Low Dimensional Manifold Model in Image Reconstruction

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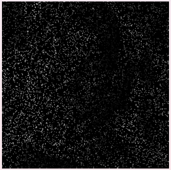
General Image Processing Problems

Many image processing problems can be formulated as recovering an image $f \in \mathbb{R}^{m \times n}$ from its noisy and linear measurements:

$$b = \Phi f + \epsilon$$

Original

Inpainting



Denoising

Deblurring

- Inpainting: $\Phi = \Phi_\Omega$ is the subsample operator, and $\epsilon = 0$.
- Denoising: $\Phi = Id$, and ϵ is the corresponding noise type.
- Deblurring: Φ is a convolution kernel.

Variational Model for Image Processing

Reconstructing f from b is an ill-posed problem, and some regularization is needed in a variational model:

$$\min_f R(f) \quad \text{subject to:} \quad b = \Phi f + \epsilon$$

- Total variation (TV):

$$R(f) = \|\nabla f\|_{L^1}$$

- Nonlocal total variation (NLTV):

$$R(f) = \|\nabla_w f\|_{L^1}$$

- Wavelet sparsity:

$$R(f) = \|Wf\|_{L^1}$$

- ......

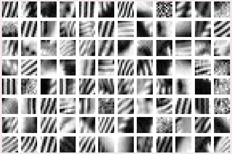
LDMM: dimension of the patch manifold.

Patch Set and Patch Manifold of an Image

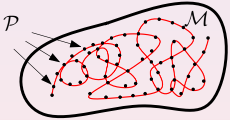
Image patches have been widely used in image processing.

Original Image f

Patch Set $\mathcal{P}(f)$



Patch Manifold $\mathcal{M}(f)$



- $\mathcal{P}(f) \subset \mathbb{R}^d$ is the collection of of all patches in the image f.
- $\mathcal{M}(f) \subset \mathbb{R}^d$ is the underlying patch manifold, discretely sampled by the point cloud $\mathcal{P}(f)$.

Low Dimensionality of the Patch Manifold $\mathcal{M}$

For most natural images, the dimension of the patch manifold $\mathcal{M}$ is usually much lower than that of the ambient space.

- If f is a smooth image, the patch at coordinate x, $p_x(f)$ can be approximated by a linear function

  $$p_x(f)(y) \approx f(x) + (y - x) \cdot \nabla f(x).$$

  This implies that $\dim \mathcal{M} \approx 3$.
- If f is a piecewise constant function corresponding to a cartoon image, then each patch is characterized by the location and the orientation of the edge. This means $\dim \mathcal{M} \approx 2$.
- If f is an oscillatory function corresponding to a texture, then

  $$f(x) \approx a(x) \cos \theta(x), \quad p_x f \approx a_L \cos \theta_L,$$

  where a_L and θ_L are linear approximation of a and θ. Hence $\dim M \approx 6$.

Low Dimensional Manifold Model

The idea of the low dimensional manifold model (LDMM) in image processing is to use the dimension of the patch manifold $\mathcal{M}$ as a regularization.

$$\min_{f,\mathcal{M}} \dim(\mathcal{M}), \quad \text{subject to:} \quad b = \Phi f + \epsilon, \mathcal{P}(f) \subset \mathcal{M}$$

Question: How to compute $\dim \mathcal{M}$?

Dimension of a Manifold

Proposition

Let $\mathcal{M}$ be a smooth submanifold embedded in $\mathbb{R}^d$. For any $x \in \mathcal{M}$,

$$\dim(\mathcal{M}) = \sum_{j=1}^{d} \|\nabla_{\mathcal{M}} \alpha_j(\mathbf{x})\|^2,$$

where $\alpha_i, i = 1, \ldots, d$ are coordinate functions,

$$\forall \mathbf{x} \in \mathcal{M}, \quad \alpha_i(\mathbf{x}) = x_i.$$

Dimension of a Manifold

Sanity check:
If $\mathcal{M} = S^1$, then $k = \dim(\mathcal{M}) = 1$, $d = \dim(\mathbb{R}^2) = 2$, and $\mathbf{x} = \psi(\theta) = (\cos\theta, \sin\theta)^t$ is the coordinate chart.
The metric tensor $g = g_{11} = \left\langle \frac{\partial\psi}{\partial\theta}, \frac{\partial\psi}{\partial\theta} \right\rangle = 1 = g^{11}$.
The gradient of α_i, $\nabla_{\mathcal{M}}\alpha_i = g^{11}\partial_1\alpha_i\partial_1 = \partial_1\alpha_i\partial_1$ can be viewed as a vector in the ambient space $\mathbb{R}^2$:

$$\nabla^j_{\mathcal{M}}\alpha_i = \partial_1\psi^j\partial_1\alpha_i$$

Therefore, we have

$$\nabla_{\mathcal{M}}\alpha_1 = \left\langle \partial_1\psi^1\partial_1\alpha_1, \partial_1\psi^2\partial_1\alpha_1 \right\rangle = \left\langle \sin^2\theta, -\cos\theta\sin\theta \right\rangle,$$
$$\nabla_{\mathcal{M}}\alpha_2 = \left\langle \partial_1\psi^1\partial_1\alpha_2, \partial_1\psi^2\partial_1\alpha_2 \right\rangle = \left\langle -\sin\theta\cos\theta, \cos^2\theta \right\rangle.$$

Hence $\|\nabla_{\mathcal{M}}\alpha_1\|^2 + \|\nabla_{\mathcal{M}}\alpha_2\|^2 = \sin^2\theta + \cos^2\theta = 1$

Low Dimensional Manifold Model

The original optimization problem can be rewritten as:

$$\min_{\substack{f \in \mathbb{R}^{m \times n} \\ \mathcal{M} \subset \mathbb{R}^d}} \sum_{i=1}^{d} \|\nabla_{\mathcal{M}} \alpha_i\|^2_{L^2(\mathcal{M})} + \lambda \|y - \Phi f\|_2^2, \quad \text{subject to: } \mathcal{P}(f) \subset \mathcal{M},$$

where

$$\|\nabla_{\mathcal{M}} \alpha_i\|_{L^2(\mathcal{M})} = \left(\int_{\mathcal{M}} \|\nabla_{\mathcal{M}} \alpha_i(\mathbf{x})\|^2 d\mathbf{x} \right)^{1/2}$$

This optimization problem is nonconvex. It can be solved by alternating the direction of minimization with respect to f and $\mathcal{M}$. We also perturb the coordinate function $\boldsymbol{\alpha}$ at each step.

