Sensitive Microstructural Descriptors of Disordered Heterogeneous Materials Across Length Scales for Materials Discovery

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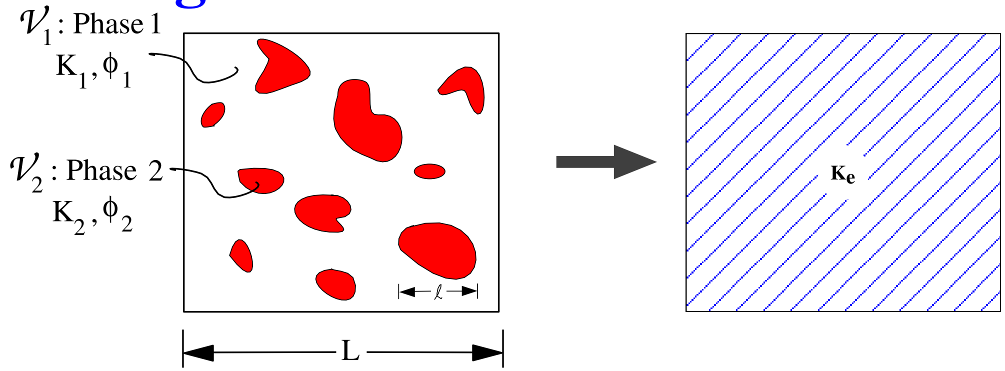
Heterogeneous Materials: Preliminaries



- **Examples: Porous and composite media,, foams, colloids, battery materials , geologic media, biological membranes, animal and plant tissue, etc.**

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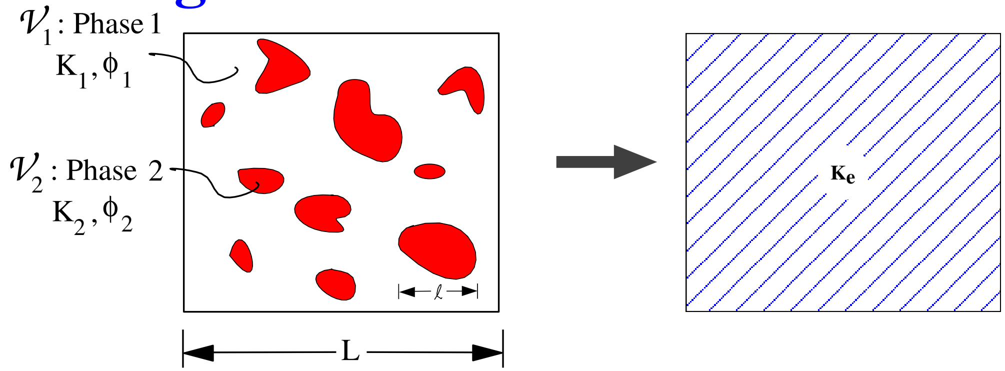

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Effective Properties

1. **Effective Conductivity Tensor,** σ_e
2. **Effective Diffusion Tensor,** $\mathbf{D}_e$
3. **Effective Stiffness Tensor,** $\mathbf{C}_e$
4. **Effective Viscosity,** μ_e
5. **Mean Survival Time,** τ
6. **Diffusion and Viscous Relaxation Times,** T_1, Θ_1
7. **Fluid Permeability Tensor,** $\mathbf{k}$
8. **Nonlinear Mechanical Properties**
9. **Optical and Other Wave Properties**

Seemingly different properties are interrelated: Cross-Property Relations

- **Effective properties are sensitive to the details of the microstructure, i.e., volume fractions; orientations, sizes and shapes of the phase domains; spatial distribution of the domains; connectedness of the phases, etc.**

OUTLINE

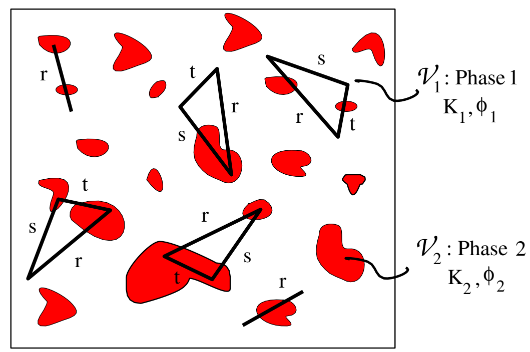
- Statistical Descriptors of Two-Phase Media
- Microstructure-Dependent Estimates on Transport Properties
- Hyperuniform Two-Phase Media
- Quantifying Order/Disorder Across Length Scales

Statistical Descriptors of Two-Phase Media

- A **two-phase medium** in $\mathbb{R}^d$ is fully statistically characterized by the n**-point correlation (probability) functions**:

$$S_n^{(i)}(\mathbf{x}_1, ..., \mathbf{x}_n) \equiv \left\langle \mathcal{I}^{(i)}(\mathbf{x}_1) \ldots \mathcal{I}^{(i)}(\mathbf{x}_n) \right\rangle,$$

where $n = 1, 2, 3, \ldots$**, angular brackets denote an ensemble average, and** $\mathcal{I}^{(i)}(\mathbf{x})$ **is the indicator function for phase** $i = 1, 2$**:**



- **The** $S_n^{(i)}$**'s arise in rigorous expressions (exact expansions and bounds) for the effective conductivity, elastic moduli, diffusion properties, fluid permeability, and optical properties.**

Statistical Descriptors of Two-Phase Media

Interfacial n-point Correlation Functions

- The interface between the phases of a realization of a two-phase medium is characterized the interface indicator function $\mathcal{M}(\mathbf{x})$ defined as

$$\mathcal{M}(\mathbf{x}) = |\nabla \mathcal{I}^{(1)}(\mathbf{x})| = |\nabla \mathcal{I}^{(2)}(\mathbf{x})| \tag{1}$$

- One type of interfacial n-point correlation function is defined as

$$F_{sss\cdots}(\mathbf{x}_1, ..., \mathbf{x}_n) \equiv \langle \mathcal{M}(\mathbf{x}_1) \ldots \mathcal{M}(\mathbf{x}_n) \rangle \, .$$

- The specific surface s (interface area per unit volume) is the one-point correlation function:

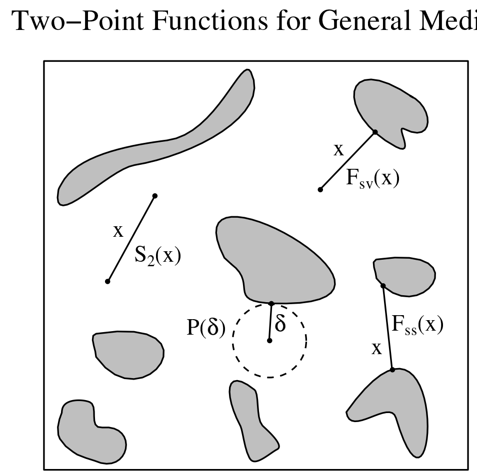
$$s = \langle \mathcal{M}(\mathbf{x}) \rangle .$$

- The interfacial two-point or surface-surface correlation function is particularly important:

$$F_{ss}(\mathbf{r}) = \langle \mathcal{M}(\mathbf{x}) \mathcal{M}(\mathbf{x} + \mathbf{r}) \rangle$$

- Interfacial n-point correlation functions naturally arise in rigorous bounds on effective properties in which the interface plays a major role, e.g., diffusion-controlled reactions and fluid permeability.

