On the Non-isothermal Electrodiffusion and Nernst-Planck-Boussinesq System

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IPAM: Embracing Stochasticity in Electrochemical Modeling

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Outline

- Review of Nernst-Planck-Navier-Stokes system
- Introduction of Nernst-Planck-Boussinesq system
- Global existence of weak solutions in $3D$
- Long-time dynamics and convergence to the steady state

- Electrodiffusion is a phenomenon describing the diffusion of ions under the influence of an electric field induced by ion charges themselves.
- It has various real-world applications in neuroscience, semiconductor theory, water purification, desalination, ion separation, etc.
- Suppressing dendrite growth will significantly improve the performance of batteries. Electrodiffusion near the dendrite nucleation site can change the local mass transport and ultimately affect dendrite growth (Tan–Ryan, 2016).

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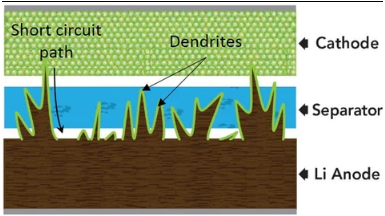

- Electrodiffusion of ions is described by the Nernst-Planck (NP) equations:

$$\partial_t c_i + \nabla \cdot j_i = 0, \quad j_i = uc_i - D_i \nabla c_i - D_i \frac{ez_i}{k_B T} c_i \nabla \Psi, \quad i = 1, \dots, N,$$

$$-\varepsilon \Delta \Psi = eN_A \sum_{i=1}^{N} z_i c_i = F \sum_{i=1}^{N} z_i c_i =: \rho.$$

- c_i: non-negative functions, i-th ionic species concentrations.
- j_i: the flux, three components: advection, diffusion, and electromigration.
- Ψ: electric potential, ρ: charge density, u: the velocity field, T: absolute temperature.
- $D_i > 0$ (constant): diffusivities, $z_i \in \mathbb{R}$ (constant): valences, k_B: Boltzmann constant, e: elementary charge, ε: the dielectric permittivity of the solvent, N_A: Avogadro constant, $F = eN_A$: Faraday constant.
- The electromigration term $-D_i \frac{ez_i}{k_B T} c_i \nabla \Psi = -D_i \frac{Fz_i}{RT} c_i \nabla \Psi$ where $R = k_B N_A$ is the molar gas constant.
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- When coupled to the incompressible homogeneous Navier-Stokes equations:

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = -\rho \nabla \Psi,$$
$$\nabla \cdot u = 0,$$

the system is called the Nernst-Planck-Navier-Stokes (NPNS) system.

- The NPNS system is typically studied in an open smooth bounded domain $\mathcal{D} \subseteq \mathbb{R}^d$ or in torus $\mathbb{T}^d$ with $d = 2, 3$.
- For bounded domain case, the boundary conditions: $u|_{\partial\mathcal{D}} = 0$ and $\Psi|_{\partial\mathcal{D}} = V(x)$, imposed voltage.
- For c_i, there are primarily three types of boundary conditions:
  1. Blocking boundary condition: $(j_i \cdot \hat{n})|_{\partial\mathcal{D}} = 0$ for $i = 1, \ldots, N$. Boundaries function as impermeable barriers, preventing ions from crossing them.
  2. Dirichlet: $c_i|_{\partial\mathcal{D}} = \gamma_i(x)$ for $i = 1, \ldots, N$ with $\gamma_i(x)$ being nonnegative functions. Ion-selective (or perm-selective) membranes that uphold a fixed ion concentration.
  3. Selective boundary condition: a mixed type of the previous two.

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Under different boundary conditions and different ionic settings: number of species; relation of z_i; relation of D_i,

- Local/Global in time existence of weak/strong solutions;
- Uniqueness, continuous dependence on the initial data (well-posedness);
- Long time dynamics;
- Stability of steady states (Boltzmann states);
- Justification of interior electroneutrality.
- ....