Alternating Direction of Minimization

$$\min_{\substack{f\in\mathbb{R}^{m\times n}\\ \mathcal{M}\subset\mathbb{R}^d}} \sum_{i=1}^{d} \|\nabla_{\mathcal{M}}\alpha_i\|^2_{L^2(\mathcal{M})} + \lambda\|y-\Phi f\|_2^2, \quad \text{subject to: } \mathcal{P}(f)\subset\mathcal{M},$$

- With a guess $\mathcal{M}^n$ and f^n of the manifold and image, update the coordinate function $\alpha_i^{n+1}, i=1,\cdots,d$ and f^{n+1}:

$$(f^{n+1},\boldsymbol{\alpha}^{n+1}) = \arg \min_{\substack{f\in\mathbb{R}^{m\times n},\\ \alpha_1,\ldots,\alpha_d\in H^1(\mathcal{M}^n)}} \sum_{i=1}^{d} \|\nabla_{\mathcal{M}^n}\alpha_i\|^2_{L^2(\mathcal{M}^n)} + \lambda\|b-\Phi f\|_2^2,$$

$$\text{subject to:} \quad \boldsymbol{\alpha}(\mathcal{P}(f^n)) = \mathcal{P}(f)$$

- Update $\mathcal{M}$ by setting

$$\mathcal{M}^{n+1} = \boldsymbol{\alpha}(\mathcal{M}^n) = \left\{(\alpha_1^{n+1}(\mathbf{x}),\ldots,\alpha_d^{n+1}(\mathbf{x}))^T : \mathbf{x}\in\mathcal{M}^n\right\}.$$

Question: How to update f and $\boldsymbol{\alpha}$

Split Bregman (ADMM) Iteration

- Solve $\alpha_i^{n+1,k+1}, i = 1, \cdots, d$ with fixed $f^{n+1,k}$,

$$\min_{\alpha_1,\cdots,\alpha_d \in H^1(\mathcal{M}^n)} \sum_{i=1}^{d} \|\nabla \alpha_i\|^2_{L^2(\mathcal{M}^n)} + \mu \|\boldsymbol{\alpha}(\mathcal{P}(f^n)) - \mathcal{P}(f^{n+1,k}) + d^k\|_F^2.$$

- Update $f^{n+1,k+1}$ as

$$\min_{f \in \mathbb{R}^{m\times n}} \lambda \|b - \Phi f\|_2^2 + \mu \|\boldsymbol{\alpha}^{n+1,k+1}(\mathcal{P}(f^n)) - \mathcal{P}(f) + d^k\|_F^2.$$

- Update d^{k+1}:

$$d^{k+1} = d^k + \boldsymbol{\alpha}^{n+1,k+1}(\mathcal{P}(f^n)) - \mathcal{P}(f^{n+1,k+1}).$$

Algorithm

Algorithm 1 LDMM Algorithm - Continuous version

1: **while** not converge **do**

2: **while** not converge **do**

3:
$$\alpha_i^{n+1,k+1} = \arg \min_{\alpha_i \in H^1(\mathcal{M}^n)} \|\nabla_{\mathcal{M}^n} \alpha_i\|^2_{L^2(\mathcal{M}^n)} + \mu \|\alpha_i(\mathcal{P}(f^n)) - \mathcal{P}_i(f^{n+1,k}) + d_i^k\|^2$$

4:
$$f^{n+1,k+1} = \arg \min_{f \in \mathbb{R}^{m\times n}} \lambda \|b - \Phi f\|_2^2 + \mu \|\boldsymbol{\alpha}^{n+1,k+1}(\mathcal{P}(f^n)) - \mathcal{P}(f) + d^k\|_F^2$$

5:
$$d^{k+1} = d^k + \boldsymbol{\alpha}^{n+1,k+1}(\mathcal{P}(f^n)) - \mathcal{P}(f^{n+1,k+1}).$$

6: **end while**

7:
$$\mathcal{M}^{n+1} = \left\{ (\alpha_1^{n+1}(\mathbf{x}), \cdots, \alpha_d^{n+1}(\mathbf{x})) : \mathbf{x} \in \mathcal{M}^n \right\}.$$

8: **end while**

Graph Laplacian

The key step in the previous algorithm is to solve the following optimization:

$$\min_{u\in H^1(\mathcal{M})} \|\nabla_{\mathcal{M}} u\|^2_{L^2(\mathcal{M})} + \mu \sum_{y\in\Omega} |u(\mathbf{y}) - v(\mathbf{y})|^2 \tag{1}$$

Normally, (1) is solved by discretizing $\nabla_{\mathcal{M}} u$ by the nonlocal gradient:

$$\nabla_w u(\mathbf{x},\mathbf{y}) = \sqrt{w(\mathbf{x},\mathbf{y})}\,(u(\mathbf{y}) - u(\mathbf{x}))\,.$$

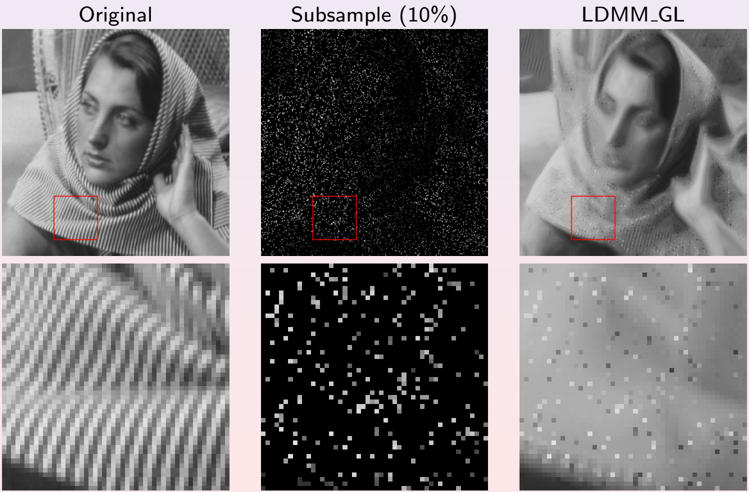
This leads to solving the following graph Laplacian (GL) problem:

$$\min_{u\in\mathbb{R}^{m\times n}} \sum_{x,y\in\Omega} w(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y}))^2 + \mu \sum_{y\in\Omega} |u(\mathbf{y}) - v(\mathbf{y})|^2.$$

Or equivalently,

$$\sum_{y\in\Omega} w(\mathbf{x},\mathbf{y})(u(\mathbf{x}) - u(\mathbf{y})) + \mu(u(\mathbf{x}) - v(\mathbf{y})) = 0, \quad \forall x \in \Omega.$$

Graph Laplacian



Laplace-Beltrami Equation

By a standard variational approach, we know that problem (1) is equivalent to the following PDE:

$$\begin{cases} -\Delta_{\mathcal{M}} u(\mathbf{x}) + \mu \sum_{\mathbf{y}\in\Omega} \delta(\mathbf{x}-\mathbf{y})(u(\mathbf{y}) - v(\mathbf{y})) = 0, & \mathbf{x} \in \mathcal{M} \\ \dfrac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x} \in \partial\mathcal{M}, \end{cases} \tag{2}$$

where $\partial\mathcal{M}$ is the boudary of $\mathcal{M}$ and $\mathbf{n}$ is the outer normal of $\partial\mathcal{M}$.

