# Existence and long time behavior of weak solutions to the Fokker-Planck-Alignment models

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IPAM: "Embracing Stochasticity in Electrochemical Modeling"

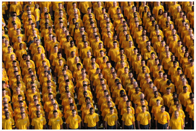
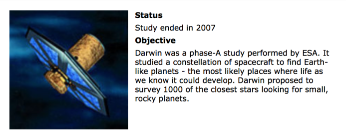
# Examples of collective behavior

- Biology – swarming of insects, bird flocking, fish schools; cell migration;

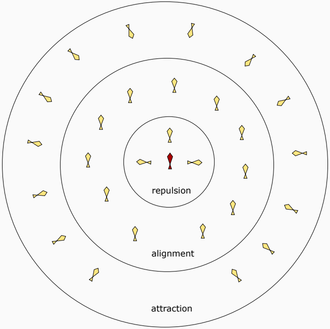


- Social science – opinion dynamics, social networks, economics

- Technology – traffic dynamics, feedback control in MAS, formations of UAVs, swarming of robots, material production, cosmology

# REYNOLD 3ZONE MODEL




1. Collision Avoidance:
avoid collisions with nearby flockmates

2. Velocity Matching: attempt
to match velocity with nearby flockmates

3. Flock Centering:
attempt to stay close to nearby flockmates

Published in *Computer Graphics*, **21**(4), July 1987, pp. 25-34.
(ACM SIGGRAPH '87 Conference Proceedings, Anaheim, California, July 1987.)

**Flocks, Herds, and Schools:
A Distributed Behavioral Model**

Craig W. Reynolds

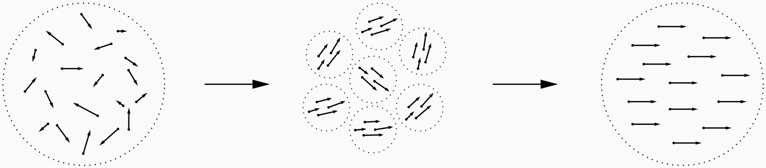
Symbolics Graphics Division

Agent-based models of collective behavior describe dynamics of a number of objects:

$$\mathbf{x}_i \in \Omega = \mathbb{R}^n \text{ or } \mathbb{T}^n, \quad i = 1, \dots, N$$
$$\dot{\mathbf{x}}_i = \mathbf{v}_i$$
$$\dot{\mathbf{v}}_i = \mathbf{F}_i(\mathbf{x}, \mathbf{v})$$

adjusting their directions relative to neighbors via communication $\phi(\mathbf{x}_i, \mathbf{x}_j)$.

*Emergence* is formation of global patterns resulting from local interactions.



# Classical Cucker-Smale Model

Cucker and Smale (2007):

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \sum_{j=1}^{N} m_j \phi(x_i - x_j)(v_j - v_i) \end{cases}$$

Key features:

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Key features:

- $\phi$ is a smooth radially symmetric decreasing communication kernel
- momentrum conservation: $\bar{v} = \sum_i m_i v_i = \bar{v}_0$
- energy dissipation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i |v_i|^2 = -\frac{1}{2} \sum_i m_i m_j |v_i - v_j|^2 \phi(x_i - x_j) \leqslant 0$$

**Theorem ( Cucker, Smale (2007) )**

*Let*

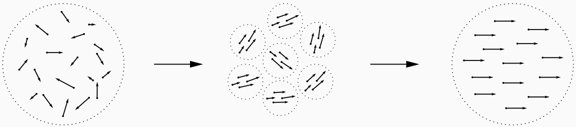
$$\phi(r) = \frac{1}{(1+r^2)^{\beta/2}}, \qquad \beta \leqslant 1$$

*or more generally* $\int_0^\infty \phi(r)dr = \infty$ *(Ha, Tadmor (2008), Ha, Liu (2009)), then there exists* $D, C, \delta > 0$ *such that*

$$\sup_{0 \leqslant t < \infty} \max_{i,j} |x_i - x_j| \leqslant D, \qquad (\textit{flocking})$$

$$\max_{i,j} |v_i - v_j| \leqslant C e^{-\delta t}, \qquad (\textit{alignment})$$

$$x_i - x_j \to \bar{x}_{ij}, \qquad (\textit{strong flocking})$$



## Statistical description

**Cucker-Smale**

$$\dot{x}_i = v_i$$
$$\dot{v}_i = \sum_{j=1}^N m_j \phi(x_i - x_j)(v_j - v_i)$$

mean-field limit: $\sum_{i=1}^N m_i \delta_{v_i} \otimes \delta_{x_i} \to f$

Ha, Tadmor (2008); Ha, Liu (2009)

