Low Dimensional Manifold Model in Image Reconstruction

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Many image processing problems can be formulated as recovering an image $f \in \mathbb{R}^{m \times n}$ from its noisy and linear measurements:

$$b = \Phi f + \epsilon$$

- **Inpainting**: $\Phi = \Phi_{\Omega}$ is the subsample operator, and $\epsilon = 0$.
- **Denoising**: $\Phi = Id$, and $\epsilon$ is the corresponding noise type.
- **Deblurring**: $\Phi$ is a convolution kernel.
Reconstructing $f$ from $b$ is an ill-posed problem, and some regularization is needed in a variational model:

$$
\min_{f} R(f) \quad \text{subject to: } \quad b = \Phi f + \epsilon
$$

- **Total variation (TV):**
  $$
  R(f) = \| \nabla f \|_{L^1}
  $$

- **Nonlocal total variation (NLTV):**
  $$
  R(f) = \| \nabla_w f \|_{L^1}
  $$

- **Wavelet sparsity:**
  $$
  R(f) = \| Wf \|_{L^1}
  $$

- ......

LDMM: dimension of the patch manifold.
Image patches have been widely used in image processing.

- $\mathcal{P}(f) \subset \mathbb{R}^d$ is the collection of all patches in the image $f$.

- $\mathcal{M}(f) \subset \mathbb{R}^d$ is the underlying patch manifold, discretely sampled by the point cloud $\mathcal{P}(f)$. 
For most natural images, the dimension of the patch manifold $\mathcal{M}$ is usually much lower than that of the ambient space.

- If $f$ is a smooth image, the patch at coordinate $x$, $p_x(f)$ can be approximated by a linear function

$$p_x(f)(y) \approx f(x) + (y - x) \cdot \nabla f(x).$$

This implies that $\dim \mathcal{M} \approx 3$.

- If $f$ is a piecewise constant function corresponding to a cartoon image, then each patch is characterized by the location and the orientation of the edge. This means $\dim \mathcal{M} \approx 2$.

- If $f$ is an oscillatory function corresponding to a texture, then

$$f(x) \approx a(x) \cos \theta(x), \quad p_x f \approx a_L \cos \theta_L,$$

where $a_L$ and $\theta_L$ are linear approximation of $a$ and $\theta$. Hence $\dim \mathcal{M} \approx 6$. 
The idea of the low dimensional manifold model (LDMM) in image processing is to use the dimension of the patch manifold $\mathcal{M}$ as a regularization.

$$\min_{f, \mathcal{M}} \dim(\mathcal{M}), \quad \text{subject to:} \quad b = \Phi f + \epsilon, \mathcal{P}(f) \subset \mathcal{M}$$

**Question:** How to compute $\dim(\mathcal{M})$?
Dimension of a Manifold

Proposition

Let $\mathcal{M}$ be a smooth submanifold embedded in $\mathbb{R}^d$. For any $x \in \mathcal{M}$,

$$\dim(\mathcal{M}) = \sum_{j=1}^{d} \| \nabla_{\mathcal{M}} \alpha_j(x) \|^2,$$

where $\alpha_i, i = 1, \ldots, d$ are coordinate functions,

$$\forall x \in \mathcal{M}, \quad \alpha_i(x) = x_i.$$
**Sanity check:**
If $\mathcal{M} = S^1$, then $k = \dim(\mathcal{M}) = 1$, $d = \dim(\mathbb{R}^2) = 2$, and $x = \psi(\theta) = (\cos \theta, \sin \theta)^t$ is the coordinate chart.
The metric tensor $g = g_{11} = \langle \frac{\partial \psi}{\partial \theta}, \frac{\partial \psi}{\partial \theta} \rangle = 1 = g^{11}$.
The gradient of $\alpha_i$, $\nabla_\mathcal{M} \alpha_i = g^{11} \partial_1 \alpha_i \partial_1 = \partial_1 \alpha_i \partial_1$ can be viewed as a vector in the ambient space $\mathbb{R}^2$:

$$\nabla^j_\mathcal{M} \alpha_i = \partial_1 \psi^j \partial_1 \alpha_i$$

Therefore, we have

$$\nabla_\mathcal{M} \alpha_1 = \langle \partial_1 \psi^1 \partial_1 \alpha_1, \partial_1 \psi^2 \partial_1 \alpha_1 \rangle = \langle \sin^2 \theta, - \cos \theta \sin \theta \rangle,$$

$$\nabla_\mathcal{M} \alpha_2 = \langle \partial_1 \psi^1 \partial_1 \alpha_2, \partial_1 \psi^2 \partial_1 \alpha_2 \rangle = \langle - \sin \theta \cos \theta, \cos^2 \theta \rangle.$$

