Kang.

This question was previously considered by Deitmar and Deitmar-Hoffman. Partial results.

Fix notation: $G = PGL_3(F)$, $K = PGL_3(\mathcal{O}_F)$, and $\mathcal{B} = G/K$ from now on.

13. Parametrizations of the simplices in $\mathcal{B}$

- $\sigma = \begin{pmatrix} & 1 & \\ & & 1 \\ \pi & & \end{pmatrix}$. Have a filtration of K:

$$K \supset E := K \ \cap \ \sigma K \sigma^{-1} \supset B := K \ \cap \ \sigma K \sigma^{-1} \cap \ \sigma^{-1} K \sigma.$$

- vertices $\leftrightarrow$ K-cosets

  Each vertex gK has a type $\tau(gK) \in \mathbb{Z}/3\mathbb{Z}$.

- The type of an edge $gK \to g'K$ is $\tau(g'K) - \tau(gK) = 1$ or 2.
- type one edges $\leftrightarrow$ E-cosets
- chambers $\leftrightarrow$ B-cosets such that gB, $g\sigma B$ and $g\sigma^2 B$ represent the same chamber.

14. Operators on $\mathcal{B}$

Hecke operators A_1 and A_2 are on K-double cosets.

The B-double cosets define Iwahori-Hecke operators acting on $L^2(G/B)$. The B-double cosets of G are represented by the Weyl group $W \ltimes < \sigma >$, where W is generated by the three reflections

$$t_1 = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}, \quad t_2 = \begin{pmatrix} & & \pi^{-1} \\ & 1 & \\ \pi & & \end{pmatrix}, \quad \text{and} \quad t_3 = \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}.$$

15. Finite quotients of $\mathcal{B}$

Γ : a torsion free discrete subgroup of G with compact quotient.
$X_\Gamma = \Gamma\backslash G/K = \Gamma\backslash\mathcal{B}$

Two assumptions on Γ:

(I) $\mathrm{ord}_\pi \det \Gamma \subset 3\mathbb{Z}$ so that Γ identifies vertices of the same type.

(II) Γ is *regular*, that is, the centralizer in G of any nonidentity element in Γ is a torus.

Division algebras of degree 3 yield many such Γ's.

16. Homotopy cycles and closed galleries in X_Γ

- $\kappa_\gamma(gK)$: the homotopy class of the backtrackless paths from gK to γgK in $\mathcal{B}$ and its image in X_Γ, where $\gamma \in \Gamma$.
- Suppose $g^{-1}\gamma g \in K \begin{pmatrix} 1 & & \\ & \pi^m & \\ & & \pi^{m+n} \end{pmatrix} K$. Then all geodesics from gK to γgK use n type one edges and m type two edges and they are homotopic in $\mathcal{B}$. Say $\kappa_\gamma(gK)$ has geometric length $n+m$ and algebraic length $n+2m$. When $m=0$ or $n=0$, we say $\kappa_\gamma(gK)$ has *type one* or *two*, accordingly. No homotopy in this case.