**Vlasov-Alignment**

$$\partial_t f + v \cdot \nabla_x f = \nabla_v Q(f,f)$$
$$Q(f,f) = \int_{\Omega \times R^n} \phi(x - x')(v - v') f f' \, \mathrm{d}v' \, \mathrm{d}x'$$

monokinetic limit: $f \to \rho \, \delta_0(v - \mathbf{u})$

Figalli, Kang (2019); RS (2020)

**Pressureless Euler Alignment System**

$$\rho_t + \nabla \cdot (\mathbf{u}\rho) = 0$$
$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = (\mathbf{u}\rho)_\phi - \mathbf{u}\rho_\phi$$

notation: $\rho_\phi = \rho * \phi$

# Statistical description with noise

**Stochastic Cucker-Smale**

$$\dot{x}_i = v_i$$
$$\dot{v}_i = CS + \sqrt{2\sigma \sum_{j=1}^{N} m_j \phi(x_i - x_j)}\, \dot{W}_i$$

↓

mean-field limit: $\sum_{i=1}^{N} m_i \delta_{v_i} \otimes \delta_{x_i} \to f$
Bolley, Canizo, Carrillo (2011)
RS (2023)

**Fokker-Planck-Alignment**

$$f_t + v \cdot \nabla_x f = \nabla_v Q(f,f) + \sigma \rho_\phi \Delta_v f$$

↓

Maxwellian limit: $f \to \frac{\rho(x,t)}{(2\pi)^{n/2}} e^{-\frac{|v - \mathbf{u}(x,t)|^2}{2}}$
Karper, Mellet, Trivisa (2014-16)

**Isothermal Euler Alignment System**

$$\rho_t + \nabla \cdot (\mathbf{u}\rho) = 0$$
$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \sigma \nabla \rho = (\mathbf{u}\rho)_\phi - \mathbf{u}\rho_\phi$$

**Theorem (Carrillo, Fornasier, Rosado, Toscani (2010))**

*Let* $\int_0^\infty \phi(r)dr = \infty$ *then for any solution to Vlasov-Alignment equation there exists* $C, \delta > 0$ *such that*

$$\int_{\Omega \times \mathbb{R}^n} |v - \bar{\mathbf{u}}|^2 f \, dv \, dx \leqslant C e^{-\delta t}.$$

**Theorem (Tan, Tadmor (2014))**

*Let* $\int_0^\infty \phi(r)dr = \infty$ *then for any solution to the pressureless EAS there exists* $D, C, \delta > 0$ *such that*

$$\sup_{0 \leqslant t < \infty} \operatorname{diam}\{\operatorname{supp} \rho\} \leqslant D,$$

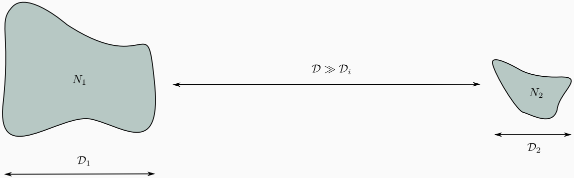
$$\max_{x, y \in \operatorname{supp} \rho} |\mathbf{u}(t,x) - \mathbf{u}(t,y)| \leqslant C e^{-\delta t}$$

## Applications


- L. Perea, P. Elosegui, and G. Gomez. Extension of the Cucker-Smale control law to space flight formations. *Journal of Guidance, Control, and Dynamics*, 32:526 – 536, 2009. (Darwin ESA mission, $\beta = 0.4$)
- M.Bongini, M.Fornasier and D.Kalise, (Un)conditional consensus emergence under perturbed and decentralized feedback controls, *Discr. Contin. Dyn. Syst. Ser. A* 35 (2015) 4071–4094.
- Y.-P. Choi, D. Kalise, J. Peszek and A. A. Peters, A collisionless singular Cucker–Smale model with decentralized formation control, *SIAM J. Appl. Dyn. Syst.* 18 (2019), no. 4, 1954–1981.
- Zhiping Mao, Zhen Li, George Em Karniadakis, Nonlocal flocking dynamics: Learning the fractional order of PDEs from particle simulations. *Commun. Appl. Math. Comput.* 1 (2019), no. 4, 597–619.
- R. Shu, E. Tadmor, Flocking hydrodynamics with external potentials. *Arch. Ration. Mech. Anal.*, 238(1):347–381, 2020.