Hence $\| \nabla_\mathcal{M} \alpha_1 \|^2 + \| \nabla_\mathcal{M} \alpha_2 \|^2 = \sin^2 \theta + \cos^2 \theta = 1$
The original optimization problem can be rewritten as:

\[
\min_{f \in \mathbb{R}^{m \times n}, \mathcal{M} \subset \mathbb{R}^d} \sum_{i=1}^{d} \| \nabla_{\mathcal{M}} \alpha_i \|_{L^2(\mathcal{M})}^2 + \lambda \| y - \Phi f \|_2^2, \quad \text{subject to: } \mathcal{P}(f) \subset \mathcal{M},
\]

where

\[
\| \nabla_{\mathcal{M}} \alpha_i \|_{L^2(\mathcal{M})} = \left( \int_{\mathcal{M}} \| \nabla_{\mathcal{M}} \alpha_i(x) \|^2 dx \right)^{1/2}
\]

This optimization problem is nonconvex. It can be solved by alternating the direction of minimization with respect to \( f \) and \( \mathcal{M} \). We also perturb the coordinate function \( \alpha \) at each step.
Alternating Direction of Minimization

\[
\min_{\substack{f \in \mathbb{R}^{m \times n} \\
\mathcal{M} \subset \mathbb{R}^d}} \sum_{i=1}^{d} \| \nabla_{\mathcal{M}} \alpha_i \|^2_{L^2(\mathcal{M})} + \lambda \| y - \Phi f \|^2_2, \quad \text{subject to: } \mathcal{P}(f) \subset \mathcal{M},
\]

With a guess \( \mathcal{M}^n \) and \( f^n \) of the manifold and image, update the coordinate function \( \alpha_{i}^{n+1}, i = 1, \ldots, d \) and \( f^{n+1} \):

\[
(f^{n+1}, \alpha^{n+1}) = \arg \min_{\substack{f \in \mathbb{R}^{m \times n}, \\
\alpha_1, \ldots, \alpha_d \in H^1(\mathcal{M}^n)}} \sum_{i=1}^{d} \| \nabla_{\mathcal{M}^n} \alpha_i \|^2_{L^2(\mathcal{M}^n)} + \lambda \| b - \Phi f \|^2_2,
\]

subject to: \( \alpha(\mathcal{P}(f^n)) = \mathcal{P}(f) \)

Update \( \mathcal{M} \) by setting

\[
\mathcal{M}^{n+1} = \alpha(\mathcal{M}^n) = \left\{ (\alpha_1^{n+1}(x), \ldots, \alpha_d^{n+1}(x))^T : x \in \mathcal{M}^n \right\}.
\]

Question: How to update \( f \) and \( \alpha \)
Split Bregman (ADMM) Iteration

- Solve $\alpha_i^{n+1,k+1}, i = 1, \cdots, d$ with fixed $f^{n+1,k},$

  $$\min_{\alpha_1, \cdots, \alpha_d \in H^1(M^n)} \sum_{i=1}^d \| \nabla \alpha_i \|^2_{L^2(M^n)} + \mu \| \alpha(P(f^n)) - P(f^{n+1,k}) + d^k \|^2_F.$$  

- Update $f^{n+1,k+1}$ as

  $$\min_{f \in \mathbb{R}^{m \times n}} \lambda \| b - \Phi f \|^2_2 + \mu \| \alpha^{n+1,k+1}(P(f^n)) - P(f) + d^k \|^2_F.$$  

- Update $d^{k+1}$:

  $$d^{k+1} = d^k + \alpha^{n+1,k+1}(P(f^n)) - P(f^{n+1,k+1}).$$
Algorithm 1 LDMM Algorithm - Continuous version