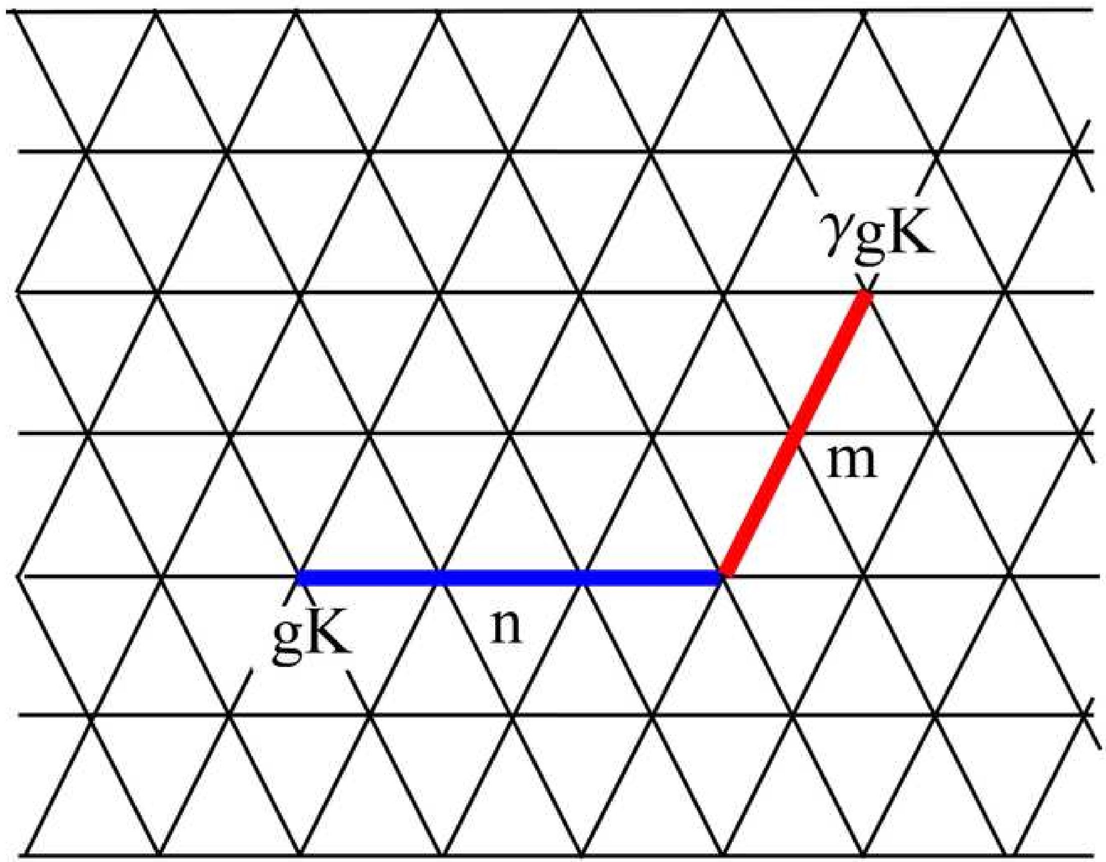

Figure 3: $\kappa_\gamma(gK)$

- A cycle $\kappa_\gamma(gK)$ is called *tailless* if $\kappa_\gamma(hK)$ has the same geometric length as $\kappa_\gamma(gK)$ for all vertices hK lying on the cycle $\kappa_\gamma(gK)$. Define primitive and equivalence as before.