- Existence of global weak solutions in 3D: Jerome–Sacco (2009), Ryham (2009), Schmuck (2009), Fischer–Saal (2017).
- Global existence and uniqueness of 2D solution in smooth bounded domain as well as the convergence to steady state: Constantin–Ignatova (2018).
- The nonlinear stability of Boltzmann states in 3D: Constantin–Ignatova–Lee (2022).
- Interior electroneutrality: Constantin–Ignatova–Lee (2021).
- With periodic boundary conditions, the long-time dynamics: Abdo–Ignatova (2021, 2024) and the analyticity of solutions: Abdo–Ignatova (2022).
- Some other electrodiffusion models, including Nernst-Planck-Euler: Zhang–Yin (2015, 2020), Ignatova–Shu (2021), Shen–Wang (2022), Abdo–Lee–Wang (2022) and Nernst-Planck-Darcy: Ignatova–Shu (2022), Abdo–Lee–Wang (2022).

- Main challenge: the electromigration term $\nabla \cdot (D_i \frac{ez_i}{k_B T} c_i \nabla \Psi)$.
- A direct L^2 estimate leads to $D_i \frac{ez_i}{k_B T} \int_{\mathcal{D}} c_i \nabla \Psi \nabla c_i$, which results in $\|c_i\|_{L^2}^k$ where the exponent $k > 2$ depends on the dimension. This only allows a local solution.
- Special case: $N = 2$, $D_1 = D_2 = D$, $z_1 = -z_2 = z > 0$. Introduce $\sigma = Fz(c_1 + c_2)$ and noticing that $\rho = Fz(c_1 - c_2)$, then (T is assumed to be constant):

$$\partial_t \sigma + u \cdot \nabla \sigma - D\Delta\sigma - \frac{Dez}{k_B T} \nabla \cdot (\rho \nabla \Psi) = 0,$$
$$\partial_t \rho + u \cdot \nabla \rho - D\Delta\rho - \frac{Dez}{k_B T} \nabla \cdot (\sigma \nabla \Psi) = 0.$$

The L^2 estimates lead to

$$\int_{\mathcal{D}} \frac{1}{T}\Big(\nabla \cdot (\sigma \nabla \Psi)\rho + \nabla \cdot (\rho \nabla \Psi)\sigma\Big) dx = -\int_{\mathcal{D}} \frac{1}{\varepsilon T} \rho^2 \sigma \, dx \leq 0,$$

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- In the general setting for ionic concentrations, study the relative entropy (Kullback-Leibler divergence)

$$\int \sum_{i=1}^{N} c_i \log\left(\frac{c_i}{c_i^*}\right) dx,$$

which can provide some good dissipations.

- Boltzmann steady states:

$$c_i^*(x) = Z_i^{-1} \exp\left(-\frac{z_i e}{k_B T}\Psi^*(x)\right),$$

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Nernst-Planck-Boussinesq System

- Limitation of existing electrodiffusion models:
  1. The exclusion of temperature variations, which simplifies the nonlinear structure of the model (in particular, the electromigration term $\nabla \cdot (\frac{z_i}{T} c_i \nabla \Psi)$).
  2. The salinity resulting from the variation of the ionic concentrations creates a buoyancy force, which has been disregarded.
- The magnitude of temperature variations and salinity might be small, but their regularity can play an important role.
- Consider Boussinesq approximation (ignores density differences except where they appear in buoyancy) for u and T:

$$\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p = -g\beta \vec{k} - \rho \nabla \Psi, \tag{1a}$$

$$\nabla \cdot u = 0, \tag{1b}$$

$$\beta = 1 - \alpha_T (T - T_r) + \alpha_S (S - S_r), \tag{1c}$$

$$\partial_t T + u \cdot \nabla T - \kappa \Delta T = 0, \tag{1d}$$

$$S = \sum_{i=1}^{N} c_i M_i. \tag{1e}$$

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- When $T \equiv T_r$ and $\alpha_S = 0$ in the NPB system, one can recover the NPNS system. NPNS system is a special case of NPB system.
- On a torus $\mathbb{T}^3$, T_r and S_r are the spatial average, and they are invariant in time.
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Challenges