1: \textbf{while} not converge \textbf{do}
2: \hspace{1em} \textbf{while} not converge \textbf{do}
3: \hspace{2em} \begin{align*}
    &\alpha_{i}^{n+1,k+1} = \arg \min_{\alpha_i \in H^1(M^n)} \| \nabla_{M^n} \alpha_i \|_{L^2(M^n)}^2 + \mu \| \alpha_i (P(f^n)) - P_i(f^{n+1,k}) + d^k_i \|_2^2 \\
    &f^{n+1,k+1} = \arg \min_{f \in \mathbb{R}^{m \times n}} \lambda \| b - \Phi f \|_2^2 + \mu \| \alpha_{i}^{n+1,k+1}(P(f^n)) - P(f) + d^k \|_F^2 \\
    &d^{k+1} = d^k + \alpha_{i}^{n+1,k+1}(P(f^n)) - P(f^{n+1,k+1}).
\end{align*}
4: \hspace{1em} \textbf{end while}
5: \textbf{end while}
6: \begin{align*}
    &\mathcal{M}^{n+1} = \left\{ (\alpha_1^{n+1}(x), \cdots, \alpha_d^{n+1}(x)) : x \in \mathcal{M}^n \right\}.
\end{align*}
7: \textbf{end while}
Graph Laplacian

The key step in the previous algorithm is to solve the following optimization:

\[
\min_{u \in H^1(M)} \| \nabla_M u \|^2_{L^2(M)} + \mu \sum_{y \in \Omega} |u(y) - v(y)|^2
\] (1)

Normally, (1) is solved by discretizing \( \nabla_M u \) by the nonlocal gradient:
\[
\nabla_w u(x, y) = \sqrt{w(x, y)} (u(y) - u(x)).
\]

This leads to solving the following graph Laplacian (GL) problem:

\[
\min_{u \in \mathbb{R}^{m \times n}} \sum_{x, y \in \Omega} w(x, y)(u(x) - u(y))^2 + \mu \sum_{y \in \Omega} |u(y) - v(y)|^2.
\]

Or equivalently,
\[
\sum_{y \in \Omega} w(x, y)(u(x) - u(y)) + \mu(u(x) - v(y)) = 0, \quad \forall x \in \Omega.
\]
By a standard variational approach, we know that problem (1) is equivalent to the following PDE:

\[
\begin{aligned}
-\Delta_M u(x) + \mu \sum_{y \in \Omega} \delta(x - y)(u(y) - v(y)) &= 0, \quad x \in \mathcal{M} \\
\frac{\partial u}{\partial n}(x) &= 0, \quad x \in \partial\mathcal{M},
\end{aligned}
\]

where \(\partial\mathcal{M}\) is the boundary of \(\mathcal{M}\) and \(n\) is the outer normal of \(\partial\mathcal{M}\).
In the point integral method (PIM), the key observation is the following integral approximation:

\[
\int_{\mathcal{M}} \Delta_{\mathcal{M}} u(y) \bar{R} \left( \frac{\|x - y\|^2}{4t} \right) dy \approx -\frac{1}{t} \int_{\mathcal{M}} (u(x) - u(y)) R \left( \frac{\|x - y\|^2}{4t} \right) dy \\
+ 2 \int_{\partial \mathcal{M}} \frac{\partial u}{\partial n}(y) \bar{R} \left( \frac{\|x - y\|^2}{4t} \right) d\tau_y.
\]

The function \( R \) is a positive function defined on \([0, +\infty)\) with compact support (or fast decay) and

\[
\bar{R} = \int_{r}^{\infty} R(s) ds.
\]
Theorem

Let $\mathcal{M}$ be a smooth manifold and $u \in C^3(\mathcal{M})$, then

$$
\left\| -\frac{1}{t} \int_{\mathcal{M}} (u(x) - u(y)) R_t(x, y) dy + 2 \int_{\partial \mathcal{M}} \frac{\partial u}{\partial n}(y) \tilde{R}_t(x, y) d\tau_y 
- \int_{\mathcal{M}} \Delta_{\mathcal{M}} u(y) \tilde{R}_t(x, y) dy \right\|_{L^2(\mathcal{M})} = O(t^{1/4}),
$$

where

$$R_t(x, y) = \frac{1}{(4\pi t)^{k/2}} R \left( \frac{\|x - y\|^2}{4t} \right), \quad \tilde{R}_t(x, y) = \frac{1}{(4\pi t)^{k/2}} \tilde{R} \left( \frac{\|x - y\|^2}{4t} \right).$$
Proof of Theorem

Using integration by part, we have

\[
\int_{\Omega} \Delta u(x) \bar{R}_t(x, y) dx = - \int_{\Omega} \nabla u(x) \cdot \nabla \bar{R}_t(x, y) dx + \int_{\partial \Omega} \frac{\partial u}{\partial n}(x) \bar{R}_t(x, y) dx
\]

\[
= \frac{1}{2t} \int_{\Omega} (x - y) \cdot \nabla u(x) R_t(x, y) dx + \int_{\partial \Omega} \frac{\partial u}{\partial n}(x) \bar{R}_t(x, y) dx
\]