# Motsch-Tadmor model (2011)

Large mass flocks can overtake the dynamics of far away, small mass flocks.




Motsch-Tadmor model renormalizes the averages

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{\sum_{j=1}^N m_j \phi(x_i - x_j)} \sum_{j=1}^N m_j \phi(x_i - x_j)(v_j - v_i) \end{cases}$$

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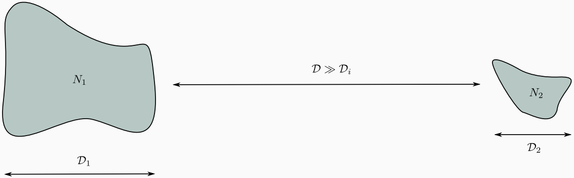


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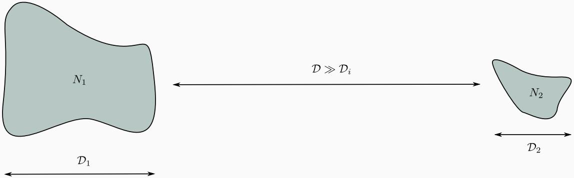


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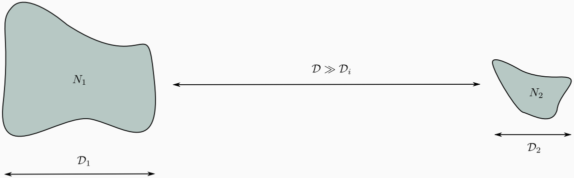


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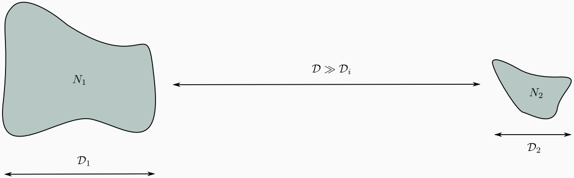


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Key features:

- the effective kernel $\phi(x-y)/\rho_\phi(x)$ is not symmetric
- no conservation of momentrum conservation, no energy dissipation
- BUT, the Cucker-Smale theorem holds!

# CS vs MT on the same time scale (T. Teolis, RS, 2025)

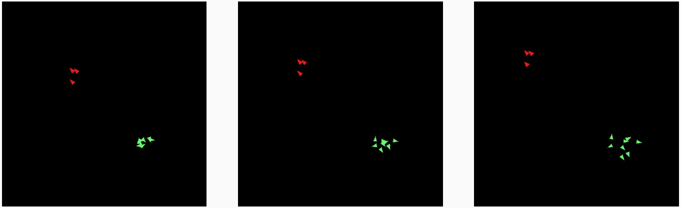

**Figure 1: Cucker-Smale** dynamics – the light green flock is hijacked by the larger flock in red.

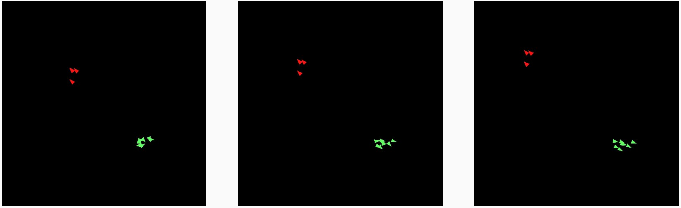

**Figure 2: Motsch-Tadmor** dynamics – the light green flock is mostly independent and conforms to the prediction of the model.

