PROPERTY τ

AND

HYPERBOLIC 3-MANIFOLDS

Alan W. Reid

Property τ

Let Γ be a group, S a finite symmetric set of generators of Γ.

Let $\mathcal{L} = \{N_i\}$ be a family of finite index (normal) subgroups of Γ.

Say Γ has **Property τ with respect to $\mathcal{L}$** if the quotient Cayley graphs $X(\Gamma/N_i, S)$ form a family of expanders.

Say Γ has **Property τ** if $\mathcal{L}$ is the family of all normal subgroups.

"Towers with τ"

1. If Γ has Property T then Γ has Property τ.

e.g. $\mathrm{SL}(\mathrm{n}, \mathbf{Z})$, $n \geq 3$.

2. Congruence arithmetic manifolds

Theorem:(Clozel) *Suppose* Γ *is an arithmetic subgroup of a semisimple Lie group and* $\mathcal{L}$ *is the family of congruence subgroups of* Γ*, then* Γ *has Property* τ *with respect to* $\mathcal{L}$*.*

e.g. The classical setting:

The congruence subgroups $\Gamma(n) < \mathrm{PSL}(2, \mathbf{Z})$ have $\lambda_1(\mathbf{H}^2/\Gamma(n)) \geq \frac{3}{16}$ (Selberg).

Hyperbolic Manifolds

Let $\mathbf{H}^n$ denote hyperbolic n-space.

The full group of isometries (resp. orientation-preserving isometries) is $\mathrm{O}_0(\mathrm{n},1)$ (resp. $\mathrm{SO}_0(\mathrm{n},1)$).

In low dimensions ($n = 2, 3$):

$$\mathrm{SO}_0(\mathrm{n},1) \cong \mathrm{PSL}(2,\mathbf{R}), \mathrm{PSL}(2,\mathbf{C}) \quad \text{respectively.}$$

Conjecture (Lubotzky-Sarnak): *Let* $\Gamma < \mathrm{O_0(n,1)}$ *be a lattice. Then* Γ *does not have Property* τ*.*

Remark: If Γ is torsion-free, using the dictionary between the discrete side (ie the Cayley graphs of quotients) and hyperbolic geometry of the covers, we need to find a tower of regular covers $\{M_i\}$ with $h(M_i) \to 0$.

How do you find such a tower.

Observation: *If* Γ *contains a finite index subgroup* Γ_0 *such that* $\Gamma_0 \to \mathbf{Z}$ *(surjecting) then* Γ *does not have Property* τ*.*

For hyperbolic manifolds in dimensions different from 3 and 7 we have using work Li-Millson, Lubotzky.

Theorem: *If* $n \neq 3, 7$ *and* Γ *is an "arithmetic" subgroup of* $\mathrm{O}_0(\mathrm{n}, 1)$ *then* Γ *does not have Property* τ*.*

Remark: Although we have seen they do have towers with τ from congruence subgroups.

Property τ

and the topology of hyperbolic 3-manifolds

One of the basic tools in 3-manifold topology is to study maps

$$f : S \longrightarrow M$$

where S is a closed orientable surface and M a compact 3-manifold.

Classical example: Heegaard splitting.

Let M be a closed orientable 3-manifold, then $M = U \cup_S V$ where

The minimal genus of such a splitting surface is called the **Heegaard genus of M**.

At the other extreme from a Heegaard surface is an **incompressible surface**.

In this case, S a closed orientable surface of genus at least 1, and $f_* : \pi_1(S) \to \pi_1(M)$ is injective.

Remark: The map f need not be an embedding.

A closed orientable hyperbolic 3-manifold M is called **Haken** if there is an embedded incompressible surface (hyperbolicity forces the surface to have genus ≥ 2).

