Moore's Bound and Ramanujan Complexes

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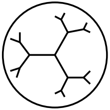
Moore's Bound for Graphs

$G = (V, E)$ a graph, $|V| = n$.
Minimal degree $\delta(G) = k \geq 3$.
$g(G) =$ Girth of $G =$ minimal length of a cycle in G.

Moore's Bound:

$(k-1)^r < |B(r)| \leq n$ for $r = \lfloor g(G)/2 \rfloor$

$\Rightarrow \quad g(G) \leq 2 \log_{k-1} n + 1$



The Lubotzky-Phillips-Sarnak Construction:

For a fixed prime p there exists a sequence of $(p+1)$-regular (Ramanujan) graphs $G_i = (V_i, E_i)$ with $|V_i| \to \infty$ such that $g(G_i) \geq \frac{4}{3} \log_p |V_i| - O(1)$.

Problem: What is the higher dimensional extension?

A Reformulation

$r_v(G) =$ maximal r such that the ball
$B_v(r) = \{u : d(u,v) \leq r\}$ is acyclic.

Acyclicity Radius:

$$r(G) = \min_v r_v(G) = \lfloor \frac{g(G)}{2} \rfloor - 1.$$

Moore's Bound: $\delta(G) = k \geq 3 \Rightarrow r(G) \leq \lfloor \log_{k-1} |V| \rfloor$.

LPS Construction: For a fixed prime p there exists a sequence of $(p+1)$-regular (Ramanujan) graphs $G_i = (V_i, E_i)$ with $|V_i| \to \infty$ such that $r(G_i) \geq \frac{2}{3} \log_p |V_i| - O(1)$.

Acyclicity Radius of a Complex

X a d-dimensional complex on the vertex set V.
$X(i) = \{\sigma \in X : \dim \sigma = i\}$, $f_i(X) = |X(i)|$.
Degree of $\sigma \in X(d-1) = \deg(\sigma) = |\{\tau \in X(d) : \sigma \subset \tau\}|$.
$\delta(X) = \min\{\deg(\sigma) : \sigma \in X(d-1)\}$.

$B_v(r)$ the ball of radius r around v.
Acyclicity Radius:

$$r(X) = \min_{v \in V} \max\{r : H_d(B_v(r)) = 0\}.$$

Higher Dimensional Extensions [LM]

A d-dimensional Moore bound

X a d-dimensional complex with $\delta(X) = k \geq d + 2$.
Then:

$$r(X) \leq \lfloor \log_{k-d} f_{d-1}(X) \rfloor \quad .$$

Ramanujan Complexes with high acyclicity

For $d \geq 1$ and q a prime power, there exists a sequence of d-dimensional $(q+1)$-regular complexes X_i on vertex sets V_i with $|V_i| \to \infty$, such that

$$r(X_i) \geq c(d) \cdot \log_q |V_i| - O(1) \quad .$$

The Upper Bound

Face Numbers and Homology

Fix $v \in V$ and let $B(t) = B_v(t)$.
If $t < m = r(X)$ then $H_d(B(t)) = 0$.
Therefore:

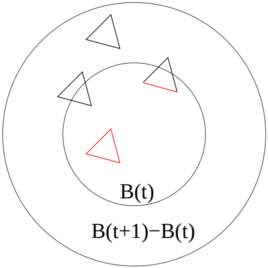
$$\beta(t) = f_d(B(t)) \leq f_{d-1}(B(t)) = \alpha(t) \ .$$

A Double Count

$\gamma(t) =$ number of pairs

$(\sigma, \tau) \in B(t)(d-1) \times X(d)$

such that $\sigma \subset \tau$



The Upper Bound (Cont.)

Lower Bound on $\gamma(t)$:

$$\gamma(t) = \sum \{\deg(\sigma) : \sigma \in B(t) \cap X(d-1)\} \geq k\alpha(t) \ .$$

Upper Bound on $\gamma(t)$:

$$\gamma(t) \leq (d+1)\beta(t) + (\beta(t+1) - \beta(t)) =$$
$$d\beta(t) + \beta(t+1) \leq d\alpha(t) + \alpha(t+1) \ .$$

Hence:

$$(k-d)\alpha(t) \leq \alpha(t+1)$$
$$(k-d)^m \leq \alpha(m) \leq f_{d-1}(X) \ .$$

Affine Buildings of Type $\tilde{A}_{d-1}$

Lattices

$\mathbb{K}$ local field with valuation ν
$\pi \in \mathbb{K}$ uniformiser: $\nu(\pi) = 1$
$\mathcal{O}$ the valuation ring, $\mathcal{O}/\pi\mathcal{O} = \mathbb{F}_q$ the residue field
A Lattice is $\mathcal{O}$-module $L \subset \mathbb{K}^d$ of rank d.
$L_1 \sim L_2$ if $L_1 = \lambda L_2$ for some $\lambda \in \mathbb{K}^*$.
$[L]$= equivalence class of L.

The Building $\mathcal{B} = \tilde{A}_{d-1}(\mathbb{K})$

Vertices: $\mathcal{B}^0 = \{[L] \ : \ L \text{ lattice in } \mathbb{K}^d\}$
Simplices: $\{[L_0], \ldots, [L_p]\}$ such that

$$\pi L_p \subset L_0 \subset L_1 \subset \cdots \subset L_p$$

Properties of $\mathcal{B} = \tilde{A}_{d-1}(\mathbb{K})$

Topology of $\mathcal{B}$

$\mathcal{B}$ is a contractible $(d-1)$-dimensional flag complex
$\text{link}([L]) \cong A_{d-1}(\mathbb{F}_q) =$ order complex of subspaces of $\mathbb{F}_q^d$.