Point Integral Method

In the point integral method (PIM), the key observation is the following integral approximation:

$$\int_{\mathcal{M}} \Delta_{\mathcal{M}} u(\mathbf{y}) \bar{R}\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right) d\mathbf{y} \approx -\frac{1}{t}\int_{\mathcal{M}} (u(\mathbf{x})-u(\mathbf{y}))\, R\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right) d\mathbf{y} + 2\int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \bar{R}\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right) d\tau_{\mathbf{y}}.$$

The function R is a positive function defined on $[0, +\infty)$ with compact support (or fast decay) and

$$\bar{R} = \int_r^\infty R(s) ds.$$

Local Truncation Error

Theorem

Let $\mathcal{M}$ be a smooth manifold and $u \in C^3(\mathcal{M})$, then

$$\left\| -\frac{1}{t}\int_{\mathcal{M}} (u(\mathbf{x}) - u(\mathbf{y}))\, R_t(\mathbf{x},\mathbf{y}) d\mathbf{y} + 2\int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}}(\mathbf{y}) \bar{R}_t(\mathbf{x},\mathbf{y}) d\tau_{\mathbf{y}} - \int_{\mathcal{M}} \Delta_{\mathcal{M}} u(\mathbf{y}) \bar{R}_t(\mathbf{x},\mathbf{y}) d\mathbf{y} \right\|_{L^2(\mathcal{M})} = O(t^{1/4}),$$

where

$$R_t(\mathbf{x},\mathbf{y}) = \frac{1}{(4\pi t)^{k/2}} R\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right), \bar{R}_t(\mathbf{x},\mathbf{y}) = \frac{1}{(4\pi t)^{k/2}} \bar{R}\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}\right).$$

Proof of Theorem

Using integration by part, we have

$$\begin{aligned}\int_\Omega \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} &= -\int_\Omega \nabla u(\mathbf{x})\cdot\nabla \bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} + \int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}\\ &= \frac{1}{2t}\int_\Omega (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{x})R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + \int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}\end{aligned}$$

We want to replace ∇u with function value u, which leads us to use the Taylor expansion

$$u(\mathbf{x}) - u(\mathbf{y}) = (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{x}) - \frac{1}{2}(\mathbf{x}-\mathbf{y})^T\mathbf{H}_u(\mathbf{x})(\mathbf{x}-\mathbf{y}) + O(\|\mathbf{x}-\mathbf{y}\|^3).$$

Proof of Theorem

$$u(\mathbf{x}) - u(\mathbf{y}) = (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{x}) - \frac{1}{2}(\mathbf{x}-\mathbf{y})^T\mathbf{H}_u(\mathbf{x})(\mathbf{x}-\mathbf{y}) + O(\|\mathbf{x}-\mathbf{y}\|^3).$$

Integrating on both sides, we have

$$\begin{aligned}
&\frac{1}{2t}\int_\Omega (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{x}) R_t(\mathbf{x},\mathbf{y}) d\mathbf{x} \\
=&\frac{1}{2t}\int_\Omega \left(u(\mathbf{x}) - u(\mathbf{y})\right) R_t(\mathbf{x},\mathbf{y}) dx \\
&+\frac{1}{4t}\int_\Omega (\mathbf{x}-\mathbf{y})^T\mathbf{H}_u(\mathbf{x})(\mathbf{x}-\mathbf{y}) R_t(\mathbf{x},\mathbf{y}) d\mathbf{x} + O(t^{1/2}),
\end{aligned}$$

where $O(t^{1/2})$ is uniform with respect to $\mathbf{y}$. Next we need to estimate the $\mathbf{H}_u$ term.

Proof of Theorem

$$
\begin{aligned}
&\frac{1}{4t}\int_\Omega (\mathbf{x}-\mathbf{y})^T\mathbf{H}_u(\mathbf{x})(\mathbf{x}-\mathbf{y})R_t(\mathbf{x},\mathbf{y})dx\\
=&\frac{1}{4t}\int_\Omega (\mathbf{x}_i-\mathbf{y}_i)(\mathbf{x}_j-\mathbf{y}_j)\partial_{ij}u(\mathbf{x})R_t(\mathbf{x},\mathbf{y})dx\\
=&-\frac{1}{2}\int_\Omega (\mathbf{x}_i-\mathbf{y}_i)\partial_{ij}u(\mathbf{x})\partial_j(\bar{R}_t(\mathbf{x},\mathbf{y}))dx\\
=&\frac{1}{2}\int_\Omega \partial_j(\mathbf{x}_i-\mathbf{y}_i)\partial_{ij}u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})dx+\frac{1}{2}\int_\Omega (\mathbf{x}_i-\mathbf{y}_i)\partial_{ijj}u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})dx\\
&-\frac{1}{2}\int_{\partial\Omega} (\mathbf{x}_i-\mathbf{y}_i)\mathbf{n}_j\partial_{ij}u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})dx\\
=&\frac{1}{2}\int_\Omega \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})dx-\frac{1}{2}\int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}):\mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})dx+O(t^{1/2}).
\end{aligned}
$$