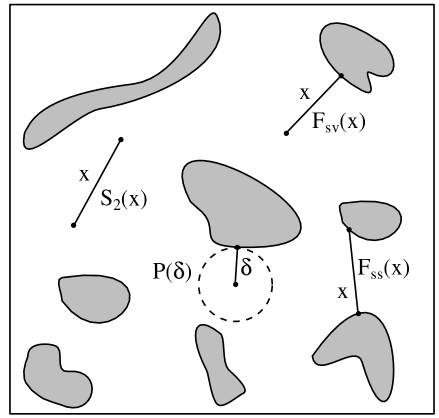
Some 2-Point Correlation Functions



- $S_2^{(i)}(r)$: **two-point correlation function for phase i.**
- $F_{sv}(r)$: **surface-void correlation function**
- $F_{ss}(r)$: **surface-surface correlation function**
- $P(\delta)$: **pore-size probability density function**

Some 2-Point Correlation Functions

Two–Point Functions for General Media



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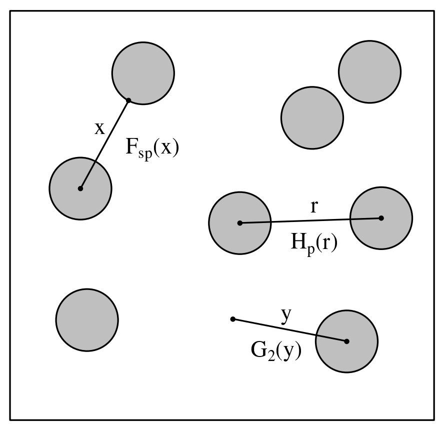
Two–Point Functions for Particulate Media



- $F_{sp}(r)$: **surface-particle correlation function**
- $G_2(r)$: **void-particle surface-void correlation function**
- $H_P(r)$: **nearest-neighbor probability density function**

Spectral Density

- The **autocovariance** function $\chi_V(\mathbf{r})$ is defined by

$$\chi_V(\mathbf{r}) \equiv S_2^{(1)}(\mathbf{r}) - \phi_1^2 = S_2^{(2)}(\mathbf{r}) - \phi_2^2.$$

- The nonnegative **spectral density** $\tilde{\chi}_V(\mathbf{k})$ at wavevector $\mathbf{k}$, which can be obtained from scattering experiments, is the **Fourier transform** of $\chi_V(\mathbf{r})$.

- In the large-k limit, it decays as an inverse power law with coefficient prop. to specific surface s:

$$\tilde{\chi}_V(\mathbf{k}) \sim \frac{\gamma(d)\, s}{k^{d+1}}, \qquad k \to \infty.$$

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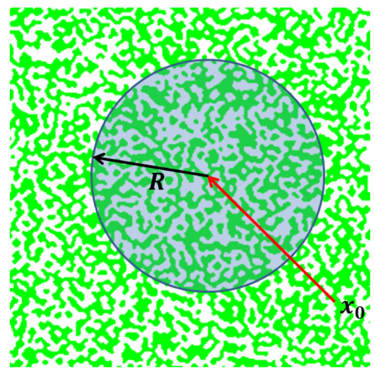


- **Local volume-fraction variance** $\sigma_V^2(R)$ within a spherical window of radius R is given in terms of $\chi_V(\mathbf{r})$ or $\tilde{\chi}_V(\mathbf{k})$ (Lu and Torquato, 1990; Zachary and Torquato, 2009):

$$\sigma_V^2(R) = \frac{1}{v_1(R)} \int_{\mathbb{R}^d} \chi_V(\mathbf{r})\, \alpha_2(r;R) d\mathbf{r}$$
$$= \frac{1}{v_1(R)(2\pi)^d} \int_{\mathbb{R}^d} \tilde{\chi}_V(\mathbf{k}) \tilde{\alpha}_2(k;R) d\mathbf{k},$$

where $v_1(R)$ is the volume of a d-dimensional sphere of radius R, $\alpha_2(r;R)$ is the scaled volume common to 2 windows separated by a distance r and $\tilde{\alpha}_2(k;R)$ is its Fourier transform.

Transport Properties of Fluid Saturated Porous Media

NMR Measurements

- Nuclear magnetic resonance (NMR) relaxation times of porous media provide useful probes of the pore-phase microstructure.
  - Mean survival time, τ
  - Diffusion relaxation times, $T_1, T_2, T_3, \ldots$, where T_1 is the largest or principal relaxation time.

Conduction Measurements

- Consider a porous medium whose pore space is filled with an electrically conducting fluid of conductivity σ_1 and a solid phase that is perfectly insulating (σ_2 = 0).
  - We consider dimensionless effective conductivity

$$\frac{\sigma_e}{\sigma_1} = \mathcal{F}^{-1}$$

where $\mathcal{F}$ is the formation factor, which a measure of the tortuosity or degree of "windiness for electrical transport pathways in the pore phase.

Stokes Flow and Fluid Permeability

- Consider Stokes flow through a porous medium.
  - Fluid permeability, k
  - Viscous relaxation times, $\Theta_1, \Theta_2, \Theta_3, \ldots$, where Θ_1 is the largest or principal relaxation time.

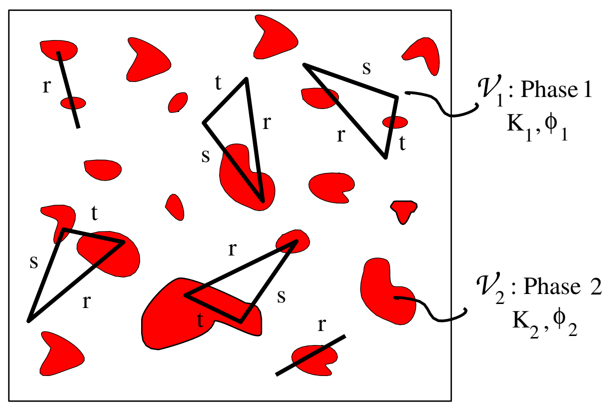
Microstructure-Dependent Estimates on Transport Properties

Exact Representation of Effective Dynamic Dielectric Tensor

- **Nonlocal strong-contrast expansions** for effective dynamic dielectric tensor $\boldsymbol{\varepsilon}_e(\mathbf{k}_q)$ that exactly account for **complete microstructural information** and hence **multiple scattering to all orders** for the range of wavenumbers for which our extended homogenization theory applies, i.e., $0 \le |\mathbf{k}_q|\ell \lesssim 1$ (Torquato and Kim, PRX, 2021):

$$\phi_p{}^2\beta_{pq}^2[\boldsymbol{\varepsilon}_e(\mathbf{k}_q) + (d-1)\varepsilon_q \boldsymbol{I}] \cdot [\boldsymbol{\varepsilon}_e(\mathbf{k}_q) - \varepsilon_q \boldsymbol{I}]^{-1} = \phi_p \beta_{pq} \boldsymbol{I} - \sum_{n=2}^{\infty} \boldsymbol{A}_n^{(p)}(\mathbf{k}_q)\beta_{pq}^n,$$

where $\beta_{pq} = f(\varepsilon_p, \varepsilon_q)$ is a **generalized "polarizability"**, $\boldsymbol{A}_n^{(p)}(\mathbf{k}_q)$ is a wavevector-dependent second-rank tensor that is a **functional involving the set of correlation functions** $S_1^{(p)}, S_2^{(p)}, \cdots, S_n^{(p)}$ and products of the **dyadic Green's function** $\boldsymbol{H}^{(q)}(\mathbf{r})$.



- Due to the **rapid convergence**, even for large contrasts, their **lower-order truncations yield accurate closed-form approximate formulas** for $\boldsymbol{\varepsilon}_e(\mathbf{k}_q)$.
- Such nonlocal formulas are **resummed representations** of the expansions that still **accurately capture multiple scattering to all orders**.

Microstructure-Dependent Bounds on Transport Properties

NMR Time Scales

- The mean survival time τ in the diffusion-controlled limit is bounded from above by an integral over the spectral density (Torquato, 2020):

$$D\tau \leq \frac{1}{\phi_1 \, \phi_2^2} \, \ell_P^2,$$

where

$$\ell_P^2 = \int_0^\infty \chi_V(r)\, r dr = \frac{1}{2\pi^2} \int_0^\infty \tilde{\chi}_V(k) dk.$$

- It is bounded from below in terms of the mean pore size (Torquato and Avellaneda, 1991):

$$\tau \geq \frac{\langle \delta \rangle^2}{D}.$$

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Formation Factor

- Derived a tight lower bound on $\mathcal{F}$ for any 3D porous medium that accounts for up to four-point correlation function information (Torquato, 2020):

$$\mathcal{F} \ge \frac{1+\frac{1}{2}\frac{\gamma_2}{\zeta_2}-\frac{1}{2}\zeta_2+\left(\frac{1}{2}+\frac{1}{2}\zeta_2+\frac{1}{4}\frac{\gamma_2}{\zeta_2}\right)\phi_2}{1+\frac{1}{2}\frac{\gamma_2}{\zeta_2}-\frac{1}{2}\zeta_2+\left(-1+\frac{1}{2}\zeta_2-\frac{1}{2}\frac{\gamma_2}{\zeta_2}\right)\phi_2},$$

$\zeta_2 = \zeta_2[S_1^{(2)}, S_2^{(2)}, S_3^{(2)}]$: three-point microstructural parameter

$\gamma_2 = \gamma_2[S_1^{(2)}, S_2^{(2)}, S_3^{(2)}, S_4^{(2)}]$: four-point microstructural parameter

Microstructure-Dependent Bounds on Transport Properties

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Fluid Permeability

- Permeability is bounded from above in terms of the **same length scale** ℓ_P (Torquato, 2020):

$$k \leq \frac{2}{3\phi_2^2}\,\ell_P^2,$$

- While this **two-point "void" bound** is **not tight**, it correctly **rank orders the permeabilities of different models**.