We want to replace \( \nabla u \) with function value \( u \), which leads us to use the Taylor expansion

\[
u(x) - u(y) = (x - y) \cdot \nabla u(x) - \frac{1}{2} (x - y)^T H_u(x) (x - y) + O(\|x - y\|^3).
\]
Proof of Theorem

\[ u(x) - u(y) = (x - y) \cdot \nabla u(x) - \frac{1}{2} (x - y)^T H_u(x)(x - y) + O(\|x - y\|^3). \]

Integrating on both sides, we have

\[
\frac{1}{2t} \int_\Omega (x - y) \cdot \nabla u(x) R_t(x, y) dx
\]

\[ = \frac{1}{2t} \int_\Omega (u(x) - u(y)) R_t(x, y) dx \]

\[ + \frac{1}{4t} \int_\Omega (x - y)^T H_u(x)(x - y) R_t(x, y) dx + O(t^{1/2}), \]

where \( O(t^{1/2}) \) is uniform with respect to \( y \). Next we need to estimate the \( H_u \) term.

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LDMM in Image Reconstruction
Proof of Theorem

\[
\frac{1}{4t} \int_{\Omega} (x - y)^T H_u(x)(x - y)R_t(x, y)dx \\
= \frac{1}{4t} \int_{\Omega} (x_i - y_i)(x_j - y_j)\partial_{ij}u(x)R_t(x, y)dx \\
= -\frac{1}{2} \int_{\Omega} (x_i - y_i)\partial_{ij}u(x)\partial_j(\bar{R}_t(x, y))dx \\
= \frac{1}{2} \int_{\Omega} \partial_j(x_i - y_i)\partial_{ij}u(x)\bar{R}_t(x, y)dx + \frac{1}{2} \int_{\Omega} (x_i - y_i)\partial_{ijj}u(x)\bar{R}_t(x, y)dx \\
- \frac{1}{2} \int_{\partial\Omega} (x_i - y_i)n_j\partial_{ij}u(x)\bar{R}_t(x, y)dx \\
= \frac{1}{2} \int_{\Omega} \Delta u(x)\bar{R}_t(x, y)dx - \frac{1}{2} \int_{\partial\Omega} ((x - y) \otimes n) : H_u(x)\bar{R}_t(x, y)dx + O(t^{1/2}).
\]
Proof of Theorem

\[
\int_{\Omega} \Delta u(x) \bar{R}_t(x, y) \, dx = \frac{1}{2t} \int_{\Omega} (x - y) \cdot \nabla u(x) R_t(x, y) \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial n}(x) \bar{R}_t(x, y) \, dx
\]

\[
= \frac{1}{2t} \int_{\Omega} (u(x) - u(y)) R_t(x, y) \, dx + \frac{1}{4t} \int_{\Omega} (x - y)^T H_u(x)(x - y) R_t(x, y) \, dx + O(t^{1/2})
\]

\[
+ \int_{\partial \Omega} \frac{\partial u}{\partial n} \bar{R}_t(x, y) \, dx
\]

\[
= \frac{1}{2t} \int_{\Omega} (u(x) - u(y)) R_t(x, y) \, dx + \frac{1}{2} \int_{\Omega} \Delta u \cdot \bar{R}_t(x, y) \, dx
\]

\[
- \frac{1}{2} \int_{\partial \Omega} ((x - y) \otimes n) : H_u(x) \bar{R}_t(x, y) \, dx + \int_{\partial \Omega} \frac{\partial u}{\partial n} \bar{R}_t(x, y) \, dx + O(t^{1/2})
\]

This implies that:

\[
\int_{\Omega} \Delta u(x) \bar{R}_t(x, y) \, dx = \frac{1}{t} \int_{\Omega} (u(x) - u(y)) R_t(x, y) \, dx + 2 \int_{\partial \Omega} \frac{\partial u}{\partial n} \bar{R}_t(x, y) \, dx
\]

\[
- \int_{\partial \Omega} ((x - y) \otimes n) : H_u(x) \bar{R}_t(x, y) \, dx + O(t^{1/2})
\]
\[
\int_{\Omega} \Delta u(x) \tilde{R}_t(x, y) dx = \frac{1}{t} \int_{\Omega} (u(x) - u(y)) R_t(x, y) dx + 2 \int_{\partial \Omega} \frac{\partial u}{\partial n} \tilde{R}_t(x, y) dx \\
- \int_{\partial \Omega} ((x - y) \otimes n) : H_u(x) \tilde{R}_t(x, y) dx + O(t^{1/2})
\]