Cucker-Smale ($\mathcal{M}_{\mathrm{CS}}$)

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \rho_\phi \left( \frac{(\mathbf{u}\rho)_\phi}{\rho_\phi} - \mathbf{u} \right)$$

Motsch-Tadmor ($\mathcal{M}_{\mathrm{MT}}$)

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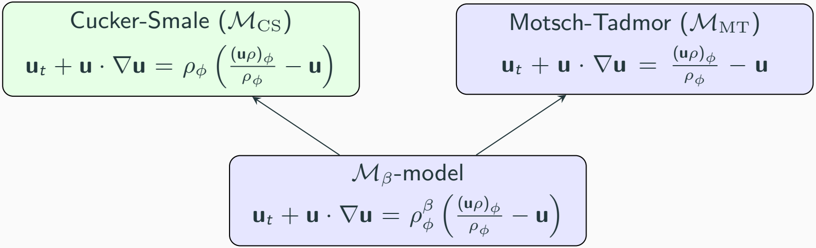
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$\mathcal{M}_\beta$-model
$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \rho_\phi^\beta\left(\frac{(\mathbf{u}\rho)_\phi}{\rho_\phi} - \mathbf{u}\right)$

### Cucker-Smale ($\mathcal{M}_{\mathrm{CS}}$)

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### Motsch-Tadmor ($\mathcal{M}_{\mathrm{MT}}$)

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \frac{(\mathbf{u}\rho)_\phi}{\rho_\phi} - \mathbf{u}$$

### $\mathcal{M}_\beta$-model

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \rho_\phi^\beta\left(\frac{(\mathbf{u}\rho)_\phi}{\rho_\phi} - \mathbf{u}\right)$$

### $\mathcal{M}_\phi$-model

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \left(\frac{(\mathbf{u}\rho)_\phi}{\rho_\phi}\right)_\phi - \mathbf{u}$$

### Segregation ($\mathcal{M}_{\mathrm{seg}}$)

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \sum_{i=1}^{L} g_i\,\frac{\int_\Omega \mathbf{u} g_i \rho\, dy}{\int_\Omega g_i \rho\, dy} - \mathbf{u}$$

Cucker-Smale ($\mathcal{M}_{\mathrm{CS}}$)

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Motsch-Tadmor ($\mathcal{M}_{\mathrm{MT}}$)

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multi-flocks, multi-species,
topological models, etc etc

# Environmental Averaging

– RS, Environmental Averaging, *EMS Surveys in Mathematical Sciences*, (2024) 134 pp.

- $[\mathbf{u}]_\rho$ := an averaging operator
- $\mathrm{s}_\rho$ := the strength of interactions

The weighted averages

$$\mathrm{s}_\rho[\mathbf{u}]_\rho = \int_\Omega \phi_\rho(x,y) u(y) \, \mathrm{d}\rho(y), \quad \rho\text{-a.e.} \tag{1}$$

are represented by a non-negative communication kernel $\phi_\rho \in L^1(\mathrm{d}\rho \otimes \mathrm{d}\rho)$, $\rho \in \mathcal{P}$, satisfying

$$\int_\Omega \phi_\rho(x,y) \, \mathrm{d}\rho(y) = \mathrm{s}_\rho(x), \quad \rho\text{-a.e.} \tag{2}$$

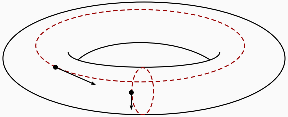
| MODEL | $\mathcal{M}_{\mathrm{CS}}$ | $\mathcal{M}_{\mathrm{MT}}$ | $\mathcal{M}_\phi$ | $\mathcal{M}_{\mathrm{seg}}$ |
|---|---|---|---|---|
| $\phi_\rho$ | $\phi(x-y)$ | $\dfrac{\phi(x-y)}{\rho * \phi(x)}$ | $\displaystyle\int_\Omega \frac{\phi(x-z)\phi(y-z)}{\rho * \phi(z)} \, \mathrm{d}z$ | $\displaystyle\sum_{l=1}^{L} \frac{g_l(x) g_l(y)}{\int_\Omega \rho g_l \, \mathrm{d}z}$ |

## LOCAL communication – the problem of Emergence

Let us assume only local communication:

$$\phi(x-y) \geqslant c_0 \mathbb{1}_{|x-y|\leqslant r_0}$$

The problem is "locked states":



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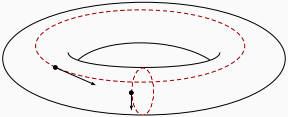

Possible ways around it:

- assume $r_0$-connectivity for all time or connectivity initially but strong communication Morales, Peszek, Tadmor (2019);
- Generic Alignment Conjecture: For almost every initial data $(\mathbf{x}_0, \mathbf{v}_0) \in \mathbb{T}^{nN} \times \mathbb{R}^{nN}$ solutions to the agent based system align.
  RS (2023): YES if $N = 2$ or $n = 1$ or for sticky particle model.