Type Function on $\mathcal{B}^0$

$\tau([g\mathcal{O}^d]) = \nu(\det g) (\text{mod } d)$.
Color Classes: $V_i = \{[L] : \tau([L]) = i\}$.
$\mathcal{B}$ is d-partite on $V_0, \ldots, V_{d-1}$.

Metric on $\mathcal{B}$

$\text{dist}(\cdot,\cdot) =$ graph distance on the 1-skeleton of $\mathcal{B}$.
E.g. for a basis $e_1, \ldots, e_d$ of $\mathbb{K}^d$

$$\text{dist}([\oplus_{i=1}^d \mathcal{O} e_i], [\oplus_{i=1}^d \pi^{a_i} \mathcal{O} e_i]) = \max_i a_i - \min_i a_i \ .$$

The Cartwright-Steger Group Γ

Local Field: $\mathbb{K} = \mathbb{F}_q((x))$, Valuation Ring: $\mathcal{O} = \mathbb{F}_q[[x]]$.

The Group Γ

$\Gamma < PGL_d(\mathbb{K})$ is generated by elements $\{g_u : u \in \mathbb{F}^*_{q^d}\}$.

- Γ acts simply-transitively on $\mathcal{B}^0$.
- For all $g \in \Gamma$

$$\mathrm{dist}([\mathcal{O}^d], g[\mathcal{O}^d]) \geq \frac{1}{d-1} \min\{t : g = g_{u_1} \cdots g_{u_t}\} .$$

Properties of the Generators

Each g_u is the image of an $f_u \in GL_d(\mathbb{F}_q[x])$ such that:

- The entries of each f_u are polynomials of degree $\leq d$.
- $\det(f_u) = D(x)$ does not depend on u.

Finite Quotients of $\tilde{A}_{d-1}(\mathbb{K})$

Constructions of Cartwright, Solé and Żuk
and Lubotzky, Samuels and Vishne:

Let $p(x) \in \mathbb{F}_q[x]$ be irreducible, $(p(x), D(x)) = 1$.
$\mathrm{I} = (p)$, $R = \mathbb{F}_q[x]/\mathrm{I}$.
$h_u =$ image of $f_u \in GL_d(\mathbb{F}_q[x])$ in $PGL_d(R)$.
The map $f_u \to h_u$ extends to a homomorphism $\Gamma \to PGL_d(R)$.
$\Gamma(\mathrm{I}) = \ker[\Gamma \to PGL_d(R)]$.

The Quotient Complex $\mathcal{B}_{\mathrm{I}} = \Gamma(\mathrm{I})\backslash\mathcal{B}$:

Vertices: $\mathcal{B}_{\mathrm{I}}^0 = \{\Gamma(\mathrm{I})[L] \ : \ [L] \in \mathcal{B}^0\}$.
Simplices: $\{\Gamma(\mathrm{I})[L_0], \ldots, \Gamma(\mathrm{I})[L_k]\}$
such that $\{g_0[L_0], \ldots, g_k[L_k]\} \in \mathcal{B}$
for some $g_0, \ldots, g_k \in \Gamma(\mathrm{I})$.

Injectivity Radius of $\mathcal{B} \to \mathcal{B}_{\mathrm{I}}$

Claim:

$$\ell_{\mathrm{I}} = \min\{\mathrm{dist}([\mathcal{O}^d], g[\mathcal{O}^d]) : \ 1 \neq g \in \Gamma(\mathrm{I}) \ \} \ \geq$$
$$c'(d) \log_q |\mathcal{B}_{\mathrm{I}}^0| - O(1) \quad .$$

Proof:
Let t be minimal such that $g = g_{u_1} \cdots g_{u_t}$.
$C = (c_{ij}) = f_{u_1} \cdots f_{u_t} \Rightarrow \deg(c_{ij}) \leq td$.
$g \in \Gamma(\mathrm{I}) \Rightarrow C = r(x)\mathrm{I}_{d\times d} + p(x)B \Rightarrow td \geq \deg(p)$.
Therefore

$$\mathrm{dist}([\mathcal{O}^d], g[\mathcal{O}^d]) \geq \frac{t}{d-1} \geq$$

$$\frac{\deg(p)}{d(d-1)} \geq c'(d) \log_q |\mathcal{B}_{\mathrm{I}}^0| - O(1) \quad .$$

Complexes with Large Acyclicity Radius

Construction of X_i

Choose irreducible polynomials $p_i(x) \in \mathbb{F}_q[x]$
such that $(p_i, D(x)) = 1$ and $\deg p_i \to \infty$.
Let $\mathrm{I}_i = (p_i)$ and let $X_i = \mathcal{B}_{\mathrm{I}_i}$.

Acyclicity Radius of X_i

$\dim X_i = d - 1$, $\delta(X_i) = \delta(\mathcal{B}) = q + 1$.

$\mathcal{B} \to X_i$ an isomorphism on balls of radius at most $\frac{\ell_{\mathrm{I}_i}}{2} - 1$ in $\mathcal{B}$.
Since $\mathcal{B}$ is contractible, it follows that

$$r_v(X_i) \geq \frac{\ell_{\mathrm{I}_i}}{2} - 1 \geq c(d) \log_q |X_i^0| - O(1) \quad .$$