Although \( \left\| \int_{\partial \Omega} ((x - y) \otimes n) : H_u(x) \tilde{R}_t(x, y) dx \right\|_{L^\infty(\Omega)} = O(1) \), it can be easily estimated in \( L^2(\Omega) \):

\[
\left\| \int_{\partial \Omega} ((x - y) \otimes n) : H_u(x) \tilde{R}_t(x, y) dx \right\|_{L^2(\Omega)} = O(t^{1/4}).
\]

Therefore

\[
\left\| -\frac{1}{t} \int_{\mathcal{M}} (u(x) - u(y)) R_t(x, y) dy + 2 \int_{\partial \mathcal{M}} \frac{\partial u}{\partial n}(y) \tilde{R}_t(x, y) d\tau_y \\
- \int_{\mathcal{M}} \Delta_{\mathcal{M}} u(y) \tilde{R}_t(x, y) dy \right\|_{L^2(\mathcal{M})} = O(t^{1/4}),
\]

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The Laplace-Beltrami equation is:

\[
\begin{cases}
-\Delta_{\mathcal{M}} u(x) + \mu \sum_{y \in \Omega} \delta(x - y)(u(y) - v(y)) = 0, & x \in \mathcal{M} \\
\frac{\partial u}{\partial n}(x) = 0, & x \in \partial \mathcal{M},
\end{cases}
\]

The integral approximation is:

\[
\int_{\mathcal{M}} \Delta u(x) \bar{R}_t(x, y) dx \approx \frac{1}{t} \int_{\mathcal{M}} (u(x) - u(y)) R_t(x, y) dx + 2 \int_{\partial \mathcal{M}} \frac{\partial u}{\partial n} \bar{R}_t(x, y) dx
\]

The integral equation is:

\[
\int_{\mathcal{M}} (u(x) - u(y)) R_t(x, y) dx + \mu t \sum_{y \in \Omega} \bar{R}_t(x, y)(u(y) - v(y)) = 0.
\]
Discretization

\[
\frac{|\mathcal{M}|}{N} \sum_{j=1}^{N} R_t(x_i, x_j)(u_i - u_j) + \mu t \sum_{j=1}^{N} \bar{R}_t(x_i, x_j)(u_j - v_j) = 0.
\]

The matrix form is:

\[
(\mathbf{L} + \bar{\mu} \mathbf{\bar{W}}) \mathbf{U} = \bar{\mu} \mathbf{\bar{W}} \mathbf{V},
\]

where \( \bar{\mu} = \mu t N / |\mathcal{M}| \),

\[
\mathbf{L} = \mathbf{D} - \mathbf{W}, \quad \mathbf{W} = (w_{ij}), \quad \mathbf{\bar{W}} = (\bar{w}_{ij}),
\]

and

\[
w_{ij} = R_t(x_i, x_j), \quad \bar{w}_{ij} = \bar{R}_t(x_i, x_j), \quad x_i, x_j \in \mathcal{P}(f^n), \quad i, j = 1, \ldots, N.
\]
Algorithm 2 LDMM_PIM

1: while not converge do
2:   Compute the matrices $\mathbf{W} = (w_{ij})_{1 \leq i, j \leq N}$ from $\mathcal{P}(f^n)$
3:   for $k = 1 : K$ do
4:     $$(L + \bar{\mu} \bar{\mathbf{W}}) \mathbf{U}_k = \bar{\mu} \bar{\mathbf{W}} \mathbf{V}_{k-1}.$$ 
5:     where $\mathbf{V}_k = (\mathcal{P}(f^n) - d^k)^T$.
6:     Update $f$ by solving a least square problem
7:     $$f^{n+1,k} = \arg \min_{f \in \mathbb{R}^{m \times n}} \lambda \|b - \Phi f\|_2^2 + \bar{\mu} \|\mathbf{U}_k^T - \mathcal{P}(f) + d^{k-1}\|_F^2$$
8:     $$d^k = d^{k-1} + \mathbf{U}_k^T - \mathcal{P}(f^{n+1,k})$$
9:   end for
10: $f^{n+1} = f^{n+1,K}$
11: end while
LDMM_PIM in Image Inpainting