Proof of Theorem

$$
\begin{aligned}
&\int_\Omega \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})dx = \frac{1}{2t}\int_\Omega (\mathbf{x}-\mathbf{y})\cdot\nabla u(\mathbf{x})R_t(\mathbf{x},\mathbf{y})dx + \int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})dx\\
=&\frac{1}{2t}\int_\Omega \left(u(\mathbf{x})-u(\mathbf{y})\right)R_t(\mathbf{x},\mathbf{y})dx + \frac{1}{4t}\int_\Omega (\mathbf{x}-\mathbf{y})^T\mathbf{H}_u(\mathbf{x})(\mathbf{x}-\mathbf{y})R_t(\mathbf{x},\mathbf{y})dx + O(t^{1/2})\\
&+\int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}\bar{R}_t(\mathbf{x},\mathbf{y})dx\\
=&\frac{1}{2t}\int_\Omega (u(\mathbf{x})-u(\mathbf{y}))R_t(\mathbf{x},\mathbf{y})dx + \frac{1}{2}\int_\Omega \Delta u\cdot\bar{R}_t(\mathbf{x},\mathbf{y})dx\\
&-\frac{1}{2}\int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}):\mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})dx + \int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}\bar{R}_t(\mathbf{x},\mathbf{y})dx + O(t^{1/2})
\end{aligned}
$$

This implies that:

$$
\begin{aligned}
\int_\Omega \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})dx =& \frac{1}{t}\int_\Omega (u(\mathbf{x})-u(\mathbf{y}))R_t(\mathbf{x},\mathbf{y})dx + 2\int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}\bar{R}_t(\mathbf{x},\mathbf{y})dx\\
&-\int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}):\mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})dx + O(t^{1/2})
\end{aligned}
$$

Proof of Theorem

$$\int_\Omega \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} = \frac{1}{t}\int_\Omega (u(\mathbf{x})-u(\mathbf{y}))R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + 2\int_{\partial\Omega}\frac{\partial u}{\partial \mathbf{n}}\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} - \int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}) : \mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} + O(t^{1/2})$$

Although $\left\|\int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}) : \mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}\right\|_{L^\infty(\Omega)} = O(1)$, it can be easily estimated in $L^2(\Omega)$:

$$\left\|\int_{\partial\Omega}((\mathbf{x}-\mathbf{y})\otimes\mathbf{n}) : \mathbf{H}_u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}\right\|_{L^2(\Omega)} = O(t^{1/4}).$$

Therefore

$$\left\| -\frac{1}{t}\int_{\mathcal{M}} (u(\mathbf{x})-u(\mathbf{y}))\, R_t(\mathbf{x},\mathbf{y})d\mathbf{y} + 2\int_{\partial\mathcal{M}}\frac{\partial u}{\partial \mathbf{n}}(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})d\tau_{\mathbf{y}} - \int_{\mathcal{M}}\Delta_{\mathcal{M}} u(\mathbf{y})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{y}\right\|_{L^2(\mathcal{M})} = O(t^{1/4}),$$

Integral Equation

The Laplace-Beltrami equation is:

$$\begin{cases} -\Delta_{\mathcal{M}} u(\mathbf{x}) + \mu \sum_{\mathbf{y}\in\Omega} \delta(\mathbf{x}-\mathbf{y})(u(\mathbf{y})-v(\mathbf{y})) = 0, & \mathbf{x}\in\mathcal{M} \\ \dfrac{\partial u}{\partial \mathbf{n}}(\mathbf{x}) = 0, & \mathbf{x}\in\partial\mathcal{M}, \end{cases}$$

The integral approximation is:

$$\int_{\mathcal{M}} \Delta u(\mathbf{x})\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x} \approx \frac{1}{t}\int_{\mathcal{M}} (u(\mathbf{x})-u(\mathbf{y}))R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + 2\int_{\partial\mathcal{M}} \frac{\partial u}{\partial \mathbf{n}}\bar{R}_t(\mathbf{x},\mathbf{y})d\mathbf{x}$$

The integral equation is:

$$\int_{\mathcal{M}} (u(\mathbf{x})-u(\mathbf{y}))R_t(\mathbf{x},\mathbf{y})d\mathbf{x} + \mu t \sum_{\mathbf{y}\in\Omega} \bar{R}_t(\mathbf{x},\mathbf{y})(u(\mathbf{y})-v(\mathbf{y})) = 0.$$

Discretization

$$\frac{|\mathcal{M}|}{N}\sum_{j=1}^{N} R_t(\mathbf{x}_i,\mathbf{x}_j)(u_i-u_j)+\mu t\sum_{j=1}^{N}\bar{R}_t(\mathbf{x}_i,\mathbf{x}_j)(u_j-v_j)=0.$$

The matrix form is:

$$(\text{Ł}+\bar{\mu}\bar{\mathbf{W}})\mathbf{U}=\bar{\mu}\bar{\mathbf{W}}\mathbf{V},$$

where $\bar{\mu}=\mu tN/|\mathcal{M}|$,

$$\text{Ł}=\mathbf{D}-\mathbf{W},\quad \mathbf{W}=(w_{ij}),\quad \bar{\mathbf{W}}=(\bar{w}_{ij}),$$

and

$$w_{ij}=R_t(\mathbf{x}_i,\mathbf{x}_j),\quad \bar{w}_{ij}=\bar{R}_t(\mathbf{x}_i,\mathbf{x}_j),\quad \mathbf{x}_i,\mathbf{x}_j\in\mathcal{P}(f^n),\quad i,j=1,\cdots,N.$$

Algorithm (LDMM_PIM)

Algorithm 2 LDMM_PIM

while not converge **do**

  Compute the matrices $\boldsymbol{W} = (w_{ij})_{1\le i,j\le N}$ from $\mathcal{P}(f^n)$

  for $k = 1 : K$ **do**

$$(\boldsymbol{L} + \bar{\mu}\bar{\boldsymbol{W}})\boldsymbol{U}_k = \bar{\mu}\bar{\boldsymbol{W}}\boldsymbol{V}_{k-1}.$$

    where $\boldsymbol{V}_k = \left(\mathcal{P}(f^n) - d^k\right)^T$.