Link Between Permeability and Diffusion Parameters

- **Exact relation (Avellaneda and Torquato, 1991):**

$$k = \frac{\mathcal{L}^2}{8\mathcal{F}},$$

where is $\mathcal{L}$ is a length parameter that is a sum over the times $\Theta_1.\Theta_2, \ldots$.

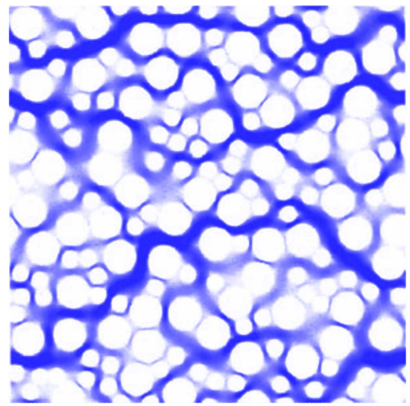


Figure 1: Laminar flow through the void space of a two-dimensional hard-disk packing. The brightness of the color indicates the magnitude of the flow velocity.

- **Rigorous bounds (Torquato, 1990; Avellaneda and Torquato, 1991):**

$$k \le D\phi_1\tau,$$

$$k \le \frac{DT_1}{\mathcal{F}}$$

- **Approximation valid when pore space is well connected (Torquato, 2020)**

$$k \approx \phi_1 \frac{\langle \delta^2 \rangle}{\mathcal{F}}$$

Predictions of Permeability Approximation

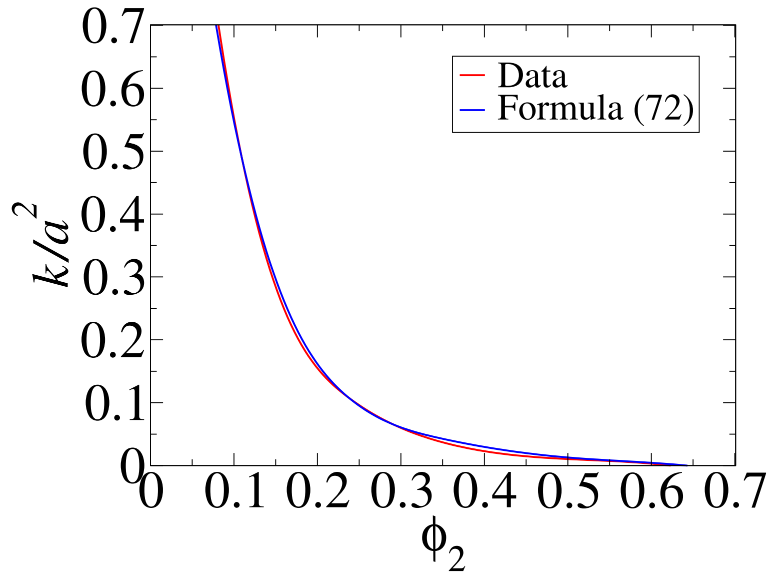


Figure 2: Comparison of computer-simulation data for the fluid permeability of BCC packings as a function of ϕ_2 to the predictions of the approximation formula.

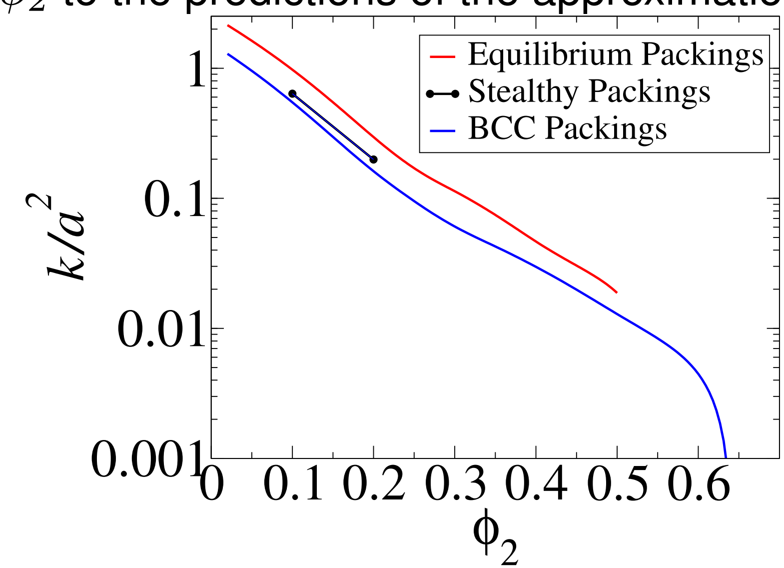
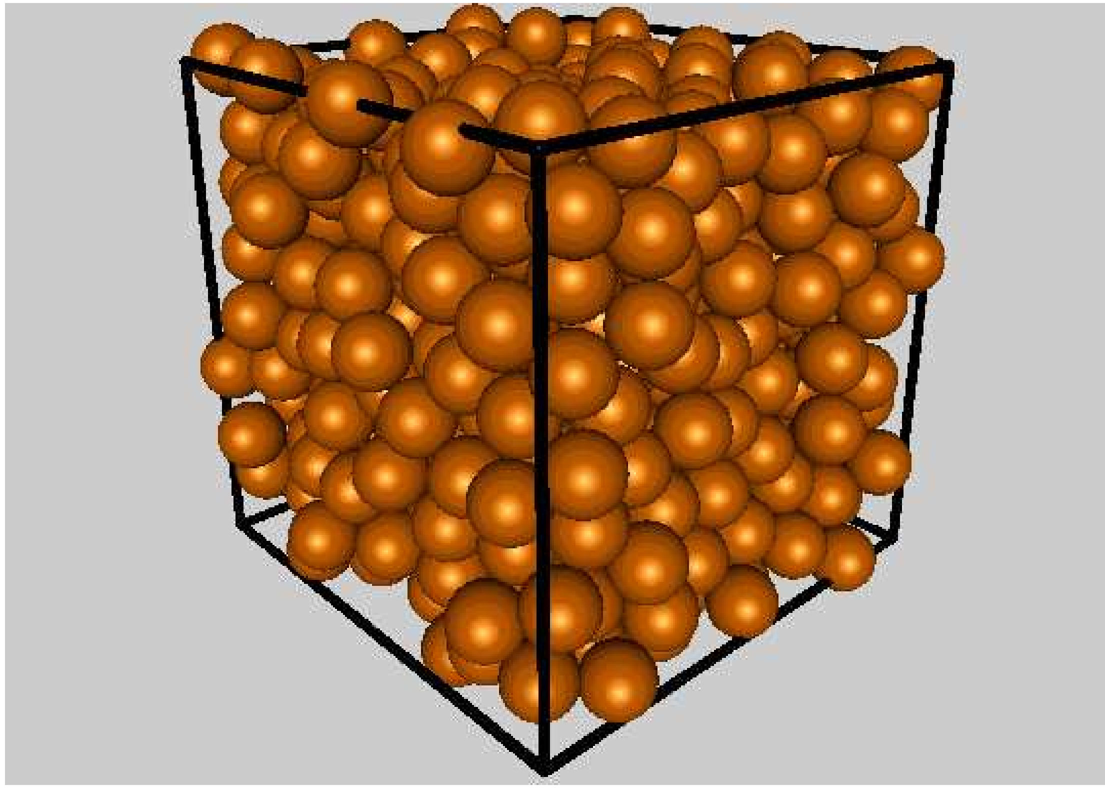


Figure 3: Estimates of the dimensionless fluid permeability k/a^2 as a function of ϕ_2 predicted by the approximation formula for the disordered stealthy, equilibrium, and BCC packings.