- Goal: Global existence of (weak) solutions: need global in time estimates.
- The appearance of T in the denominator: assume $T_0(x) \geq T^* > 0$ for all $x \in \mathcal{D}$ according to the Third Law of Thermodynamics. By maximum principle, $T(x,t) \geq T^* > 0$.
- All challenges in the NPNS system remain in the NPB system, but things can be worse! For example, in the case of $N = 2$, $D_1 = D_2 = D$, $z_1 = -z_2 = z$ case,

$$\int_{\mathcal{D}} \frac{1}{T}\Big(\nabla \cdot (\sigma \nabla \Psi)\rho + \nabla \cdot (\rho \nabla \Psi)\sigma\Big) dx = -\int_{\mathcal{D}} \frac{1}{\varepsilon T}\rho^2 \sigma dx + \int_{\mathcal{D}} \frac{\rho\sigma}{T^2} \nabla T \cdot \nabla \Psi dx.$$

- The dissipation structure in many NPNS works is no longer valid because the temperature is no longer a constant. Integration by parts will introduce some bad terms involving with T as well as its derivatives.
- The presence of salinity $S = \sum c_i M_i$ (through buoyancy force) in the momentum equation requires some control of c_i.

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$$\int_{\mathcal{D}} \frac{1}{T}\Big(\nabla\cdot(\sigma\nabla\Psi)\rho + \nabla\cdot(\rho\nabla\Psi)\sigma\Big)dx = -\int_{\mathcal{D}} \frac{1}{\varepsilon T}\rho^2\sigma dx + \int_{\mathcal{D}} \frac{\rho\sigma}{T^2}\nabla T\cdot\nabla\Psi dx.$$

- The dissipation structure in many NPNS works is no longer valid because the temperature is no longer a constant. Integration by parts will introduce some bad terms involving with T as well as its derivatives.
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Estimates

- The estimate of relative entropy $E(t) := \sum_{i=1}^N \int_{\mathbb{T}^3} c_i(t) \log \frac{c_i(t)}{\bar{c}_i} dx$:

$$\frac{d}{dt}E + \underbrace{\sum_{i=1}^N D_i \left\| \frac{1}{\sqrt{c_i}} \left(\nabla c_i + \frac{ez_i c_i \nabla \Psi}{k_B T} \right) \right\|_{L^2}^2}_{:=D_A} = \underbrace{\sum_{i=1}^N D_i \int_{\mathbb{T}^3} \left(\nabla c_i + \frac{ez_i}{k_B T} c_i \nabla \Psi \right) \cdot \left(\frac{ez_i}{k_B T} \nabla \Psi \right) dx}_{:=\mathcal{N}}, \quad (3)$$

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Challenges and Strategies

- (continue) The nonlinear term $\int_{\mathbb{T}^3} u \cdot \nabla \Psi \rho dx$ is cancelled by the electric force $\rho \nabla \Psi$ in momentum equation when estimating $\|u\|_{L^2}^2$.
- In the estimate of $\|u\|_{L^2}^2$, we need to control the term $\int_{\mathbb{T}^3} u c_i dx$. Keep in mind we don't have L^2 control of c_i.
- By triangle inequality, D_A and D_B lead to the dissipation $\|\nabla \sqrt{c_i}\|_{L^2}^2$.
- As $c_i \geq 0$, write $u c_i = u \sqrt{c_i} \sqrt{c_i}$, and do L^3 for each term using Hölder inequality, and then interpolation. $\|\nabla \sqrt{c_i}\|_{L^2}$ can be controlled by the dissipation, and

$$\|\sqrt{c_i}\|_{L^2}^2 = \|c_i\|_{L^1} = \overline{c_i} = \overline{c_i(0)}.$$

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Global Existence of Weak Solutions in 3D

- Let $\{\phi_\eta\}_{\eta\in(0,1)}$ be a family of standard periodic mollifiers with $\int_{\mathbb{T}^3}\phi_\eta(x)dx = 1$. Denote by $\mathcal{J}_\eta$ the operator $\mathcal{J}_\eta f = \phi_\eta * f = \int_{\mathbb{T}^3}\phi_\eta(x-y)f(y)dy$.
- A regularization scheme of the NPB system:

$$\partial_t c_i^\eta + \mathcal{J}_\eta u^\eta \cdot \nabla c_i^\eta - D\Delta c_i^\eta - D\frac{e}{k_B}\nabla\cdot(z_i\frac{1}{T^\eta}c_i^\eta\nabla\mathcal{J}_\eta\mathcal{J}_\eta\Psi^\eta) = 0, \tag{4a}$$