    Update f by solving a least square problem

$$f^{n+1,k} = \arg \min_{f\in\mathbb{R}^{m\times n}} \lambda\|b - \Phi f\|_2^2 + \bar{\mu}\|\boldsymbol{U}_k^T - \mathcal{P}(f) + d^{k-1}\|_F^2$$

$$d^k = d^{k-1} + \boldsymbol{U}_k^T - \mathcal{P}(f^{n+1,k})$$

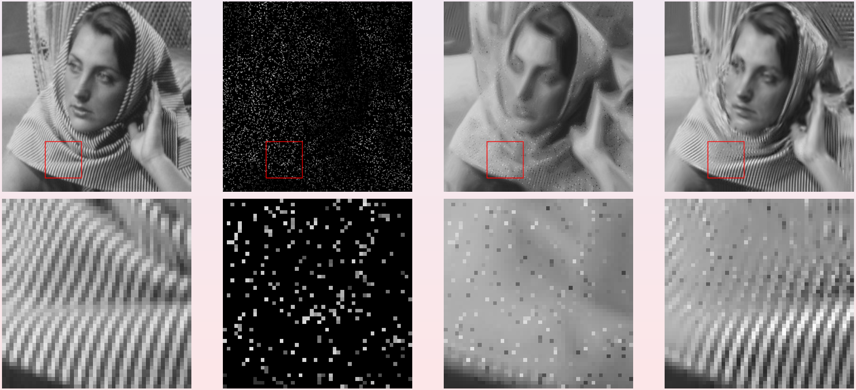
  end for

  $f^{n+1} = f^{n+1,K}$

end while

LDMM_PIM in Image Inpainting

Original | Subsample | LDMM_GL | LDMM_PIM



Another Reason Why Graph Laplacian Fails

Consider an unknown function u defined on a discrete set $\bar{\Omega} \subset \mathcal{M}$. Assume that we know the function value of u on a subset $\Omega \subset \bar{\Omega}$, $u(x) = b(x), \forall x \in \Omega$. Assume also that $|\Omega| \ll |\bar{\Omega}|$. The harmonic extension of u onto Ω is modeled as

$$\min_{u \in H^1(\mathcal{M})} \|\nabla_{\mathcal{M}} u\|^2, \quad \text{subject to:} \quad u(x) = b(x), \forall x \in \Omega$$

If we discretize the objective function above using graph Laplacian, we have

$$\begin{aligned} \|\nabla_{\mathcal{M}} u\|^2 &= \sum_{x \in \bar{\Omega}} \sum_{y \in \bar{\Omega}} w(x,y) \left(u(x) - u(y)\right)^2 \\ &= \sum_{x \in \Omega} \sum_{y \in \bar{\Omega}} w(x,y) \left(u(x) - u(y)\right)^2 + \sum_{x \in \bar{\Omega} \setminus \Omega} \sum_{y \in \bar{\Omega}} w(x,y) \left(u(x) - u(y)\right)^2 \end{aligned}$$

The first term on the right is of order $|\Omega|$, which is much smaller than that of the second term $|\bar{\Omega} \setminus \Omega|$. This causes the first term to be neglected in the minimization, and the algorithm sacrifices the continuity of u on Ω for small variation in $\bar{\Omega} \setminus \Omega$.

Weighted Graph Laplacian (WGL)

An easy fix for the aforementioned problem is to put an extra weight μ in front of the first term.

$$\|\nabla_{\mathcal{M}} u\|^2 = \mu \sum_{x \in \Omega} \sum_{y \in \bar{\Omega}} w(x,y) \left(u(x) - u(y)\right)^2 + \sum_{x \in \bar{\Omega} \setminus \Omega} \sum_{y \in \bar{\Omega}} w(x,y) \left(u(x) - u(y)\right)^2$$

To balance the orders of the two terms, μ is chosen to be $\frac{|\bar{\Omega}|}{|\Omega|}$.
Notice that if $\bar{\Omega} = \Omega$, the weighted graph Laplacian is just the graph Laplacian.

LDMM_WGL for Image Inpainting

Notice that a key step in LDMM for image inpainting is to solve the following optimization problem:

$$\min_{f \in \mathbb{R}^{m\times n}} \sum_{i=1}^{d} \|\nabla_{\mathcal{M}} \alpha_i\|^2_{L^2(\mathcal{M}^k)},$$

$$\text{subject to:} \quad \alpha_i\left(\mathcal{P}(f^k)(x)\right) = \mathcal{P}_i f(x), \qquad \forall x \in \bar{\Omega}, i = 1, \cdots, d,$$

$$f(x) = b(x), \qquad \forall x \in \Omega \subset \bar{\Omega},$$

where $\mathcal{P}_i f(x)$ is the i-th element of the patch $\mathcal{P}f(x)$. We use the notation $x_{\widehat{i-1}}$ to denote the $(i-1)$-th element after x in a patch, i.e. $\mathcal{P}_i f(x) = f(x_{\widehat{i-1}})$.
If we use periodic padding near the boundary, the ajoint operator $\mathcal{P}_i^* = \mathcal{P}_i^{-1}$

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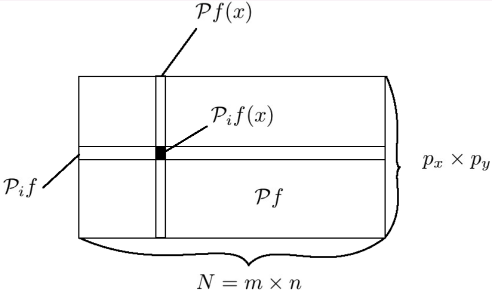
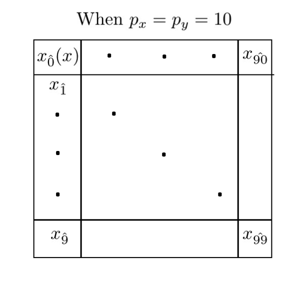




LDMM_WGL for Image Inpainting

$$\min_{f\in\mathbb{R}^{m\times n}} \sum_{i=1}^{d} \|\nabla_{\mathcal{M}}\alpha_i\|^2_{L^2(\mathcal{M}^k)},$$

$$\begin{aligned} \text{subject to:}\quad & \alpha_i\left(\mathcal{P}(f^k)(x)\right) = \mathcal{P}_i f(x), && \forall x\in\bar{\Omega}, i=1,\cdots,d,\\ & f(x) = b(x), && \forall x\in\Omega\subset\bar{\Omega}, \end{aligned}$$

Applying WGL, we have the following discretized optimization problem:

$$\min_{f\in\mathbb{R}^{m\times n}} \sum_{i=1}^{d} \left(\sum_{x\in\bar{\Omega}\setminus\Omega_i} \sum_{y\in\bar{\Omega}} \bar{w}(x,y)((\mathcal{P}_i f(x) - \mathcal{P}_i f(y))^2 \right.$$

$$\left. + \frac{mn}{|\Omega|} \sum_{x\in\Omega_i} \sum_{y\in\bar{\Omega}} \bar{w}(x,y)((\mathcal{P}_i f(x) - \mathcal{P}_i f(y))^2 \right) \quad \text{subject to: } f(x) = b(x), \quad x\in\Omega\subset\bar{\Omega}$$

where $\Omega_i = \left\{x\in\bar{\Omega}: \quad \mathcal{P}_i f(x) \text{ is sampled}\right\}$, and $\bar{w}(x,y) = w(\mathcal{P}f(x), \mathcal{P}f(y))$