Permeability of MRJ Sphere Packings



- **The approximation formula for the permeability of the MRJ packing yields $k/a^2 \approx 0.010$ at porosity $\phi_1 = 0.374$, where a is the radius of a sphere. This is to be contrasted with the corresponding BCC packing with $k/a^2 \approx 0.0016$, which is much lower.**
- **This is consistent with the fact that the pore space in the MRJ packing is more localized and less uniformly dynamically connected than that of the BCC packing.**

Hyperuniformity of Two-Phase Media

Hyperuniformity and Large-Scale Structural Fluctuations

- A hyperuniform many-particle system is one in which large-scale density fluctuations are greatly suppressed compared to those of typical disordered systems (e.g., liquids). Torquato and Stillinger, Phys. Rev. E (2003)

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- Hyperuniformity provides a unified means of categorizing and characterizing crystals, quasicrystals and special disordered systems. Thus, hyperuniformity concept generalizes our traditional notions of long-range order.
- This enables a new structural classification scheme that encompasses all of these states according to this generalization of long-range order.

Disordered Hyperuniform States

- **Disordered hyperuniform many-particle systems can be regarded to be new ideal states of matter in that they**

  1. ***behave more like crystals or quasicrystals in the way they suppress large-scale density fluctuations, and yet are also like liquids and glasses, since they are statistically isotropic structures with no Bragg peaks*;**
  2. ***can exist as both as equilibrium and nonequilibrium phases*;**
  3. ***come in quantum-mechanical and classical varieties*;**
  4. ***and, are endowed with unique bulk physical properties*.**

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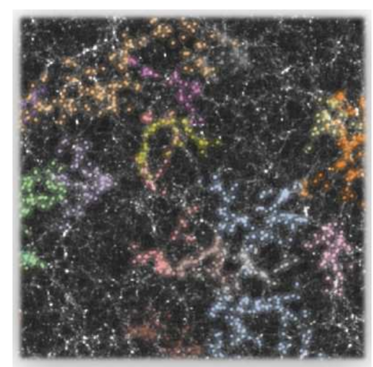
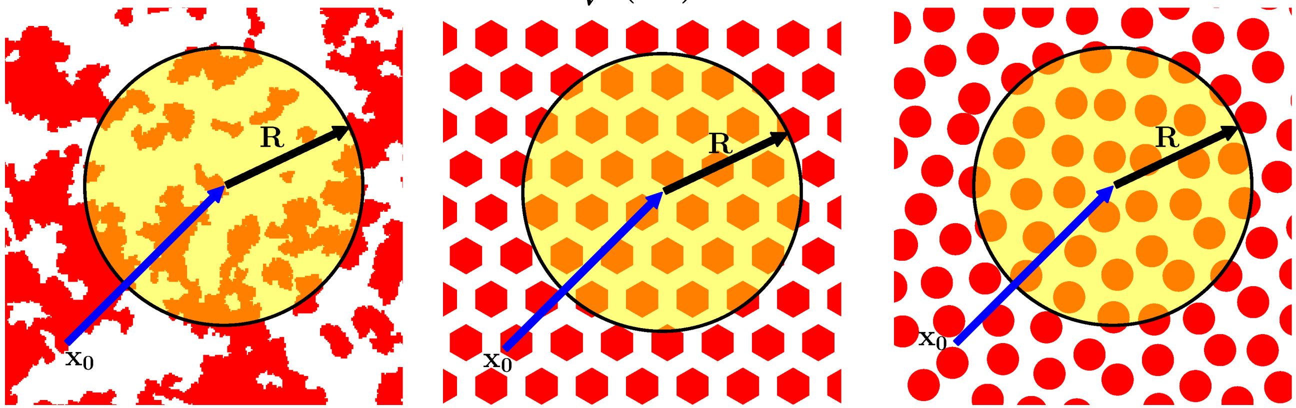
- Existence of disordered hyperuniform states forces us to re-think what we mean by “disorder” and on what length scales.
- Such exotic correlated disordered states arise in a myriad of contexts:
  - Classical equilibrium liquids and ground states
  - Classical nonequilibrium systems
  - Quantum systems
  - Sphere packings
  - Random matrices
  - Dynamical systems and quantum chaos
  - Number theory (e.g., prime numbers; Torquato et al. J. Phys. A, 2019)
  - Biological systems
  - Novel materials
  - Large-scale structure of the Universe (Philcox & Torquato, PRX, 2023)



Hyperuniformity of Disordered Two-Phase Materials

- **Hyperuniformity concept was generalized to the case of heterogeneous materials: suppression of large-scale volume fraction fluctuations, as measured by the local variance $\sigma_V^2(R)$ (Zachary and Torquato, 2009).**



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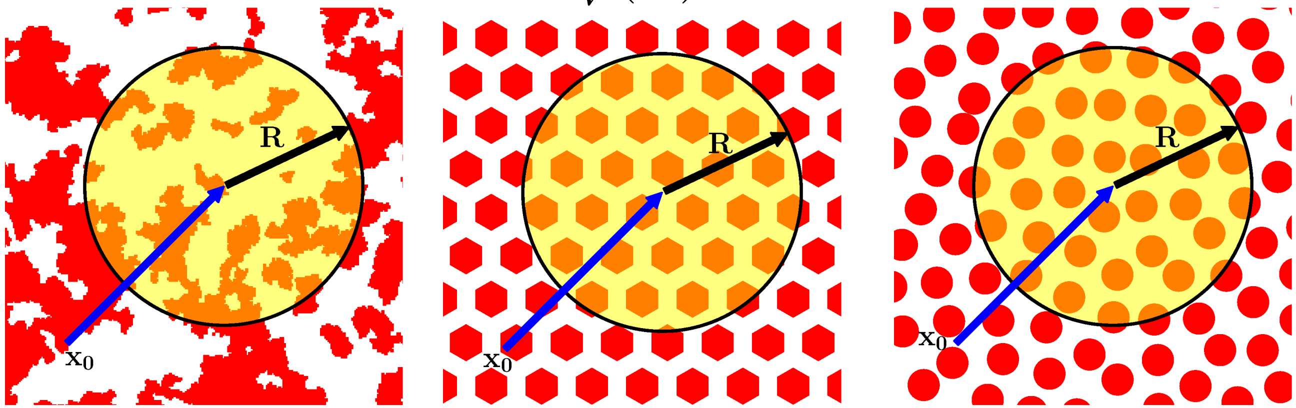

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- For typical disordered media, volume-fraction variance $\sigma_V^2(R)$ for large R goes to zero like R^{-d}.
- For hyperuniform two-phase media, $\sigma_V^2(R)$ goes to zero faster than R^{-d}, equivalent to following condition on spectral density $\tilde{\chi}_V(\mathbf{k})$:

$$\lim_{|\mathbf{k}|\to 0} \tilde{\chi}_V(\mathbf{k}) = 0.$$

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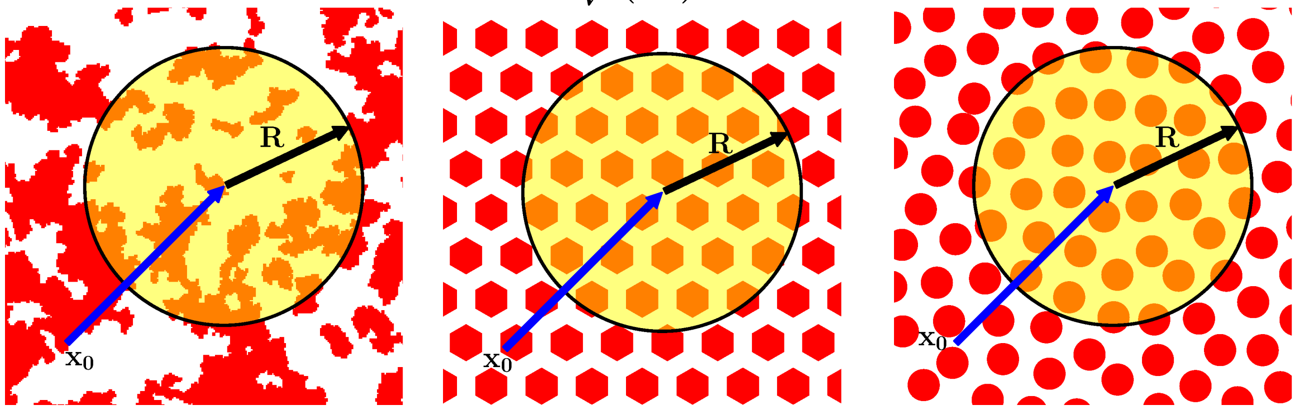