Post Perelman, the fundamental conjectures about the topology of (hyperbolic) 3-manifolds are:

Let M be a closed orientable hyperbolic 3-manifold. Then:

1. *M is virtually Haken.*

2. *M has a finite a finite sheeted cover N for which $b_1(N) > 0$ (i.e. N has positive first Betti number).*

Strong version of 2. *M has a finite sheeted cover N that fibers over the circle.*

Improved versions of **2**:

3. $\sup\{b_1(X) : X \to M \textit{ a finite sheeted cover}\} = \infty$.

4. *M has a finite a finite sheeted N for which $\pi_1(N)$ surjects a non-abelian free group (say $\pi_1(M)$ is large).*

Comments on these conjectures and not τ

1. Note that a positive solution to 2, 3 or 4 would imply that the fundamental groups of hyperbolic 3-manifolds do not have Property τ

i.e. the Lubotzky-Sarnak Conjecture holds!

Remark: If M is not hyperbolic and $\pi_1(M)$ is infinite then (assuming geometrization), $\pi_1(M)$ does not have Property τ.

2. In the converse direction we have

Theorem:(Lackenby-Long-Reid)*Suppose that for every compact orientable 3-manifold M with infinite fundamental group, $\pi_1(M)$ fails to have Property τ. Then any arithmetic hyperbolic 3-manifold has large fundamental group.*

3. LERF and Property τ.

Suppose M is virtually Haken, then $\pi_1(M)$ contains a surface subgroup.

4. *Does* $\pi_1(M)$ *contain a surface subgroup?*

If it does, you get a surface and to prove virtually Haken you need to embed a surface in finite cover.

How do you do that?

Definiton: *G a f.g. group. Say G is LERF if the following holds for every f.g. subgroup H of G. Given any $g \in G \setminus H$, there is a finite index subgroup N of G such that $H < N$ but $g \notin N$.*

i.e. H is closed in the profinite topology on G.

LERF + surface group implies a positive answer to Lubotzky-Sarnak conjecture (in fact implies large).

Question: *Does LERF imply not τ?*

Heegaard splittings and Property τ

Let $M = \mathbf{H}^3/\Gamma$ and $\mathcal{L} = \{M_i = \mathbf{H}^3/\Gamma_i\}$ a family of finite sheeted covers of M ($d_i = [\Gamma : \Gamma_i]$).

If M has Heegaard genus g then a minimal genus splitting of M lifts to determine a splitting of M_i of genus ($d_i(g-1)+1$ by Riemann-Hurwitz).

This is almost never a minimal genus splitting!

The study of how Heegaard splittings behave in towers of finite sheeted covers has connections to:

The virtual Haken question raised earlier (Casson-Gordon).

The question of “rank of $\pi_1(M)$ versus Heegaard genus of M”.

Work of Lackenby connects Property τ to the behavior of Heegaard genus in towers.

Define the **Infimal Heegaard Gradient** of M with respect to $\mathcal{L}$ is:

$$\mathrm{inf_i} \frac{\chi_-^{\mathrm{h}}(\mathrm{M_i})}{[\pi_1(\mathrm{M}) : \pi_1(\mathrm{M_i})]},$$

where

$\chi_-^h(M_i)$ denotes the minimal value for the negative of the Euler characteristic of a Heegaard surface in M_i.

The previous remark says that $\chi_-^h(M_i) \leq d_i \chi_-^h(M)$. In fact, some 3-manifold topology implies that Infimal Heegaard Gradient of M with respect to $\mathcal{L}$ is stricly less than $\chi_-^h(M)$.

Example: If M is fibered then the infimal Heegaard gradient of M with respect to the tower of finite cyclic covers dual to the fiber is zero.

Theorem (Lackenby): *Let* $M = \mathbf{H}^3/\Gamma$ *be a closed orientable hyperbolic 3-manifold and*

$$\mathcal{L} = \{M_i = \mathbf{H}^3/\Gamma_i\}$$

a family such that Γ *has Property* τ *with respect to* $\mathcal{L}$*.*

Then the infimal Heegaard gradient of M *with respect to* $\mathcal{L}$ *is positive.*

Corollary: *If* M *is arithmetic then* M *has positive infimal Heegaard gradient with respect to its family of congruence covers.*

From Additive Combinatorics to Heegaard Splittings

Want to extend the previous corollary to general hyperbolic 3-manifolds.