LDMM_WGL

Using a standard variational approach, the equivalent Euler-Lagrange equation is

$$\begin{cases} \left[\sum_{i=1}^{d} \mathcal{P}_i^*(h_i) + \mu \sum_{i=1}^{d} \mathcal{P}_i^*(g_i) \right](x) = 0, & x \in \bar{\Omega} \setminus \Omega \\ f(x) = b(x), & x \in \Omega \end{cases}$$

where

$$h_i(x) = \sum_{y \in \bar{\Omega}} 2\bar{w}(x,y)(\mathcal{P}_i f(x) - \mathcal{P}_i f(y))$$

$$g_i(x) = \sum_{y \in \Omega_i} \bar{w}(x,y)(\mathcal{P}_i f(x) - \mathcal{P}_i f(y))$$

LDMM_WGL

$$h_i(x) = \sum_{y \in \bar{\Omega}} 2\bar{w}(x,y)(\mathcal{P}_i f(x) - \mathcal{P}_i f(y))$$

$$\begin{aligned}\mathcal{P}_i^* h_i(x) = h_i(x_{\widehat{1-i}}) &= \sum_{y \in \bar{\Omega}} 2\bar{w}(x_{\widehat{1-i}}, y)\left(\mathcal{P}_i f(x_{\widehat{1-i}}) - \mathcal{P}_i f(y)\right) \\ &= \sum_{y \in \bar{\Omega}} 2\bar{w}(x_{\widehat{1-i}}, y)\left(f(x) - f(y_{\widehat{i-1}})\right) \\ &= \sum_{y \in \bar{\Omega}} 2\bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}})\left(f(x) - f(y)\right)\end{aligned}$$

Therefore

$$\sum_{i=1}^{d} \mathcal{P}_i^*(h_i)(x) = \sum_{i=1}^{d} \sum_{y \in \bar{\Omega}} 2\bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}})\left(f(x) - f(y)\right)$$

Similarly,

$$\sum_{i=1}^{d} \mathcal{P}_i^*(g_i)(x) = \sum_{i=1}^{d} \sum_{y \in \Omega} \bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}})\left(f(x) - f(y)\right)$$

LDMM_WGL

The Euler-Lagrange equation becomes:

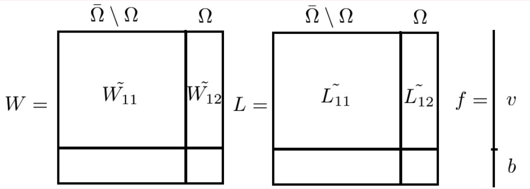
$$\begin{cases} \displaystyle\sum_{y\in\bar{\Omega}} \left(\sum_{i=1}^{d} 2\bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}})\right)(f(x)-f(y)) \\ \displaystyle + \mu \sum_{y\in\Omega} \left(\sum_{i=1}^{d} \bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}})\right)(f(x)-f(y)) = 0, & x \in \bar{\Omega}\setminus\Omega \\ f(x) = b(x), & x \in \Omega \end{cases}$$

Let $\tilde{w}(x,y) = \sum_{i=1}^{d} \bar{w}(x_{\widehat{1-i}}, y_{\widehat{1-i}})$, then

$$2\sum_{y\in\bar{\Omega}} \tilde{w}(x,y)(f(x)-f(y)) + \mu \sum_{y\in\Omega} \tilde{w}(x,y)(f(x)-f(y)) = 0, \quad x \in \bar{\Omega}\setminus\Omega$$

LDMM_WGL

$$2\sum_{y\in\bar{\Omega}} \tilde{w}(x,y)\,(f(x)-f(y)) + \mu \sum_{y\in\Omega} \tilde{w}(x,y)\,(f(x)-f(y)) = 0, \quad x\in\bar{\Omega}\setminus\Omega$$



Let $\Delta = \mathrm{diag}(\mathrm{sum}(\tilde{W}_{12}, 2))$, then

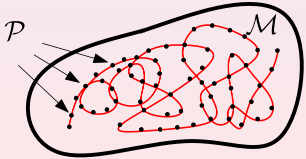
$$2\tilde{L}_{11}v + 2\tilde{L}_{12}b + \mu(\Delta v - \tilde{W}_{12}b) = 0$$
$$(2\tilde{L}_{11} + \mu\Delta)v = \mu\tilde{W}_{12}b - 2\tilde{L}_{12}b$$

Semi-local Patches

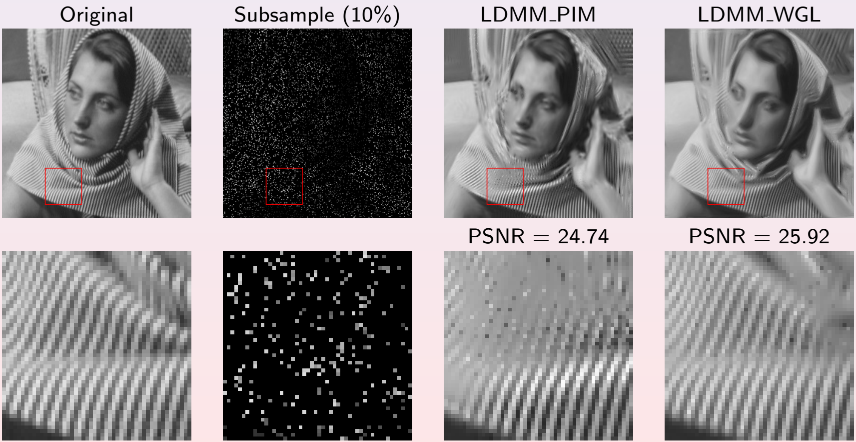
The semi-local patches are obtained by adding local coordinates to the nonlocal patches with a weight λ, i.e.

$$\bar{\mathcal{P}}f(x) = [\mathcal{P}f(x), \lambda x] \,.$$

When $\lambda = 0$, semi-local patches are just nonlocal patches. When $\lambda \to \infty$, the patches are completely determined by local coordinates. We choose a proper λ to help LDMM update the "true" metric on the manifold $\mathcal{M}$ faster and more reliably.