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- Similarly, variance in surface-area fluctuations $\sigma_S^2(R)$ also goes to zero faster than R^{-d} for hyperuniform media or, equivalently (Torquato, PRE, 2016)

$$\lim_{|\mathbf{k}|\to 0} \tilde{\chi}_S(\mathbf{k}) = 0.$$

General Hyperuniform Scaling Behaviors

- Consider **power-law spectral density**

$$\tilde{\chi}_V(\mathbf{k}) \sim |\mathbf{k}|^{\alpha}, \qquad (|\mathbf{k}| \to \mathbf{0})$$

- Can prove following large-R scalings (Torquato, Phys. Rep. 2018):

$$\sigma_V^2(R) \sim \begin{cases} R^{-(d+1)}, & \alpha > 1 \qquad \textbf{(Class I)} \\ R^{-(d+1)} \ln R, & \alpha = 1 \qquad \textbf{(Class II)} \\ R^{-(d+\alpha)}, & 0 < \alpha < 1 \qquad \textbf{(Class III).} \end{cases}$$

- Classes **I** and **III** are the **strongest and weakest** forms of hyperuniformity, respectively. Class I media include all **crystals**, many **quasicrystals** and **exotic disordered media**.

- **Stealthy hyperuniform** media are also of **class I**:

$$\tilde{\chi}_V(\mathbf{k}) = 0 \qquad \text{for } 0 \le |\mathbf{k}| \le K.$$

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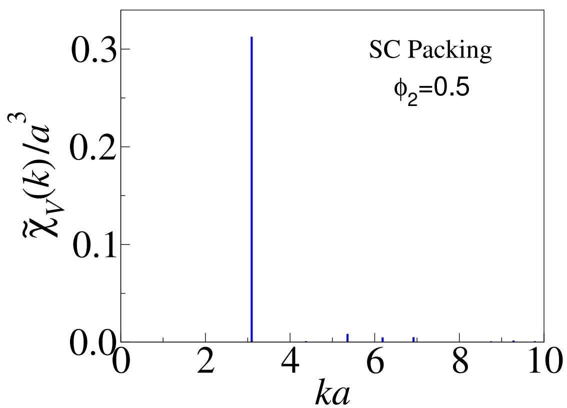
General Nonhyperuniform Scaling Behaviors

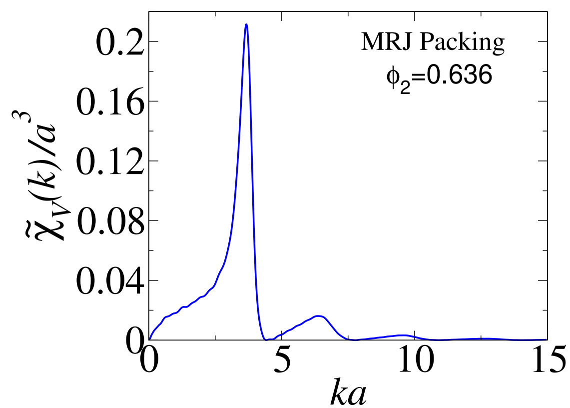
$$\sigma_V^2(R) \sim \begin{cases} R^{-d}, & \alpha = 0 \quad \text{(typical nonhyperuniform)} \\ R^{-(d+\alpha)}, & -d < \alpha < 0 \quad \text{(antihyperuniform).} \end{cases}$$

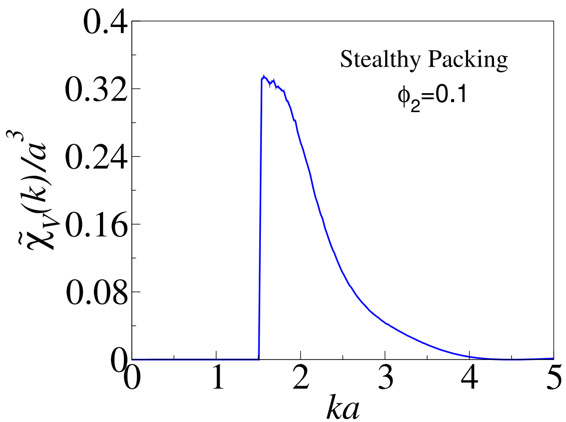
- For a "typical" nonhyperuniform media, $\tilde{\chi}_V(0)$ is bounded, but is unbounded for antihyperuniform media (e.g., fractal structures).
- Thus, can classify all translationally invariant states of matter according to their large-scale fluctuations (Torquato, PRE, 2021).