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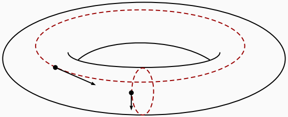

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- disrupt locked states by small noise $\to$ relaxation problem.

# Fokker-Planck-Alignment model

Locked states are disrupted by stochastic noise

$$\dot{v}_i = \mathrm{s}_i([v]_i - v_i) + \sqrt{2\sigma \mathrm{s}_i}\dot{W}_i,$$

where $W_i$'s are independent Brownian motions in $\mathbb{R}^n$.

$$f^\sigma_t + v \cdot \nabla_x f^\sigma = \sigma \mathrm{s}_\rho \Delta_v f^\sigma + \mathrm{s}_\rho \nabla_v((v - [u^\sigma]_\rho) f^\sigma) \qquad \text{(FPA)}$$

So, the expected behavior as $t \to \infty$ would be the same as for the linear Fokker-Planck equation which is a relaxation to the global Maxwellian

$$f^\sigma \to \mu_{\sigma,\bar{u}} = \frac{1}{(2\pi\sigma)^{n/2}} e^{-\frac{|v-\bar{u}|^2}{2\sigma}},$$

where $\bar{u}$ is the mean velocity. If such a convergence holds true, then the alignment of the original system can be recovered in the limit of vanishing noise $\sigma \to 0$:

$$\lim_{\sigma\to 0} \lim_{t\to\infty} f^\sigma(t) = \delta_{v=\bar{u}} \otimes 1 \, dx.$$

– Duan, Fornasier, and Toscani (2010): relaxation in the Cucker-Smale case

$$f_t + v \cdot \nabla_x f = \sigma \rho_\phi \Delta_v f + \nabla_v((\rho_\phi v - (u\rho)_\phi) f),$$

for perturbation data,

$$f = \mu_{\sigma,\bar{u}} + g\sqrt{\mu_{\sigma,\bar{u}}}, \qquad \|g_0\|_{H^m(\mathbb{T}^n \times \mathbb{R}^n)} \leqslant \varepsilon,$$

for some small $\varepsilon > 0$.

– Choi (2016): relaxation for purely local model

$$f_t + v \cdot \nabla_x f = \sigma \Delta_v f + \nabla_v((v - u) f),$$

in the perturbative settings also.

These results are inspired by techniques from collisional models (Landau, Boltzmann).

– RS (2021): global relaxation for the $\mathcal{M}_\phi$-model:

$$\mathrm{s}_\rho = 1, \qquad [u]_\rho(x) = \left( \frac{(u\rho)_\phi}{\rho_\phi} \right)_\phi,$$

$$f_t + v \cdot \nabla_x f = \sigma \Delta_v f + \nabla_v((v - [u]_\rho) f). \tag{3}$$

Relaxation holds provided

$$\sup_t \|u(t)\|_\infty < \infty.$$

# Kinetic Cucker-Smale.

Consider FPA based on Cucker-Smale protocol

$$f_t + v \cdot \nabla_x f = \sigma \rho_\phi \Delta_v f + \nabla_v((\rho_\phi v - (u\rho)_\phi) f),$$

**Theorem (RS, 2023)**

*Suppose $f$ is a classical solution with sufficiently fast algebraic decay in $v$. Suppose $\phi = \psi * \psi$. Then for any $t_0 > 0$,*

$$\inf \rho \geqslant c(t_0) > 0$$

*uniformly for all $t > t_0$, and $f$ relaxes to the corresponding Maxwellian at an exponential rate*

$$\|f(t) - \mu_{\sigma,\bar{u}}\|_{L^1(\mathbb{T}^n \times \mathbb{R}^n)} \leqslant c_1 \sigma^{-1/2} e^{-c_2 \sigma^{1/2} t},$$

*for some $c_1$ depending on the initial data, and $c_2 > 0$ depending only on the parameters of the system.*

For the $\mathcal{M}_{\mathrm{MT}}$ and other non-symmetric models we have perturbative result in terms of Fisher information: if

$$\mathcal{I}(f_0) \leqslant c\sigma\delta_0, \tag{4}$$

then there exists a global classical solution $f$ and there exists $u_\infty \in \mathbb{R}^n$ such that

$$\|f(t) - \mu_{\sigma,u_\infty}\|_{L^1(\mathbb{T}^n\times\mathbb{R}^n)} \leqslant c_9 e^{-c_{10}t}. \tag{5}$$

Main ingredients:

– uniform gain of positivity, $f \geqslant a e^{-b|v|^2}$, where $a, b$ are time-independent for $t > t_0$.