$$-\varepsilon\Delta\Psi^\eta = \rho^\eta = \sum_{i=1}^N eN_A z_i c_i^\eta, \tag{4b}$$

$$\partial_t u^\eta + \mathcal{J}_\eta u^\eta \cdot \nabla u^\eta - \nu\Delta u^\eta + \nabla p^\eta$$
$$= g\left(\alpha_T(T^\eta - T_r^\eta) - \alpha_S\left(\sum_{i=1}^N c_i^\eta M_i - S_r^\eta\right)\right)\vec{k} - \mathcal{J}_\eta\left(\rho^\eta\nabla\mathcal{J}_\eta\mathcal{J}_\eta\Psi^\eta\right), \tag{4c}$$

$$\nabla\cdot u^\eta = 0, \tag{4d}$$

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- Initial conditions $(c_i^\eta(0), u_0^\eta, T_0^\eta) = (\mathcal{J}_\eta c_i(0), \mathcal{J}_\eta u_0, \mathcal{J}_\eta T_0)$.
- The conditions $c_i(0)\geq 0$, $T_0 \geq T^* > 0$, and $\overline{u_0} = 0$ imply that $c_i^\eta(0)\geq 0$, $T_0^\eta \geq T^* > 0$, and $\overline{u_0^\eta} = 0$. Also $T_r^\eta = \overline{T^\eta} = T_r$ and $S_r^\eta = \sum_{i=1}^N \overline{c_i^\eta} M_i = S_r$.

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- Assume the initial conditions of the original NPB system satisfy:
  1. $c_i(0)$ is non-negative and satisfying the compatibility condition $\sum_{i=1}^N eN_A z_i \overline{c_i(0)} = 0$;
  2. T_0 is bounded below by $T_0 \geq T^* > 0$;
  3. $\nabla\Psi_0 \in L^2,\ c_i(0) \in L^1,\ c_i(0)\log c_i(0) \in L^1,\ T_0 \in L^2$, and $u_0 \in H = \overline{\left\{u \in C^\infty(\mathbb{T}^3) : \nabla \cdot u = 0,\ \int_{\mathbb{T}^3} u = 0\right\}}^{\|\cdot\|_{L^2}}$.

Proposition 1

For each $\eta > 0$, consider system (4) with initial conditions $(c_i^\eta(0), u_0^\eta, T_0^\eta) = (\mathcal{J}_\eta c_i(0), \mathcal{J}_\eta u_0, \mathcal{J}_\eta T_0)$. Then for any $\mathcal{T} > 0$, there exists a unique strong solution $(c_i^\eta, u^\eta, T^\eta)$ to system (4) on $[0, \mathcal{T}]$ such that

$$c_i^\eta, T^\eta \in C([0,\mathcal{T}]; C^\infty(\mathbb{T}^3)), \quad \textit{and} \quad u^\eta \in C([0,\mathcal{T}]; C^\infty(\mathbb{T}^3) \cap H). \tag{5}$$

Moreover, $c_i^\eta(t) \geq 0$ and $T^\eta(t) \geq T^*$ for any $t \in [0, \mathcal{T}]$.

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- The proof is based on iteration of the linearization of the regularization scheme, and prove uniform bound in terms of the iteration steps.

Proposition 2

Assume the same initial conditions as before, but now suppose $T_0 \in L^p$ for some $p \geq 2$. Then the sequence of solutions $(u^\eta, c_i^\eta, T^\eta)$ and the corresponding Ψ^η and ρ^η satisfy the following uniform-in-η bounds:

$$\sqrt{c_i^\eta} \text{ are uniformly bounded in } L^\infty(0,\mathcal{T};L^2) \cap L^2(0,\mathcal{T};H^1),$$
$$T^\eta \text{ are uniformly bounded in } L^\infty(0,\mathcal{T};L^p) \cap L^2(0,\mathcal{T};H^1),$$
$$u^\eta \text{ are uniformly bounded in } L^\infty(0,\mathcal{T};H) \cap L^2(0,\mathcal{T};V),$$
$$c_i^\eta \text{ and } \rho^\eta \text{ are uniformly bounded in } L^2(0,\mathcal{T};L^{\frac{3}{2}}) \cap L^{\frac{4}{3}}(0,\mathcal{T};W^{1,\frac{6}{5}}).$$