LDMM_WGL with Semi-local Patches

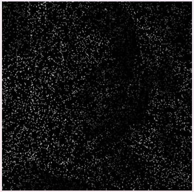


Numerical Results

2D Image Inpainting

Original

Subsample (10%)

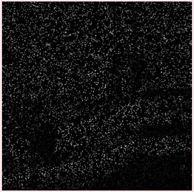


BPFA (23.44dB)

LDMM (25.92dB)

Original

Subsample (10%)



BPFA (24.71dB)

LDMM (25.70dB)

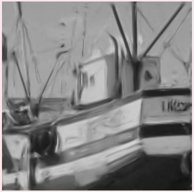
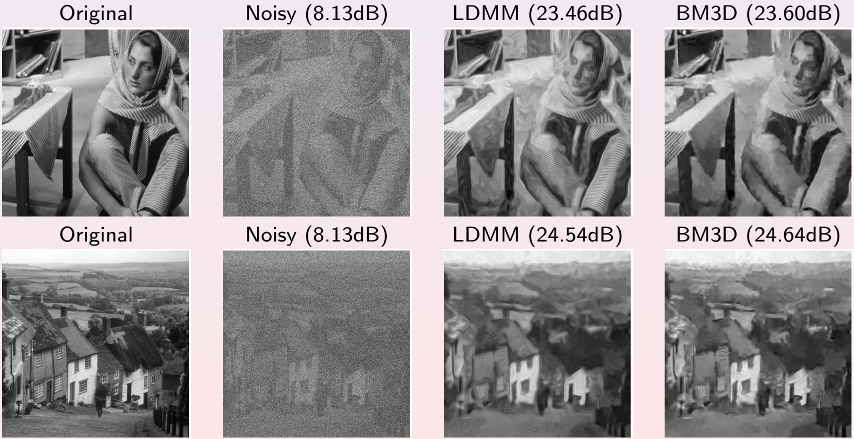
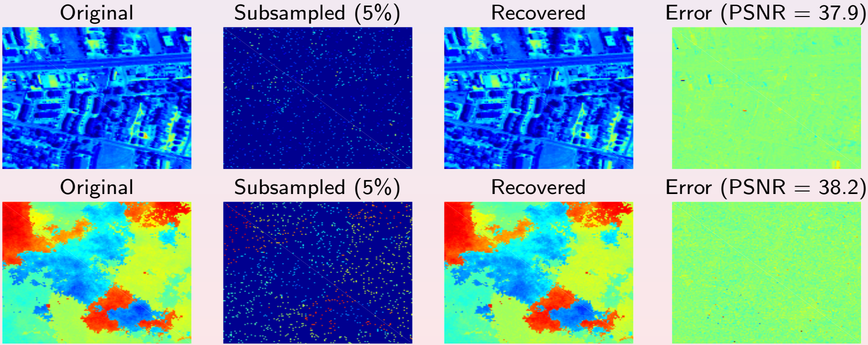


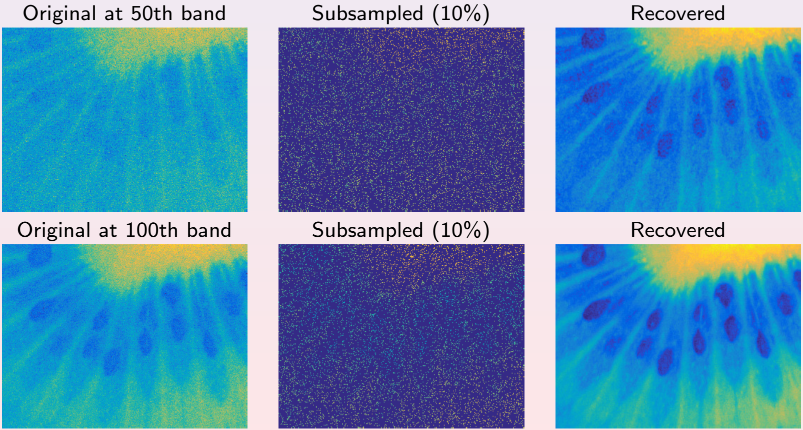
Image Denoising



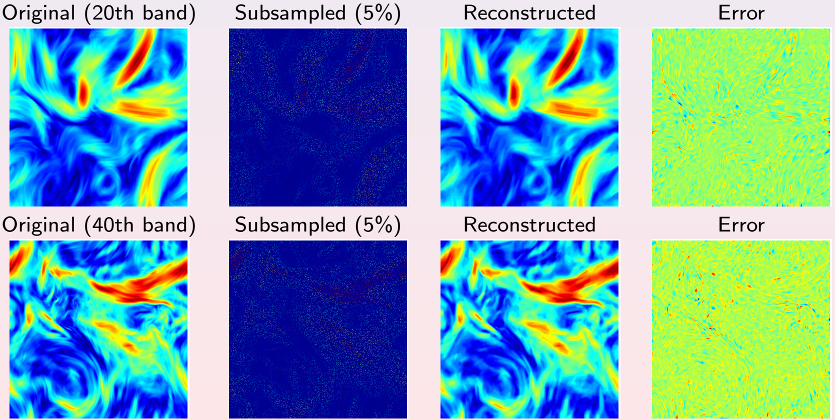
Hyperspectral Image Inpainting



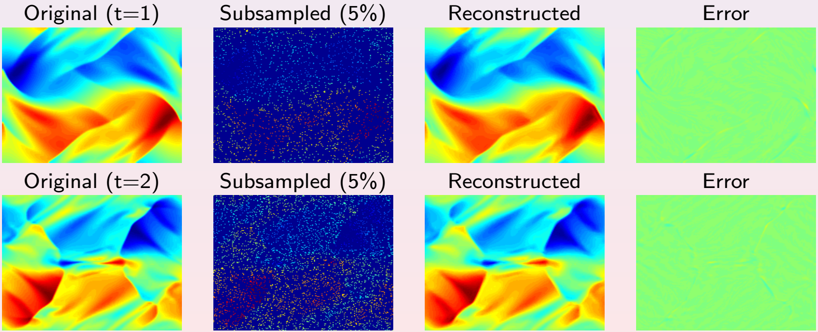
Noisy and Incomplete Hyperspectral Images