– estimate on the spectral gap of $[\cdot]_\rho$: $\varepsilon \sim (\inf \rho)^3$. This is where we use $\phi = \psi * \psi$.

– hypocoercivity and Bakry-Emery-type method.

## Hypocoercivity via Villani's ($A^*A+B$)-formulation

Assuming $\bar{u}=0$ by Galilean invariance and $\sigma=1$, consider $h=f/\mu$:

$$\partial_t h = -\rho_\phi A^* A h - Bh + A^*((u\rho)_\phi h),$$

where

$$A=\nabla_v, \quad A^* = v - \nabla_v, \quad B = v\cdot\nabla_x.$$

We have the entropy

$$\mathcal{H} = \int_{\mathbb{T}^n\times\mathbb{R}^n} h\log h \, \mathrm{d}\mu,$$

which obeys two forms of entropy law:

- non-dissipative

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} = -\mathcal{I}_{vv}(h) + (u, [u]_\rho)_{\rho\rho_\phi},$$

where

$$\mathcal{I}_{vv}(h) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \frac{|\nabla_v h|^2}{h} \, \mathrm{d}\mu, \qquad (u, [u]_\rho)_{\rho\rho_\phi} = \int_{\Omega^n} (u\rho)_\phi u\rho \, \mathrm{d}x,$$

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**Lemma (Spectral gap estimate for $\mathcal{M}_{\mathrm{CS}}$)**

*We have*

$$(u, u)_{\rho\rho_\phi} - (u, [u]_\rho)_{\rho\rho_\phi} \gtrsim (\inf \rho)^3 (u, u)_{\rho\rho_\phi}.$$

So, if we control $\inf \rho$ uniformly from below for $t > t_0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} \lesssim -\mathcal{I}_{vv}(h) - (u, u)_{\rho\rho_\phi}.$$

# Bakry-Emery method a la Villani

Next: use the full Fischer information

$$\mathcal{I} = \mathcal{I}_{vv} + \varepsilon \mathcal{I}_{xv} + \mathcal{I}_{xx} \gtrsim \mathcal{H} \quad \text{(log-Sobolev inequality)},$$

$$\mathcal{I}_{xv}(h) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \frac{\nabla_x h \cdot \nabla_v h}{h} \, \mathrm{d}\mu, \quad \mathcal{I}_{xx}(h) = \int_{\mathbb{T}^n \times \mathbb{R}^n} \frac{|\nabla_x h|^2}{h} \, \mathrm{d}\mu.$$

Then one computes

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{I} \leqslant c_1 \mathcal{I}_{vv} - c_2 \mathcal{I}_{xx} + c_3 (u, u)_{\rho \rho_\phi}.$$

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So,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[ c_4 \mathcal{H} + \mathcal{I} \right] \leqslant -c_5 \left[ c_4 \mathcal{H} + \mathcal{I} \right].$$

In particular, by the Csiszár-Kullback inequality

$$\|f - \mu\|_1^2 \leqslant \mathcal{H} \leqslant c e^{-ct}.$$

# Gain of positivity

Kolmogorov, Carleman (1930's); Pulvirenti, Wennberg (1997); Villani, Desvillettes (2000); Imbert, Mohout, Silvestre (2020); Henderson, Snelson, Tarfulea (2020), F. Anceschi, Y. Zhu (2021): Boltzmann, Landau, Fokker-Planck;

**Theorem**

*There exist time-independent constants $a, b > 0$ which depend only on the initial entropy $\mathcal{H}_0$ such that*

$$f(t,x,v) \geqslant b e^{-a|v|^2}, \quad \forall x \in \mathbb{T}^n,\ v \in \mathbb{R}^n,\ t > 1. \tag{6}$$

Consequently,

$$\inf \rho \geqslant c(a,b).$$

Hence, the spectral gap is uniform in time and the hypocoercivity estimates apply.

– initial plateau and domain of ellipticity; Harnack chains based on recent J. Guerand, C. Imbert (2022) weak Harnack inequality for supersolutions.

# Why do we need a good regularity theory?

Both the hypocoercivity analysis and the Gain of Positivity result rely on regularity of solutions.