Spectral Densities of Hyperuniform and Nonhyperuniform Media

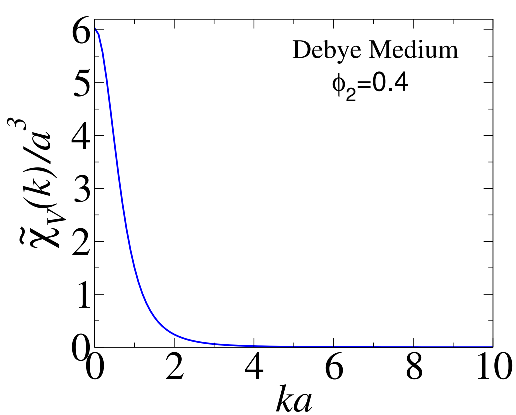
3 Different Hyperuniform Models

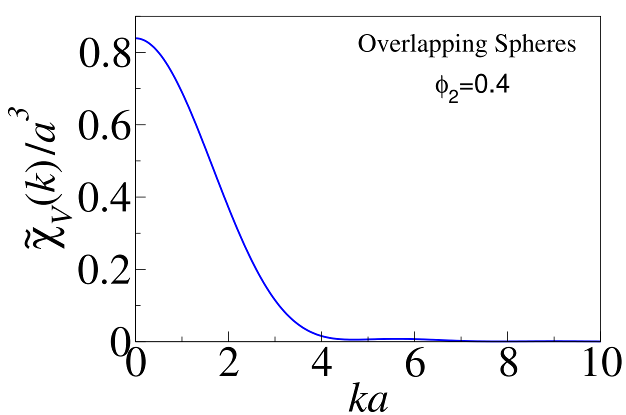


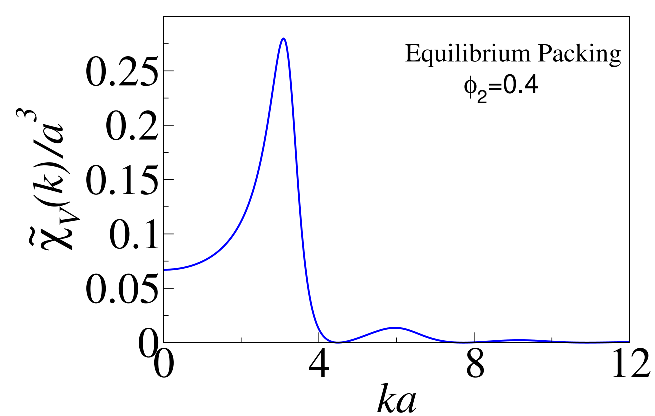




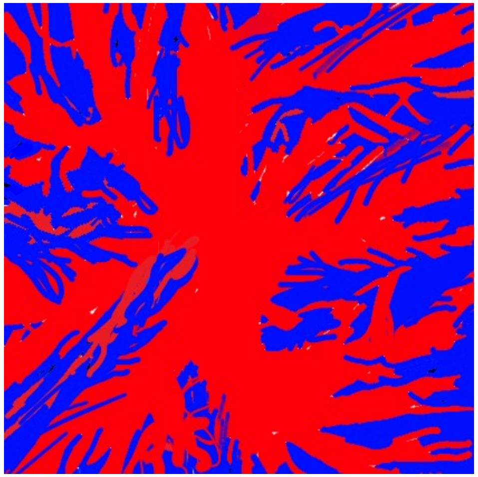
3 Different Nonhyperuniform Models



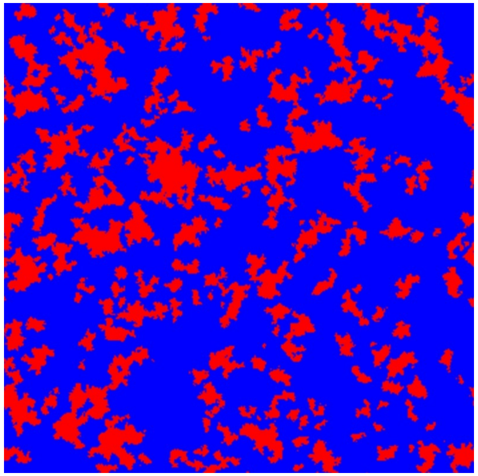




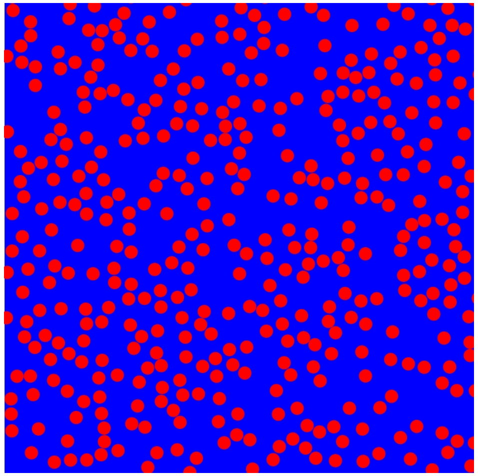
Examples of Microstructures Spanning Across the Possible Spectrum



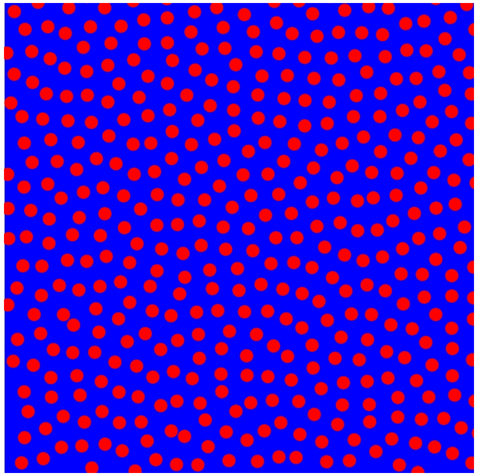
Antihyper.: $-d < \alpha < 0$



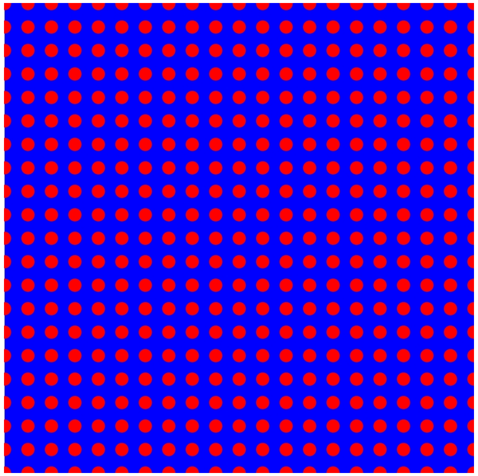
Typical nonhyper.: $\alpha = 0$



Nonstealthy hyper.: $0 < \alpha < \infty$



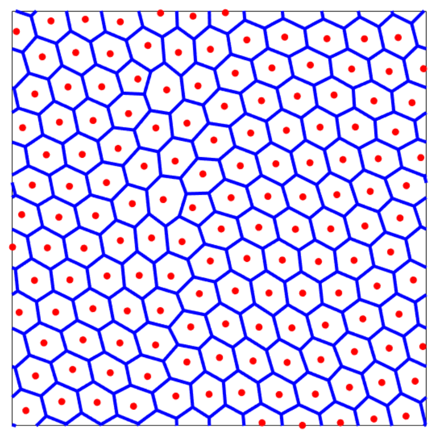
Disordered stealthy hyperuniform: $\alpha = \infty$



Ordered stealthy hyperuniform: $\alpha = \infty$

Optimality of Disordered Hyperuniform Two-Phase Media

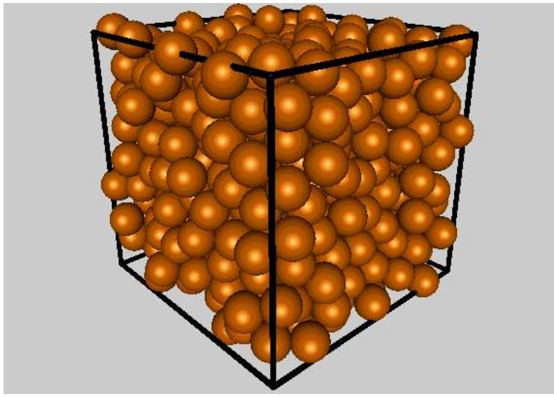
- Disordered hyperuniform dielectric networks have the largest isotropic photonic band gaps: Florescu, Steinhardt and Torquato, PNAS (2009).
- Disordered hyperuniform media have a rich optical "phase diagram" - transparency, diffusive, PBG and localization regimes: Froufe-Pérez et al. PNAS (2017).
- Disordered hyperuniform networks have maximal effective thermal (electrical) conductivities as well as maximal effective bulk and shear moduli: Chen & Torquato, Multifunctional Materials (2018)

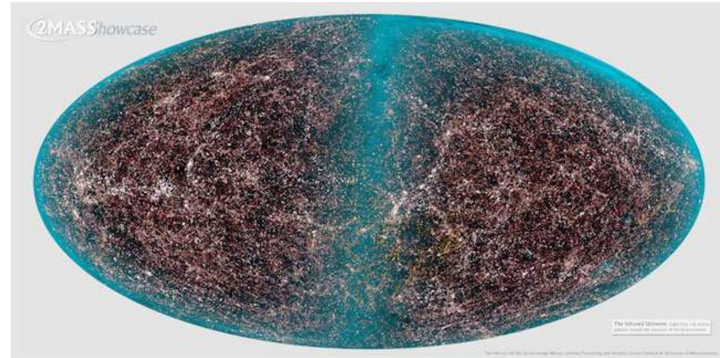


- Disordered hyperuniform materials are nearly optimal wave absorbers: Bigourdan et al. Opt. Exp. (2018).
- Disordered hyperuniform composite lenses can dramatically reduce back scattering relative to its periodic counterparts: Zhang et al. APL (2019).