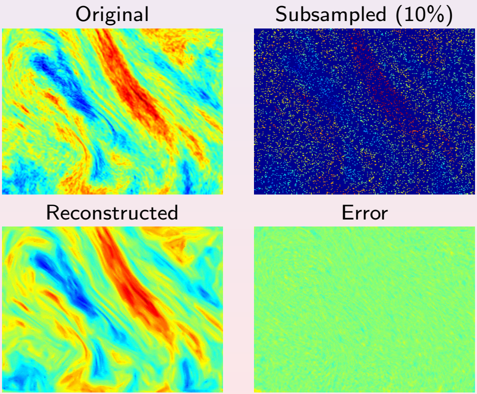
3D Plasma Reconstruction



Shock Reconstruction

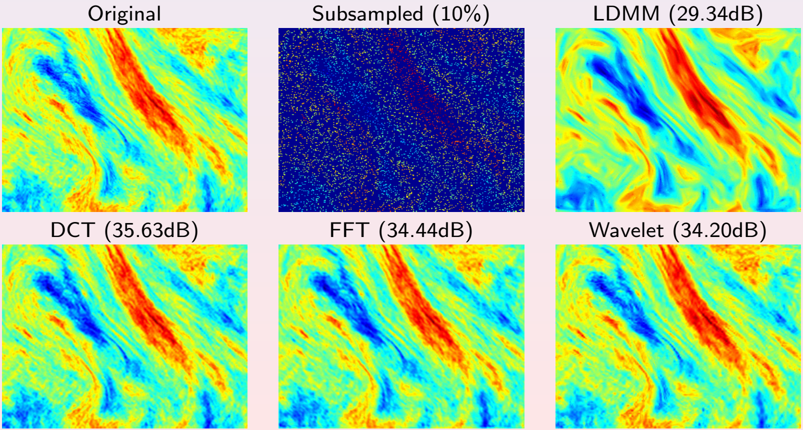


Shock Reconstruction with Granular Structure



Comparison of Compression Results

Every algorithm except for LDMM has access to the entire image. The budge is set to be 10% of the original data size.

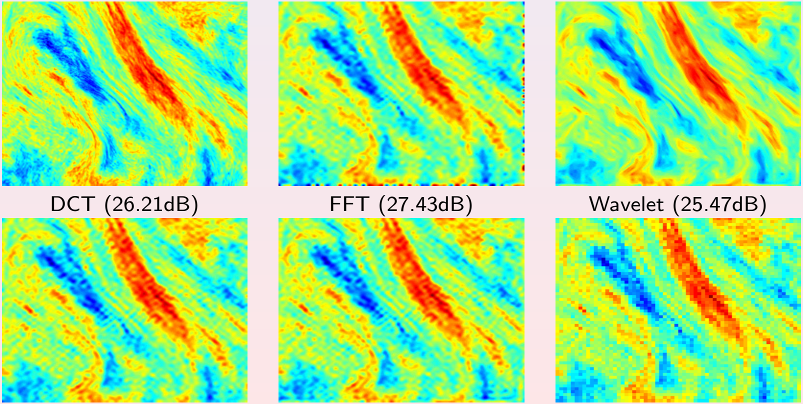


Comparison of Super-resolution Results

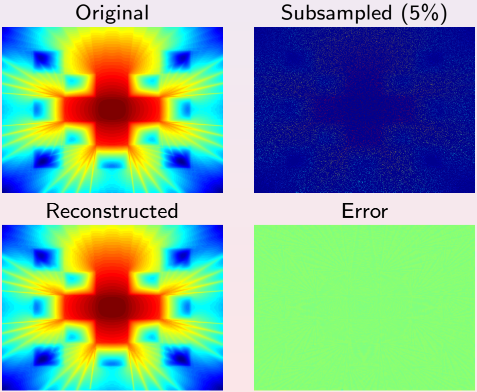
The image is downsampled with a sampling spacing of 4 in each direction.

Original | Bicubic (26.81dB) | LDMM (29.20dB)

DCT (26.21dB) | FFT (27.43dB) | Wavelet (25.47dB)



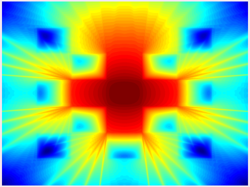
Neutron Transport



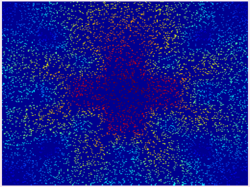
Comparison of Compression Results

Every algorithm except for LDMM has access to the entire image. The budge is set to be 10% of the original data size.

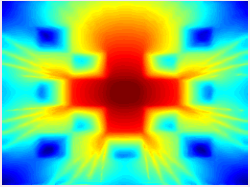
Original



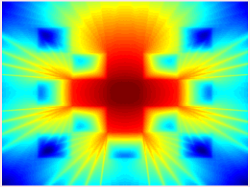
Subsampled (10%)



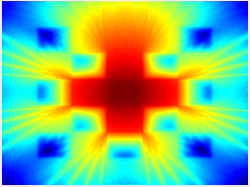
LDMM (40.06dB)



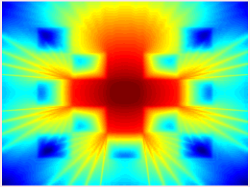
DCT (59.64dB)



FFT (47.89dB)

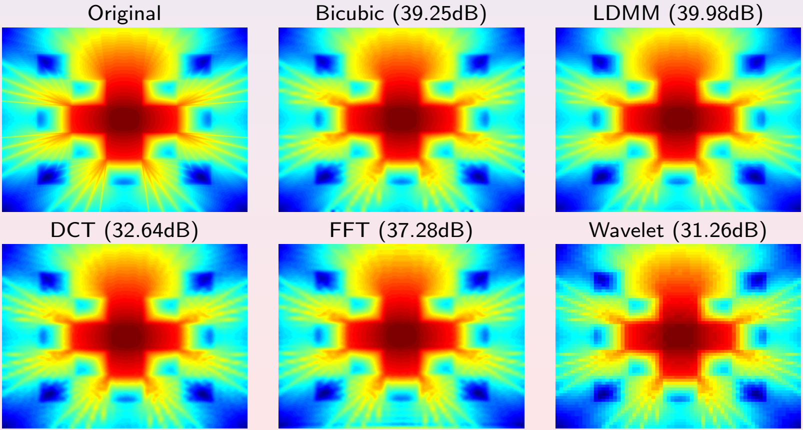


Wavelet (50.95dB)



Comparison of Super-resolution Results

The image is downsampled with a sampling spacing of 4 in each direction.



Conclusion and Future Work

Conclusion

- LDMM uses the dimension of the patch manifold to regularize the variational problem.
- The Laplace-Beltrami equation is solved via either the point integral method or the weighted graph Laplacian
- Weighted graph Laplacian is much more efficient for image inpainting, because the equation is solved on the image domain instead of the patch domain

Ongoing and future work

- Sparse dimensional manifold model (SDMM) for HSI processing.
- LDMM or SDMM with rotating patches with different resolutions.