- Fisher information is controlled by higher order weighted Sobolev norms G. Toscani and C. Villani (2002):

$$H^m_q(\mathbb{T}^n \times \mathbb{R}^n) = \left\{ f : \sum_{2|\mathbf{k}|+|\mathbf{l}| \leqslant 2m} \int_{\mathbb{T}^n \times \mathbb{R}^n} \langle v \rangle^{q-2|\mathbf{k}|-|\mathbf{l}|} \, |\partial_x^{\mathbf{k}} \partial_v^{\mathbf{l}} f|^2 \, \mathrm{d}v \, \mathrm{d}x < \infty \right\}.$$

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- Entropy law needs to be differential (not guaranteed for weak solutions).
- The gain of positivity argument relies on comparison principle on parabolic domains, known only for classical solutions or weak solutions in regularity classes with well-defined traces.

- Duan, Fornasier, Toscani (2010) classical solutions near Maxwellian.

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- Karper, Mellet, Trivisa (2013): weak solutions to FPA with constant thermolization + local alignment + forces in finite energy space

$$f \in L^\infty \cap L^1(1 + |x|^2 + |v|^2),$$

and satisfying entropy *inequality*.

# LWP for thick data

We say that the environmental averaging model $\mathcal{M} = \{(\mathrm{s}_\rho, \phi_\rho)\}_{\rho \in \mathcal{P}}$ is regular if $\mathrm{s}_\rho, \phi_\rho$ are smooth for "thick" flocks

$$\theta(x) = \int_{B_{r_0}(x)} \rho(y)\, \mathrm{d}y > 0.$$

**Theorem (2024)**

*Suppose that the model $\mathcal{M}$ is regular. Let $f_0 \in H^m_q(\mathbb{T}^n \times \mathbb{R}^n)$, $m, q \geqslant n+3$, be an initial condition such that*

$$\theta_0 > 0.$$

*Then there exists a unique local solution to*

$$\partial_t f + v \cdot \nabla_x f = \mathrm{s}_\rho \Delta_v f + \nabla_v \cdot (\mathrm{s}_\rho (v - [u]_\rho) f). \tag{7}$$

*on a time interval $[0, T)$, where $T > 0$ depends only on the initial energy $\mathcal{E}_0$ and thickness $\theta_0$, in the regularity class*

$$f \in C_w([0,T); H^m_q), \quad \nabla_v f \in L^2([0,T]; H^m_q). \tag{8}$$

# GWP for thick data

**Theorem (2024)**

*Suppose that the model $\mathcal{M}$ is regular. If, in addition,*

$$\mathrm{s}_\rho \, [u]_\rho : L^2(\rho) \to L^\infty(\rho),$$

*then the FPA is globally well-posed for thick data in $H^m_q(\mathbb{T}^n \times \mathbb{R}^n)$. Consequently, $\mathcal{M}_{\mathrm{CS}}$, and all $\mathcal{M}_\beta$-models with $\beta \geqslant \frac{1}{2}$, are globally well-posed.*

# Existence of weak solutions

**Theorem (2025)**

*Let $\mathcal{M}$ be a regular model.*

- *Existence: For any initial condition $f_0 \in L^\infty \cap L^1(1+|v|^2)$ there exists a global weak solution to FPA in the same class and*

$$\mathrm{s}_\rho \frac{|\nabla_v f|^2}{f} \in L^1([0,T) \times \mathbb{T}^n \times \mathbb{R}^n). \tag{9}$$

- *Uniqueness: holds for thick data with higher momentum $f \in L^2(1+|v|^q)$.*
- *Entropy Equality: Any weak solution in the above class satisfies*

$$\frac{d}{dt}\mathcal{H} = -\int_{\mathbb{T}^n \times \mathbb{R}^n} \mathrm{s}_\rho \frac{|\nabla_v f + vf|^2}{f} \, dv \, dx + ([u]_\rho, u)_{\rho \mathrm{s}_\rho}.$$

- *Renormalization: every weak solution satisfies*

$$\partial_t G(f) + v \cdot \nabla_x G(f) = \mathrm{s}_\rho \nabla_v \cdot (\nabla_v G(f) + vfG'(f)) \\ - \mathrm{s}_\rho \nabla_v f \cdot (\nabla_v f + vf) G''(f) - \mathrm{s}_\rho [u]_\rho \cdot \nabla_v G(f)$$

*in the weak sense for any $G$ such that $\sup |xG''(x)| < \infty$.*

- Fisher regularity $\frac{|\nabla_v f|^2}{f} \in L^1$ plays the same role in establishing Entropy Law as Onsager-1/3 regularity in establishing Energy Conservation for solutions of the Euler equation. But we have access only to the degenerate information $\mathrm{s}_\rho \frac{|\nabla_v f|^2}{f} \in L^1$.