Quantifying Order/Disorder Across Length Scales

Quantifying Disorder/Order of Microstructures

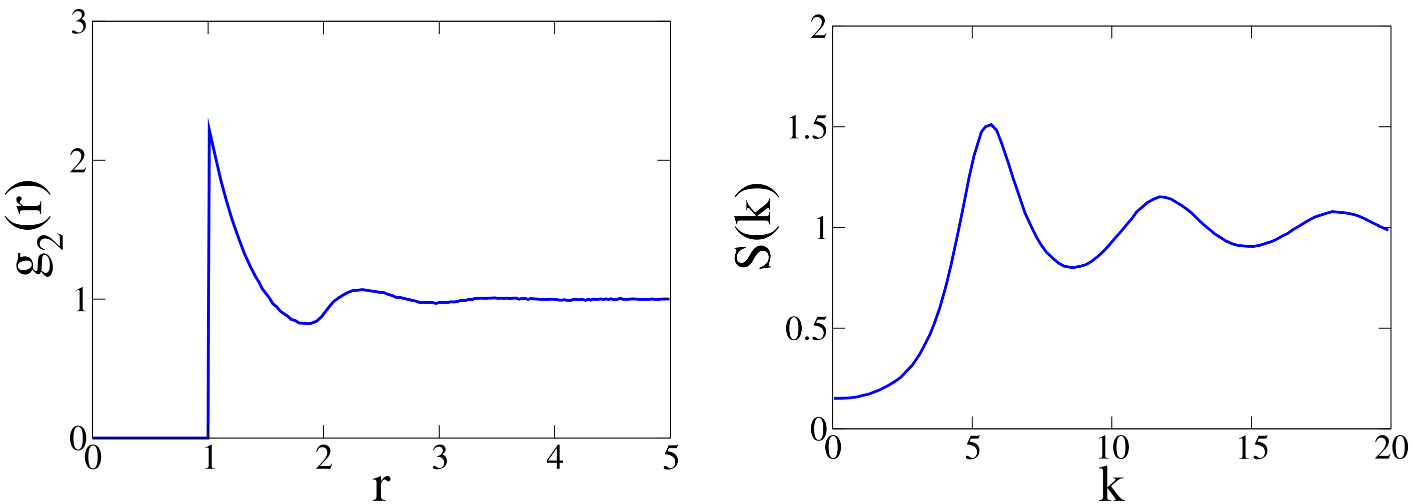




- Must settle for reduced information. For example, scalar order metrics $\psi_1, \psi_2, \psi_3, \ldots$ such that $0 \leq \psi_i \leq 1$.
- Can order metrics be devised consistent with our intuition:
  - Perfect crystals
  - Perturbed crystals
  - Quasicrystals
  - Highly defective crystals
  - Correlated random systems (e.g., "maximally random jammed" state)
  - Uncorrelated random systems (ideal gases)
- Crucially, should be able detect degree of order across length and time scales.
- Variety of useful translational and orientational order metrics have been introduced (Torquato & Stillinger, Rev. Mod. Phys. 2010).
- Order metrics generally may be tensor quantities.

Detecting Order/Disorder Across Length Scales

- Consider a many-particle system with **positive and negative correlations** as measured by the **pair correlation function $g_2(\mathbf{r})$ and structure factor S(k)**:



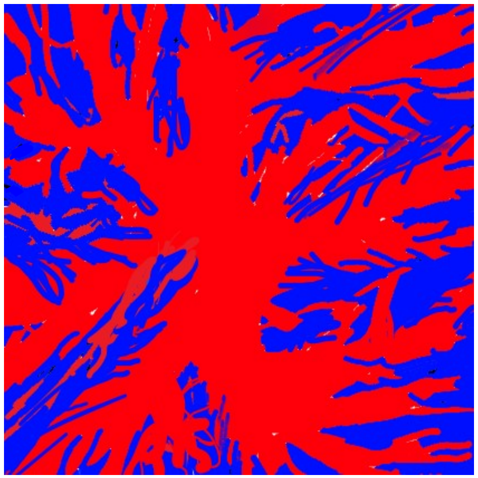
- The τ order metric characterizes translational order **(Torquato, Zhang& Stillinger, PRX, 2015)**:

$$\tau = \frac{1}{D^d}\int_{\mathbb{R}^d}[g_2(\mathbf{r})-1]^2 d\mathbf{r} = \frac{1}{(2\pi)^d D^d}\int_{\mathbb{R}^d}[S(\mathbf{k})-1]^2 d\mathbf{k}$$

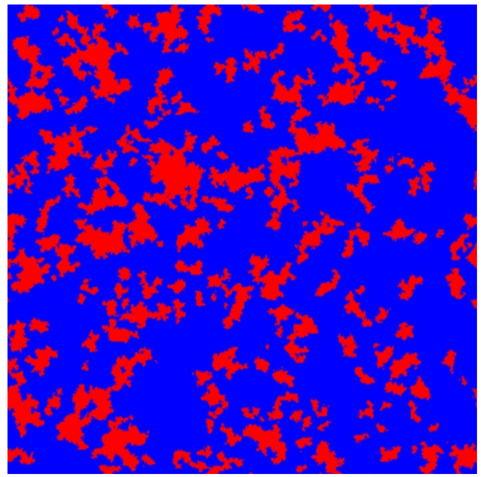
- Deviations of the pair functions from unity, whether **positive or negative**, are picked by τ, and has been fruitfully applied to describe a broad spectrum of many-particle systems.

Local Order Metrics Across the Spectrum of Two-Phase Media

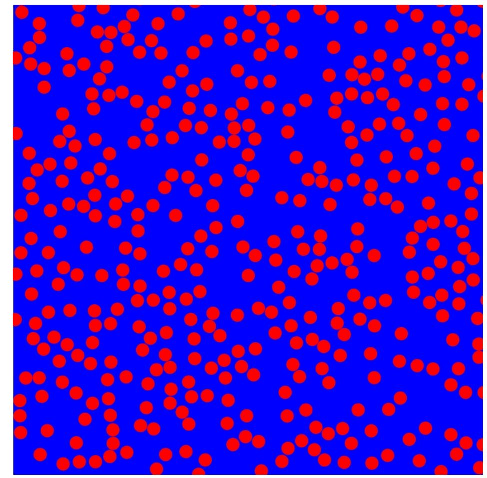
- **Devising order metrics to characterize and classify microstructures across length scales is a highly challenging task, given the richness of the possible phase geometries and topologies.**



Antihyper.: $-d < \alpha < 0$



Typical nonhyper.: $\alpha = 0$



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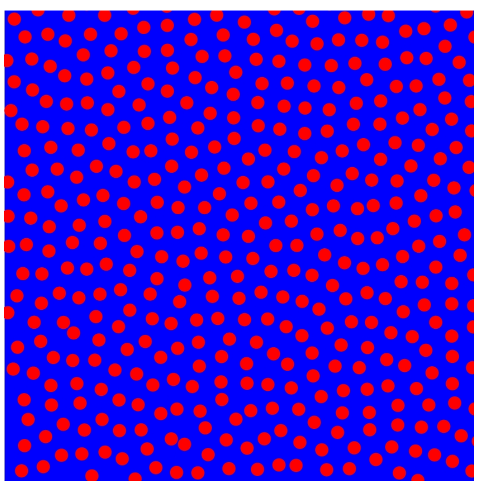
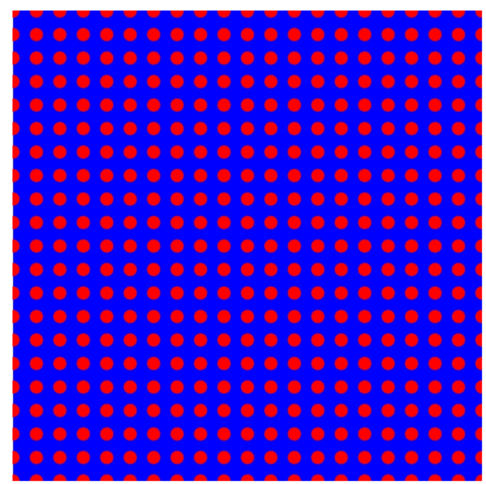


Disordered stealthy hyperuniform: $\alpha = \infty$

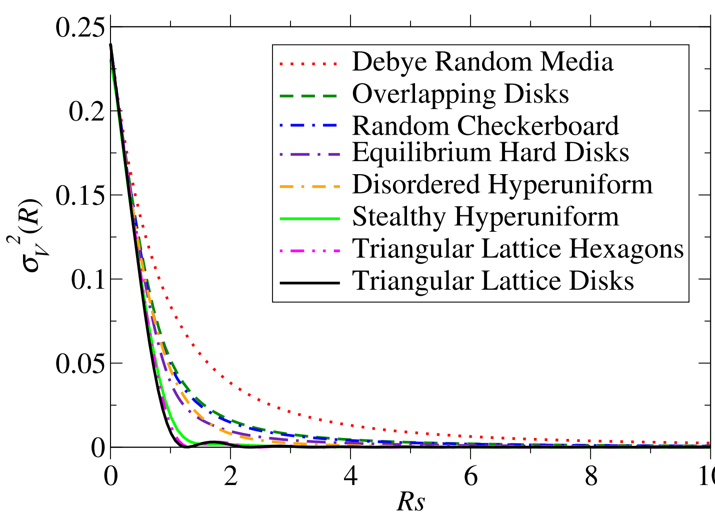


Ordered stealthy hyperuniform: $\alpha = \infty$

Local Order Metrics Across the Spectrum of Two-Phase Media

Torquato, Skolnick and Kim, J. Phys. A: Math. Theory (2022)

- For this purpose, we propose the use of the **local volume- fraction variance** $\sigma_V^2(R)$ associated with a spherical window of radius R as an **local order metric**.
- We determined $\sigma_V^2(R)$ as a function of R for 22 different models across the first three space dimensions, including **both hyperuniform and non-hyperuniform systems with varying degrees of short- and long-range order**.