Proposition 3

Suppose the same assumption on the initial conditions in Proposition 2 holds. Then the sequence of time derivatives $(\partial_t T^\eta, \partial_t u^\eta, \partial_t \sqrt{c_i^\eta + 1})$ satisfy

$$\partial_t T^\eta \text{ are uniformly bounded in } L^{\frac{4}{3}}(0,\mathcal{T};H^{-1}), \quad \partial_t u^\eta \text{ are uniformly bounded in } L^{\frac{4}{3}}(0,\mathcal{T};\mathcal{D}(A)'),$$
$$\partial_t \sqrt{c_i^\eta + 1} \text{ are uniformly bounded in } L^1(0,\mathcal{T};H^{-2}).$$

- The estimate of T^η is straightforward:

$$\|T^\eta(t)\|_{L^2}^2 + 2\kappa \int_0^t \|\nabla T^\eta(s)\|_{L^2}^2 ds = \|T_0^\eta\|_{L^2}^2, \quad \|T^\eta(t)\|_{L^p} \le \|T_0^\eta\|_{L^p},$$

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- Perform estimate of u, $\nabla\Psi$, and E simultaneously:

$$\frac{d}{dt}\left(\varepsilon\|\nabla \mathcal{J}_\eta \Psi^\eta\|_{L^2}^2 + \|u^\eta\|_{L^2}^2 + 2N_A k_B T^* E^\eta\right) + \text{dissipation} \le C\|T^\eta - T_r^\eta\|_{L^2}^2 + \overline{c_i^\eta(0)}^2 \|u^\eta\|_{L^2}^2.$$

- In order to control $E^\eta(0)$ by $E(0)$, we need:

Lemma 1

Let $f \in L^1(\mathbb{T}^3)$ be a nonnegative function with $\int_{\mathbb{T}^3} f \log f dx < \infty$. For any $\eta \in (0,1)$, it holds that

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$$\|T^\eta(t)\|_{L^2}^2 + 2\kappa \int_0^t \|\nabla T^\eta(s)\|_{L^2}^2 ds = \|T_0^\eta\|_{L^2}^2, \quad \|T^\eta(t)\|_{L^p} \le \|T_0^\eta\|_{L^p},$$

$$\|T^\eta(t) - T_r^\eta\|_{L^2}^2 \le \|T_0^\eta - T_r^\eta\|_{L^2}^2 e^{-\frac{2\kappa}{C_p} t}.$$

- Perform estimate of u, $\nabla\Psi$, and E simultaneously:

$$\frac{d}{dt}(\varepsilon \|\nabla \mathcal{J}_\eta \Psi^\eta\|_{L^2}^2 + \|u^\eta\|_{L^2}^2 + 2N_A k_B T^* E^\eta) + \text{dissipation} \le C\|T^\eta - T_r^\eta\|_{L^2}^2 + \overline{c_i^\eta(0)}^2 \|u^\eta\|_{L^2}^2.$$

- In order to control $E^\eta(0)$ by $E(0)$, we need:

Lemma 1

Let $f \in L^1(\mathbb{T}^3)$ be a nonnegative function with $\int_{\mathbb{T}^3} f \log f dx < \infty$. For any $\eta \in (0,1)$, it holds that

$$\int_{\mathbb{T}^3} \mathcal{J}_\eta f \log \mathcal{J}_\eta f dx \le \int_{\mathbb{T}^3} f \log f dx.$$

- Note that $\partial_t \sqrt{c_i^\eta} = \frac{\partial_t c_i^\eta}{2\sqrt{c_i^\eta}}$ and $\partial_t \sqrt{c_i^\eta + 1} = \frac{\partial_t c_i^\eta}{2\sqrt{c_i^\eta + 1}}$. The "$+1$" is to prevent denominator from vanishing.
- Based on Proposition 2 and 3, we have

Corollary 1

Suppose the same assumption on the initial conditions in Proposition 2 holds. Then there exists a decreasing sequence of numbers $\{\eta_k\}_{k\in\mathbb{N}}$ with $\lim_{k\to\infty} \eta_k = 0$ such that the sequence of solutions $(u^{\eta_k}, c_i^{\eta_k}, T^{\eta_k})$ satisfy

$$u^{\eta_k} \rightharpoonup u \text{ in } L^2(0,\mathcal{T};V), \quad u^{\eta_k} \overset{*}{\rightharpoonup} u \text{ in } L^\infty(0,\mathcal{T};H), \quad u^{\eta_k} \to u \text{ in } L^2(0,\mathcal{T};H),$$

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- The proof follows directly from Proposition 2, Proposition 3, the Banach-Alaoglu theorem, and the Aubin-Lions compactness theorem.