Original

Subsample

LDMM_GL

LDMM_PIM
Another Reason Why Graph Laplacian Fails

Consider an unknown function $u$ defined on a discrete set $\tilde{\Omega} \subset M$. Assume that we know the function value of $u$ on a subset $\Omega \subset \tilde{\Omega}$, $u(x) = b(x), \forall x \in \Omega$. Assume also that $|\Omega| \ll |\tilde{\Omega}|$. The harmonic extension of $u$ onto $\Omega$ is modeled as

$$\min_{u \in H^1(M)} \| \nabla_M u \|^2, \quad \text{subject to:} \quad u(x) = b(x), \forall x \in \Omega$$

If we discretize the objective function above using graph Laplacian, we have

$$\| \nabla_M u \|^2 = \sum_{x \in \tilde{\Omega}} \sum_{y \in \tilde{\Omega}} w(x, y) (u(x) - u(y))^2$$

$$= \sum_{x \in \tilde{\Omega}} \sum_{y \in \tilde{\Omega}} w(x, y) (u(x) - u(y))^2 + \sum_{x \in \tilde{\Omega} \setminus \Omega} \sum_{y \in \tilde{\Omega}} w(x, y) (u(x) - u(y))^2$$

The first term on the right is of order $|\Omega|$, which is much smaller than that of the second term $|\tilde{\Omega} \setminus \Omega|$. This causes the first term to be neglected in the minimization, and the algorithm sacrifices the continuity of $u$ on $\Omega$ for small variation in $\tilde{\Omega} \setminus \Omega$. 
An easy fix for the aforementioned problem is to put an extra weight $\mu$ in front of the first term.

$$\|\nabla_M u\|^2 = \mu \sum_{x \in \Omega} \sum_{y \in \tilde{\Omega}} w(x, y) (u(x) - u(y))^2 + \sum_{x \in \tilde{\Omega} \setminus \Omega} \sum_{y \in \tilde{\Omega}} w(x, y) (u(x) - u(y))^2$$

To balance the orders of the two terms, $\mu$ is chosen to be $\frac{|\tilde{\Omega}|}{|\Omega|}$.

Notice that if $\tilde{\Omega} = \Omega$, the weighted graph Laplacian is just the graph Laplacian.
Notice that a key step in LDMM for image inpainting is to solve the following optimization problem:

$$\min_{f \in \mathbb{R}^{m \times n}} \sum_{i=1}^{d} \| \nabla \mathcal{M} \alpha_i \|^2_{L^2(\mathcal{M}^k)},$$

subject to:

$$\alpha_i \left( \mathcal{P}(f^k)(x) \right) = \mathcal{P}_i f(x), \quad \forall x \in \bar{\Omega}, i = 1, \cdots, d,$$

$$f(x) = b(x), \quad \forall x \in \Omega \subset \bar{\Omega},$$

where $\mathcal{P}_i f(x)$ is the $i$-th element of the patch $\mathcal{P} f(x)$. We use the notation $x_{i-1}$ to denote the $(i-1)$-th element after $x$ in a patch, i.e. $\mathcal{P}_i f(x) = f(x_{i-1})$.

If we use periodic padding near the boundary, the ajoint operator $\mathcal{P}_i^* = \mathcal{P}_i^{-1}$.
$P_i f(x)$ is the $i$-th element of the patch $P f(x)$. $x_{i-1}$ denotes the $(i - 1)$-th element after $x$ in a patch, i.e. $P_i f(x) = f(x_{i-1})$. If we use periodic padding near the boundary, the ajoint operator $P_i^* = P_i^{-1}$.
LDMM_WGL for Image Inpainting

\[
\begin{aligned}
\min_{f \in \mathbb{R}^{m \times n}} & \sum_{i=1}^{d} \| \nabla M \alpha_i \|^2_{L^2(M^k)}, \\
\text{subject to:} & \quad \alpha_i (\mathcal{P}(f^k)(x)) = \mathcal{P}_i f(x), \quad \forall x \in \bar{\Omega}, i = 1, \ldots, d, \\
& \quad f(x) = b(x), \quad \forall x \in \Omega \subset \bar{\Omega},
\end{aligned}
\]