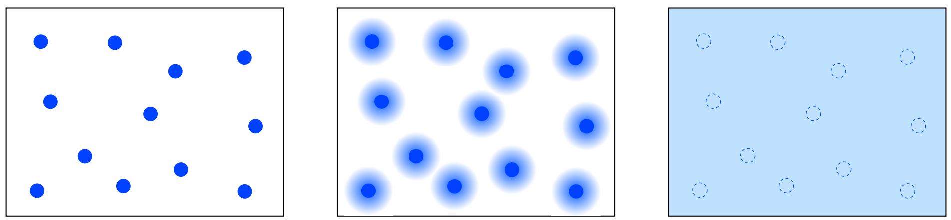
2D cases

For any particular value of R, the **lower (higher)** the value of $\sigma_V^2(Rs)$, the **greater the degree of order (disorder)**.

Diffusion Spreadability as a Dynamic Probe of Microstructure

Torquato, Phys. Rev. E (2021)

- At $t = 0$, assumed a solute is uniformly distributed throughout phase 2 with volume fraction ϕ_2, and completely absent from phase 1 with volume fraction ϕ_1, and each phase has same diffusion coefficient D.
- Problem: Calculate "spreadability" $\mathcal{S}(t)$, fraction of the total amount of solute present that has diffused into phase 1 at time t.



- Excess spreadability has following Fourier representation:

$$\mathcal{S}(\infty) - \mathcal{S}(t) = \frac{1}{(2\pi)^d \phi_2} \int_{\mathbb{R}^d} \tilde{\chi}_V(\mathbf{k}) \exp[-k^2 Dt] d\mathbf{k} \geq 0,$$

where $\tilde{\chi}_V(\mathbf{k})$ is the spectral density.

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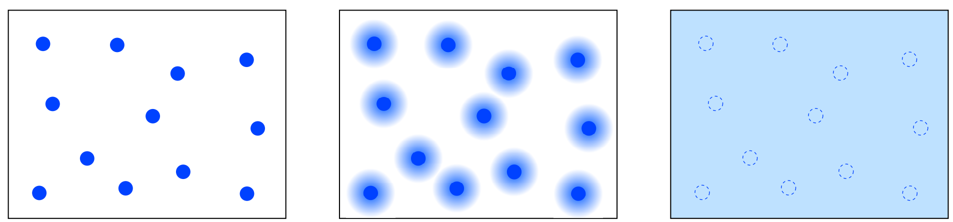

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where $\tilde{\chi}_{_V}(\mathbf{k})$ is the spectral density.

- The time-dependent diffusion spreadability $\mathcal{S}(t)$ relates its short-, intermediate- and long-time behaviors link to the small-, intermediate- and large-scale structure of heterogeneous materials.

Spreadability Phase Diagram

- **Consider power-law scaling form for spectral density:**
$$\lim_{|\mathbf{k}|\to\mathbf{0}} \tilde{\chi}_V(\mathbf{k}) = B|\mathbf{k}a|^{\alpha} \qquad (|\mathbf{k}| \to \mathbf{0}),$$
where B is a positive constant and $\alpha \in (-d, \infty)$.

- **Using this with the Fourier representation of $\mathcal{S}(t)$ yields following general asymptotic expansion:**
$$\mathcal{S}(\infty) - \mathcal{S}(t) = \frac{B\,\Gamma((d+\alpha)/2)\,\phi_2}{\pi^{d/2}\,\Gamma(d/2)\,(Dt/a^2)^{(d+\alpha)/2}} + o\left((Dt/a^2)^{-(d+\alpha)/2}\right) \ (Dt/a^2 \gg 1).$$

- **Large-t behavior is determined by exponent α and dimension d, i.e., excess spreadability decays to zero as a power-law $\frac{1}{t^{(d+\alpha)/2}}$, implying a faster decay as α increases for fixed d and finite α.**

- **In stealthy limit ($\alpha \to \infty$), predicted infinitely-fast decay rate implies exponentially fast decay.**

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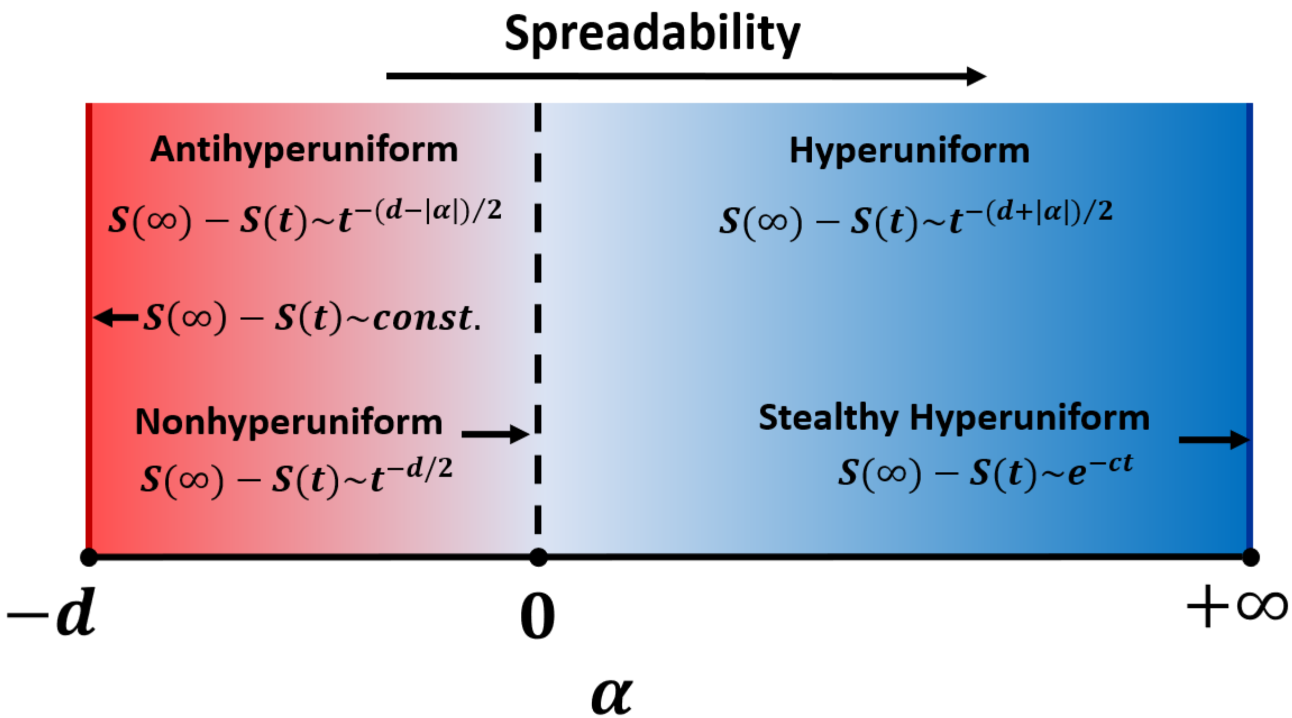

- What stealthy structure maximizes spreadability?

CONCLUSIONS

- There are a plethora of statistical descriptors of microstructures that arise in rigorous relations for various effective properties, leading to accurate estimates.
- Cross-property relations link seemingly different effective properties.
- Hyperuniformity concept provides a unified means of categorizing and characterizing crystals, quasicrystals and special correlated disordered systems according to their large-scale behaviors.
- Disordered hyperuniform materials are ideal states of amorphous matter that often are endowed with novel bulk properties that we are only beginning to discover.
- Spreadability of diffusion information $\mathcal{S}(t)$ across timescales is a powerful tool to dynamically probe and classify all translationally invariant two-phase microstructures across length scales.
- Order metrics have been devised to quantify the degree of order across length scales.
- These general findings have implications for the design of two-phase media with desirable transport, optical and mechanical properties.

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Grant Support

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Paul Steinhardt (Princeton)
Frank Stillinger (Princeton)
Ge Zhang (Princeton/Hong Kong)

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