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Theorem 1 (Abdo–Hu–L., preprint 2024)

Suppose the same assumption on the initial conditions in Proposition 2 holds, but $T_0 \in L^{3+\delta}$ for a fixed and arbitrarily small $\delta > 0$. Then for any time $\mathcal{T} > 0$ there exists a weak solution (c_i, u, T) to system (2) on $[0, \mathcal{T}]$ satisfying

$$u \in L^\infty(0,\mathcal{T};H) \cap L^2(0,\mathcal{T};V), \quad T \in L^\infty(0,\mathcal{T};L^{3+\delta}) \cap L^2(0,\mathcal{T};H^1),$$

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with the property that $c_i(x,t) \geq 0$ and $T(x,t) \geq T^*$ for a.e. $(x,t) \in \mathbb{T}^3 \times [0,\mathcal{T}]$.

- $c_i(x,t) \geq 0$ and $T(x,t) \geq T^*$ a.e. follows from the results that $c_i^{\eta_k}(x,t) \geq 0$ and $T^{\eta_k}(x,t) \geq T^*$ for all $k \in \mathbb{N}$.
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Long-time Dynamics

Theorem 2 (Abdo–Hu–L. 2024, preprint)

Suppose that (c_i, u, T) is a weak solution to system (2) obtained in Theorem 1. Then the temperature $T(t)$ decays exponentially in time to the initial average $\overline{T_0} = T_r$ in L^2. Moreover, if the initial averages of the concentrations $\overline{c_i(0)}$ satisfy the condition

$$\max_{i\in\{1,\ldots,N\}} \overline{c_i(0)} \le \frac{DRT^*\nu}{C\alpha_S^2}, \tag{6}$$

where C is some constant depending only on the domain, then $(c_i(t), u(t))$ decays exponentially in time to $(\overline{c_i(0)}, 0)$ in L^1 and L^2, respectively, and the relative entropy $\int_{\mathbb{T}^3} c_i \log \frac{c_i(t)}{\bar{c}_i} dx$ decays exponentially in time to 0. In particular, if the salinity effect in the buoyancy is negligible, i.e., $\alpha_S = 0$, then the decay holds unconditionally without any size assumptions on $\overline{c_i(0)}$.

- Recall that

$$\frac{d}{dt}(\varepsilon \|\nabla \mathcal{J}_\eta \Psi^\eta\|_{L^2}^2 + \|u^\eta\|_{L^2}^2 + 2N_A k_B T^* E^\eta)$$
$$+ \frac{2D}{\varepsilon}\|\mathcal{J}_\eta \rho^\eta\|_{L^2}^2 + \nu\|\nabla u^\eta\|_{L^2}^2 + \frac{DN_A k_B T^*}{2}\sum_{i=1}^N \|\nabla\sqrt{c_i^\eta}\|_{L^2}^2$$
$$+ \dots$$
$$\leq C\|T^\eta - T_r\|_{L^2}^2 + \overline{c_i(0)}^2\|u^\eta\|_{L^2}^2.$$

- Denote by the energy and the dissipation

$$\mathcal{E}^\eta := \varepsilon\|\nabla \mathcal{J}_\eta \Psi^\eta\|_{L^2}^2 + \|u^\eta\|_{L^2}^2 + 2N_A k_B T^* E^\eta,$$

$$\mathcal{D}^\eta := \frac{2D}{\varepsilon}\|\mathcal{J}_\eta \rho^\eta\|_{L^2}^2 + \nu\|\nabla u^\eta\|_{L^2}^2 + \frac{DN_A k_B T^*}{2}\sum_{i=1}^N \|\nabla\sqrt{c_i^\eta}\|_{L^2}^2.$$