- Fisher regularity $\frac{|\nabla_v f|^2}{f} \in L^1$ plays the same role in establishing Entropy Law as Onsager-1/3 regularity in establishing Energy Conservation for solutions of the Euler equation. But we have access only to the degenerate information $\mathrm{s}_\rho \frac{|\nabla_v f|^2}{f} \in L^1$.
- To circumvent degeneracy we consider the equation for thickness-based renormalization

$$\partial_t \left[ \frac{(\theta f)_{\varepsilon_1}}{\theta + \varepsilon_2} \right] + \frac{(\theta v \cdot \nabla_x f)_{\varepsilon_1}}{\theta + \varepsilon_2} = \frac{(\theta \mathrm{s}_\rho \Delta_v f)_{\varepsilon_1}}{\theta + \varepsilon_2} + \frac{\nabla_v \cdot (\theta \mathrm{s}_\rho v f)_{\varepsilon_1}}{\theta + \varepsilon_2}$$
$$- \frac{\nabla_v \cdot (\theta \mathrm{s}_\rho [u]_\rho f)_{\varepsilon_1}}{\theta + \varepsilon_2} + \frac{(f \partial_t \theta)_{\varepsilon_1}}{\theta + \varepsilon_2} - \frac{(\theta f)_{\varepsilon_1} \partial_t \theta}{[\theta + \varepsilon_2]^2}.$$

and test the equation with $\varphi G' \left( \frac{(\theta f)_{\varepsilon_1}}{\theta + \varepsilon_2} \right)$. The analysis relies on commutator estimates for the resulting flux terms in the limit $\varepsilon_1 \to 0$ then $\varepsilon_2 \to 0$.

# Global Hypoellipticity

**Theorem (2025)**

*Let $\mathcal{M}$ be a regular model with*

$$\mathrm{s}_\rho [u]_\rho : L^2(\rho) \to L^\infty(\rho).$$

*Then any weak solution $f$ to FPA gains instant regularization on $(0, T)$ with*

$$\|f\|_{H^m_2} \leqslant \frac{C_{\varepsilon,T,m}}{t^\kappa}, \quad \forall m \in \mathbb{N}. \tag{10}$$

*If, in addition, $f_0 \in L^2_q$, $q \geqslant 2$, then*

$$\|f\|_{H^m_q} \leqslant \frac{C_{\varepsilon,T,m}}{t^\kappa}, \quad \forall m \in \mathbb{N}. \tag{11}$$

# Full description of dynamics for the Cucker-Smale based FPA

Consider FPA based on Cucker-Smale protocol

$$f_t + v \cdot \nabla_x f = \sigma \rho_\phi \Delta_v f + \nabla_v((\rho_\phi v - (u\rho)_\phi) f).$$

**Theorem (2025)**

*Suppose $\phi$ is a Bochner-positive kernel $\phi = \psi * \psi$. For any data $f_0 \in L^\infty \cap L^1(1+|v|^q)$, $q \geqslant n+3$, there exists a global weak solution in the same class and with finite Fisher information $\rho_\phi \frac{|\nabla_v f|^2}{f} \in L^1_{t,x,v}$, satisfying the entropy equality. Every such solution gains uniform Gaussian tails*

$$f(t,x,v) \geqslant b e^{-a|v|^2}, \quad \forall x \in \mathbb{T}^n,\ v \in \mathbb{R}^n,\ t > 1. \tag{12}$$

*and regularizes into $H^m_q$ for any $m \in \mathbb{N}$ instantly. Subsequently, it relaxes to the Maxwellian exponentially fast,*

$$\|f(t) - \mu_{\sigma,\bar{u}_0}\|_{L^1(\mathbb{T}^n \times \mathbb{R}^n)} \leqslant c_1 \sigma^{-1/2} e^{-c_2 \sigma^{1/2} t}.$$

Thanks!