Theorem 1 (Long-Lubotzky-Reid): *Let Γ be a f.g. non-elementary Kleinian group.*

Then Γ has a co-final family of finite index normal subgroups $\mathcal{L} = \{N_i\}$ with respect to which Γ has Property τ.

Corollary: *Let M be a closed hyperbolic 3-manifold. Then M has a co-final family of finite sheeted covers for which the infimal Heegaard gradient is positive.*

Theorem 1 follows from:

Theorem 2: *Let k be a number field and let* Γ *be a f.g. subgroup of* $\mathrm{SL}(2, \mathrm{k})$ *which is not virtually soluble.*

Then Γ *has a co-final family of finite index normal subgroups* $\mathcal{L} = \{N_i\}$ *with respect to which* Γ *has Property* τ.

Remark: This can be viewed as providing a first step towards a generalization of Clozel's result ("Property τ for congruence arithmetic lattices in all semi-simple Lie groups") to non-arithmetic lattices.

Proof of Theorem 2 uses the following results:

Proposition A: *Let* Γ *be a group, and* $\mathcal{L} = \{N_i\}$ *a family of finite index normal subgroups of* Γ. *Suppose that* $H < \Gamma$ *(not necessarily of finite index), and assume that* H *surjects onto the finite quotients* Γ/N_i *for all but a finite number of* i.
Then if H *has Property* τ *with respect to the family* $\{H \cap N_i\}$, Γ *has Property* τ *with respect to* $\mathcal{L}$.

Theorem(Bourgain-Gamburd): *Suppose that for each* p*,* S_p *is some symmetric generating set for* $\mathrm{SL}(2,\mathrm{p})$ *of fixed size independent of* p*, such that the girth of the Cayley graph* $X(\mathrm{SL}(2,\mathrm{p}),\mathrm{S_p})$ *at least* $C\log(p)$ *(where* C *is independent of* p*). Then* $X(\mathrm{SL}(2,\mathrm{p}),\mathrm{S_p})$ *form a family of expanders.*

Girth: The *girth* of a finite graph X is the length of the shortest non-trivial closed path in X.

Lemma(Extended Margulis): *There is a constant* $C = C(a,b)$ *so that the girth of the Cayley graph of* $\mathrm{SL}(2,\mathrm{p})$ *with respect to the generating set*

$$\{\pi_{\mathcal{P}}(a^{\pm 1}), \pi_{\mathcal{P}}(b^{\pm 1})\}$$

is at least $C\log(p)$*.*

From Co-final to Nested

Say that $\{\Gamma_i\}$ is a nested sequence of normal subgroups of Γ if $\{\Gamma_i\}$ is co-final and:

$$\Gamma > \Gamma_1 > \Gamma_2 > \ldots \Gamma_n > \ldots$$

The sequences constructed in the proof of Theorem 2 (using Bourgain-Gamburd) are not nested.

Recent work of Bourgain-Gamburd-Sarnak extends some of Bourgain-Gamburd to products of primes, but this is still not enough for us.

Why care? Ideas of Abert-Nikolov have provided impetus for constructing **nested families with Property** τ in relation to the rank vs Heegaard genus problem.

Final Comments

It seems reasonable to expect the following conjecture to hold.

Conjecture: *Let* Γ *be a f.g. subgroup of* $\mathrm{GL}(\mathrm{n}, \mathbf{C})$ *whose Zariski Closure is semi-simple.*
Then Γ *has a co-final (nested) family* $\mathcal{L} = \{N_i\}$ *for which* Γ *has Property* τ *with respect to* $\mathcal{L}$.
This Conjecture would provide a far reaching generalization of Clozel's work mentioned earlier.