- Elliptic estimate: $\frac{2D}{\varepsilon}\|\mathcal{J}_\eta \rho^\eta\|_{L^2}^2 \geq \frac{2D\varepsilon}{C_e}\|\nabla \mathcal{J}_\eta \Psi^\eta\|_{L^2}^2$,
- Poincaré inequality: $\frac{\nu}{2}\|\nabla u^\eta\|_{L^2}^2 \geq \frac{\nu}{2C_p}\|u^\eta\|_{L^2}^2$.
- Logarithmic Sobolev inequality:

$$\frac{DN_A k_B T^*}{4}\sum_{i=1}^N \|\nabla\sqrt{c_i^\eta}\|_{L^2}^2 \geq \frac{DN_A k_B T^*}{4C_s}\sum_{i=1}^N \int_{\mathbb{T}^3} c_i^\eta \log\frac{c_i^\eta}{\bar{c}_i}\,dx = \frac{DN_A k_B T^*}{4C_s}E^\eta.$$

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- The term $\overline{c_i(0)}^2 \|u\|_{L^2}^2$ on the right-hand side will destroy the decay. It comes from the salinity term in the momentum equation. Revisit the estimate:

$$\left| \int_{\mathbb{T}^3} g\alpha_S (\sum_{i=1}^N c_i^\eta M_i - S_r^\eta) u_3^\eta \, dx \right| \leq \frac{\nu}{4} \|\nabla u^\eta\|_{L^2}^2 + C\nu^{-1}\alpha_S^2 \sum_{i=1}^N \overline{c_i(0)} \|\nabla \sqrt{c_i^\eta}\|_{L^2}^2.$$

- If $C\nu^{-1}\alpha_S^2 \max_{i\in\{1,\dots,N\}} \overline{c_i(0)} \leq \frac{DN_A k_B T^*}{4} \Leftrightarrow \max_{i\in\{1,\dots,N\}} \overline{c_i(0)} \leq \frac{DN_A k_B T^* \nu}{4C\alpha_S^2}$, the orange part can be absorbed by the dissipation.
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$$\mathcal{E}^\eta(t) \lesssim e^{-Mt},$$

- If (c_i, u, T) is a weak solution obtained by passing to the limit $\eta \to 0$ from $(c_i^\eta, u^\eta, T^\eta)$, then $\|u^\eta(t)\|_{L^2} \to \|u(t)\|_{L^2}$ in $L^2(0, \mathcal{T})$ and $E^\eta(t) \to E(t)$ in $L^1(0, \mathcal{T})$. By passing to a subsequence in $\eta \to 0$, it follows that for a.e. $t \in [0, \mathcal{T}]$,

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- Global existence of 3D weak solutions for the case when ionic diffusivities are identical.
- With some size constraint on $\overline{c_i(0)}$, we have the exponential in time decay of $(c_i(t), u(t), T(t))$ to $(\overline{c_i(0)}, 0, \overline{T_0})$, and $E(t) = \int_{\mathbb{T}^3} c_i(t) \log \frac{c_i(t)}{\bar{c}_i} dx$ to 0. The size condition can be dropped if the salinity effect can be ignored, i.e., $\alpha_S = 0$.
- Some interesting future questions:
  1. Global existence of 3D weak solutions when ionic diffusivities are different.
  2. The study in bounded domain with physical boundary conditions.
  3. Decay of solutions to steady state without size constraint on $\overline{c_i(0)}$.
  4. The corresponding Boltzmann states and their stability.

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  3. Decay of solutions to steady state without size constraint on $\overline{c_i(0)}$.
  4. The corresponding Boltzmann states and their stability.

Conclusion

- Main differences and challenges between NPB system and other electrodiffusion models are the non-constant temperature and the salinity.
- Global existence of 3D weak solutions for the case when ionic diffusivities are identical.
- With some size constraint on $\overline{c_i(0)}$, we have the exponential in time decay of $(c_i(t), u(t), T(t))$ to $(\overline{c_i(0)}, 0, \overline{T_0})$, and $E(t) = \int_{\mathbb{T}^3} c_i(t) \log \frac{c_i(t)}{\bar{c}_i} dx$ to 0. The size condition can be dropped if the salinity effect can be ignored, i.e., $\alpha_S = 0$.
- Some interesting future questions:
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Thank you!