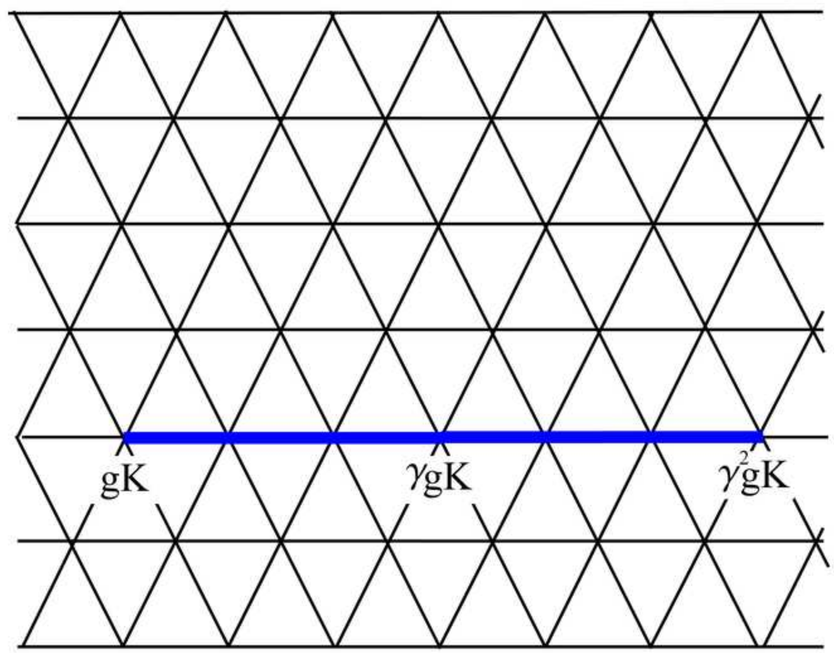

Figure 4: a tailless type one cycle

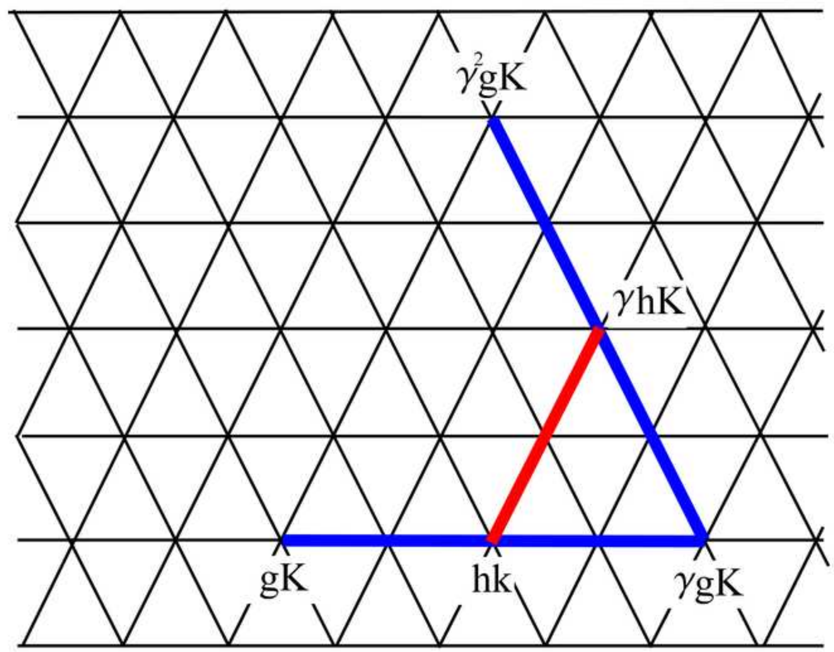

Figure 5: a type one cycle with tail

A sequence of edge-adjacent chambers is called a gallery. The least number of reflections needed to go from the chamber gB to the chamber γgB is the length of the backtrackless gallery $\kappa_\gamma(gB)$.

Interested in tailless type one backtrackless closed galleries in X_Γ. Such a gallery has length $n = 3m$ and is uniquely represented by $\kappa_\gamma(gB)$, resulting from repeating m times the reflection sequence $t_2 t_1 t_3$.