Applying WGL, we have the following discretized optimization problem:

\[
\min_{f \in \mathbb{R}^{m \times n}} \sum_{i=1}^{d} \left( \sum_{x \in \bar{\Omega} \setminus \Omega} \sum_{y \in \bar{\Omega}} \tilde{w}(x, y)((\mathcal{P}_i f(x) - \mathcal{P}_i f(y))^2 \right) \\
+ \frac{mn}{|\Omega|} \sum_{x \in \Omega} \sum_{y \in \bar{\Omega}} \tilde{w}(x, y)((\mathcal{P}_i f(x) - \mathcal{P}_i f(y))^2 \right) \\
\text{subject to:} \quad f(x) = b(x), \quad x \in \Omega \subset \bar{\Omega}
\]

where \( \Omega_i = \{ x \in \bar{\Omega} : \mathcal{P}_i f(x) \text{ is sampled} \} \), and \( \tilde{w}(x, y) = w(\mathcal{P} f(x), \mathcal{P} f(y)) \).
Using a standard variational approach, the equivalent Euler-Lagrange equation is

\[
\begin{cases}
\sum_{i=1}^{d} P_i^*(h_i) + \mu \sum_{i=1}^{d} P_i^*(g_i) (x) = 0, & x \in \bar{\Omega} \setminus \Omega \\
 f(x) = b(x), & x \in \Omega
\end{cases}
\]

where

\[
\begin{align*}
  h_i(x) &= \sum_{y \in \bar{\Omega}} 2\bar{w}(x, y)(P_i f(x) - P_i f(y)) \\
  g_i(x) &= \sum_{y \in \Omega_i} \bar{w}(x, y)(P_i f(x) - P_i f(y))
\end{align*}
\]
$h_i(x) = \sum_{y \in \tilde{\Omega}} 2\tilde{w}(x, y)(P_i f(x) - P_i f(y))$

$P_i^* h_i(x) = h_i(x_{1-i}) = \sum_{y \in \tilde{\Omega}} 2\tilde{w}(x_{1-i}, y) \left( P_i f(x_{1-i}) - P_i f(y) \right)$

$= \sum_{y \in \tilde{\Omega}} 2\tilde{w}(x_{1-i}, y) \left( f(x) - f(y_{1-i}) \right)$

$= \sum_{y \in \tilde{\Omega}} 2\tilde{w}(x_{1-i}, y_{1-i}) \left( f(x) - f(y) \right)$

Therefore

$\sum_{i=1}^{d} P_i^* (h_i)(x) = \sum_{i=1}^{d} \sum_{y \in \tilde{\Omega}} 2\tilde{w}(x_{1-i}, y_{1-i}) \left( f(x) - f(y) \right)$

Similarly,

$\sum_{i=1}^{d} P_i^* (g_i)(x) = \sum_{i=1}^{d} \sum_{y \in \Omega} \tilde{w}(x_{1-i}, y_{1-i}) \left( f(x) - f(y) \right)$
The Euler-Lagrange equation becomes:

\[
\begin{cases}
\sum_{y \in \Omega} \left( \sum_{i=1}^{d} 2 \tilde{w}(x_{1-i}, y_{1-i}) \right) (f(x) - f(y)) \\
+ \mu \sum_{y \in \Omega} \left( \sum_{i=1}^{d} \tilde{w}(x_{1-i}, y_{1-i}) \right) (f(x) - f(y)) = 0, & x \in \bar{\Omega} \setminus \Omega \\
f(x) = b(x), & x \in \Omega
\end{cases}
\]

Let \( \tilde{w}(x, y) = \sum_{i=1}^{d} \tilde{w}(x_{1-i}, y_{1-i}) \), then

\[
2 \sum_{y \in \bar{\Omega}} \tilde{w}(x, y) (f(x) - f(y)) + \mu \sum_{y \in \Omega} \tilde{w}(x, y) (f(x) - f(y)) = 0, \quad x \in \bar{\Omega} \setminus \Omega
\]
Low Dimensional Manifold Model

Point Integral Method

Weighted Graph Laplacian and Semi-local Patches

Results

Conclusion

LDMM_WGL

\[
2 \sum_{y \in \bar{\Omega}} \tilde{w}(x, y) (f(x) - f(y)) + \mu \sum_{y \in \Omega} \tilde{w}(x, y) (f(x) - f(y)) = 0, \quad x \in \bar{\Omega} \setminus \Omega
\]

\[
W = \begin{pmatrix} \bar{\Omega} \setminus \Omega & \Omega \\ \bar{\Omega} \setminus \Omega & \Omega \end{pmatrix}, \quad L = \begin{pmatrix} \bar{\Omega} \setminus \Omega & \Omega \\ \bar{\Omega} \setminus \Omega & \Omega \end{pmatrix}, \quad f = \begin{pmatrix} v \\ b \end{pmatrix}
\]