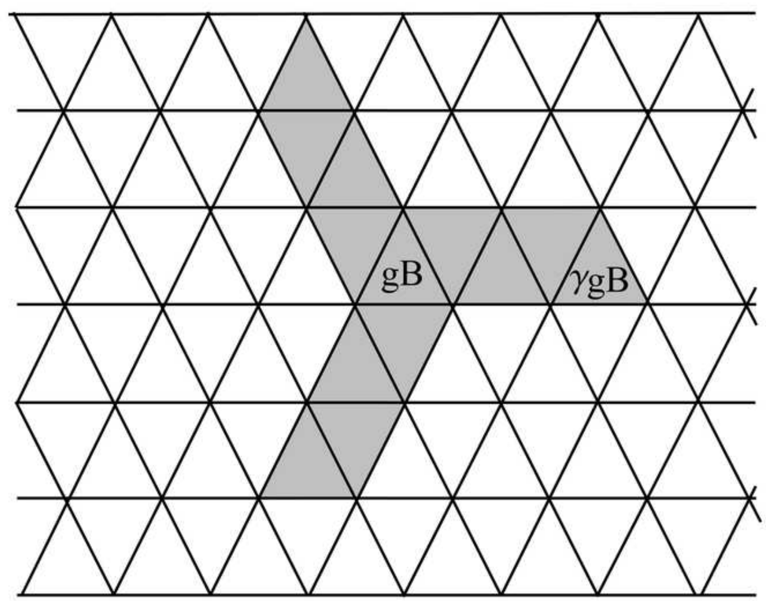

Figure 6: three type one galleries from a chamber

17. Type one chamber zeta function of X_Γ

The chamber zeta function of X_Γ is defined as

$$Z_B(X_\Gamma, u) = \prod_{[\mathfrak{G}]} \frac{1}{1 - u^{l(\mathfrak{G})}} \ ,$$

where $[\mathfrak{G}]$ runs through the equiv. classes of backtrackless primitive tailless type one closed galleries in X_Γ.

Denote by L_B the Iwahori-Hecke operator supported on $Bt_2\sigma^2 B$.

Theorem. $Z_B(X_\Gamma, u)$ *is a rational function, given by*

$$Z_B(X_\Gamma, u) = \frac{1}{\det(I - L_B u)}.$$

18. Type one edge zeta function of X_Γ

The type one edge zeta function is defined as

$$Z_E(X_\Gamma, u) = \prod_{[\mathfrak{C}]} \frac{1}{1 - u^{l_A(\mathfrak{C})}},$$

where $[\mathfrak{C}]$ runs through the equiv. classes of backtrackless primitive tailless type one cycles in X_Γ.

Theorem. $Z_E(X_\Gamma, u)$ *is a rational function, given by*

$$Z_E(X_\Gamma, u) = \frac{1}{\det(I - L_E u)}.$$

Here L_E is an operator on $L^2(G/E)$ given by the E-double coset $E(t_2\sigma^2)^2 E$.

L_E has a combinatorial interpretation: It is the "edge adjacency matrix" on the set of type one edges $\Gamma\backslash G/E$ of X_Γ such that the neighbors of a type one edge $v \to v'$ are the q^2 type one edges $v' \to v''$ with v'' not adjacent to v.

Note that the type one vertex cycles traveled in reverse direction are the type two cycles, while the algebraic length is doubled. Define the zeta function of X_Γ to be

$$Z(X_\Gamma, u) = Z_E(X_\Gamma, u) Z_E(X_\Gamma, u^2) = \prod_{[\mathfrak{C}]} \frac{1}{1 - u^{l_A(\mathfrak{C})}},$$

where $[\mathfrak{C}]$ runs through the equiv. classes of backtrackless primitive tailless type one and type two cycles in X_Γ.

19. The Main results for 2-dim'l complexes

Main Theorem (Kang-L.)

(1) $Z(X_\Gamma, u)$ *is a rational function given by*

$$Z(X_\Gamma, u) = \frac{(1-u^3)^{\chi(X)}}{\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I)\det(I + L_B u)},$$

where $\chi(X) = \#V - \#E + \#C$ *is the Euler characteristic of* X_Γ*.*

(2) X_Γ *is a Ramanujan complex if and only if* $Z(X_\Gamma, u)$ *satisfies RH.*

Remarks. (1) Ramanujan complexes are defined in terms of the eigenvalues of A_1 and A_2, which are equivalent to the nontrivial zeros of $\det(I - A_1u + qA_2u^2 - q^3u^3I)$ having absolute value q^{-1}. In this case, Kang-Li-Wang showed that the nontrivial zeros of $\det(I + L_Bu)$ have absolute value $q^{-1/2}$.

(2) The zeta identity can be reformulated in terms of operators:

$$\frac{(1-u^3)^{\chi(X)}}{\det(I - A_1u + qA_2u^2 - q^3u^3I)} = \frac{\det(I + L_Bu)}{\det(I - L_Eu)\det(I - L_Eu^2)},$$

compared with the identity for graphs:

$$\frac{(1-u^2)^{\chi(X)}}{\det(I - Au + qu^2I)} = \frac{1}{\det(I - A_eu)}.$$