Let \( \Delta = \text{diag}(\text{sum}(\tilde{W}_{12}, 2)) \), then

\[
2 \tilde{L}_{11} v + 2 \tilde{L}_{12} b + \mu (\Delta v - \tilde{W}_{12} b) = 0
\]

\[
(2 \tilde{L}_{11} + \mu \Delta) v = \mu \tilde{W}_{12} b - 2 \tilde{L}_{12} b
\]
The semi-local patches are obtained by adding local coordinates to the nonlocal patches with a weight $\lambda$, i.e.

$$\tilde{P}f(x) = [Pf(x), \lambda x].$$

When $\lambda = 0$, semi-local patches are just nonlocal patches. When $\lambda \to \infty$, the patches are completely determined by local coordinates. We choose a proper $\lambda$ to help LDMM update the “true” metric on the manifold $\mathcal{M}$ faster and more reliably.
LDMM_WGL with Semi-local Patches

Original

Subsample (10%)

LDMM_PIM

PSNR = 24.74

LDMM_WGL

PSNR = 25.92

LDMM in Image Reconstruction
Numerical Results
2D Image Inpainting

Original

Subsample (10%)

BPFA (23.44dB)

LDMM (25.92dB)

Original

Subsample (10%)

BPFA (24.71dB)

LDMM (25.70dB)
Image Denoising

Original

Noisy (8.13dB)

LDMM (23.46dB)

BM3D (23.60dB)

Original

Noisy (8.13dB)

LDMM (24.54dB)

BM3D (24.64dB)
Hyperspectral Image Inpainting

Original | Subsampled (5%) | Recovered | Error (PSNR = 37.9)
--- | --- | --- | ---
Original | Subsampled (5%) | Recovered | Error (PSNR = 38.2)
Noisy and Incomplete Hyperspectral Images

Original at 50th band

Subsampled (10%)

Recovered

Original at 100th band

Subsampled (10%)

Recovered
3D Plasma Reconstruction

Original (20th band)  
Subsampled (5%)  
Reconstructed  
Error

Original (40th band)  
Subsampled (5%)  
Reconstructed  
Error
Shock Reconstruction

Original (t=1)

Subsampled (5%)

Reconstructed

Error

Original (t=2)

Subsampled (5%)

Reconstructed

Error
Shock Reconstruction with Granular Structure

Original

Subsampled (10%)

Reconstructed

Error
Comparison of Compression Results

Every algorithm except for LDMM has access to the entire image. The budget is set to be 10% of the original data size.

Original

Subsampled (10%)

LDMM (29.34dB)

DCT (35.63dB)

FFT (34.44dB)

Wavelet (34.20dB)
Comparison of Super-resolution Results

The image is downsampled with a sampling spacing of 4 in each direction.

<table>
<thead>
<tr>
<th>Method</th>
<th>PSNR (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td></td>
</tr>
<tr>
<td>Bicubic</td>
<td>26.81dB</td>
</tr>
<tr>
<td>LDMM</td>
<td>29.20dB</td>
</tr>
<tr>
<td>DCT</td>
<td>26.21dB</td>
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<tr>
<td>FFT</td>
<td>27.43dB</td>
</tr>
<tr>
<td>Wavelet</td>
<td>25.47dB</td>
</tr>
</tbody>
</table>
Neutron Transport

Original

Subsampled (5%)

Reconstructed

Error
Comparison of Compression Results

Every algorithm except for LDMM has access to the entire image. The budget is set to be 10% of the original data size.

Original

Subsampled (10%)

LDMM (40.06dB)

DCT (59.64dB)

FFT (47.89dB)

Wavelet (50.95dB)
Comparison of Super-resolution Results

The image is downsampled with a sampling spacing of 4 in each direction.

![Original Image](original.png)
![Bicubic (39.25dB)](bicubic.png)
![LDMM (39.98dB)](ldmm.png)

![DCT (32.64dB)](dct.png)
![FFT (37.28dB)](fft.png)
![Wavelet (31.26dB)](wavelet.png)
Conclusion and Future Work

Conclusion

- LDMM uses the dimension of the patch manifold to regularize the variational problem.
- The Laplace-Beltrami equation is solved via either the point integral method or the weighted graph Laplacian.
- Weighted graph Laplacian is much more efficient for image inpainting, because the equation is solved on the image domain instead of the patch domain.

Ongoing and future work

- Sparse dimensional manifold model (SDMM) for HSI processing.
- LDMM or SDMM with rotating patches with different resolutions.