Decay of correlations for hyperbolic and normally hyperbolic trapping

Semiclassical origins of density functional approximation, IPAM

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Correlations

$$X = \mathbb{R}, \quad P = -\partial_x^2, \quad U(t) := \sin(\sqrt{P}t)/\sqrt{P}$$

$$C(f,g)(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} U(t) fg \, dx, \qquad f,g \in C_{\mathrm{c}}^{\infty}(\mathbb{R}).$$

$$U(t)g = \frac{1}{2} \int_{x-t}^{x+t} g(y) dy$$

$$C(f,g) = \frac{1}{2} \int_{\mathbb{R}} f \, dx \int_{\mathbb{R}} g \, dx, \quad t \geq T(\operatorname{supp} f, \operatorname{supp} g).$$

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This particular behaviour is due to the fact that the resolvent of P,

$$R(\lambda) \stackrel{\text{def}}{=} (P-\lambda^2)^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \operatorname{Im}\lambda > 0,$$

has a pole $\lambda = 0$. What does that mean?

$$R(\lambda)f(x) = \int G(\lambda,x,y)f(y)dy$$

In this basic case we see this from an explicit formula,

$$G(\lambda,x,y) = \frac{i}{2\lambda}e^{i\lambda|x-y|}.$$

$$P = -\partial_x^2 + V(x), \quad V \in L^\infty_{\mathrm{c}}(\mathbb{R};\mathbb{R}), \quad V \geq 0,$$

$$(P - \lambda^2)^{-1} f(x) = \int G(\lambda, x, y) f(y) dy, \quad \operatorname{Im}\lambda > 0, \quad f, u \in L^2.$$

$G(\lambda, x, y)$ continues meromorphically in λ to the complex plane

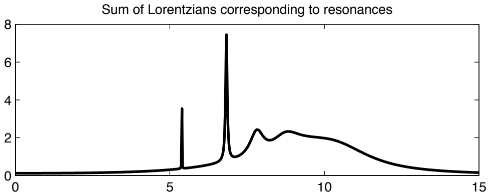
Lax-Phillips:

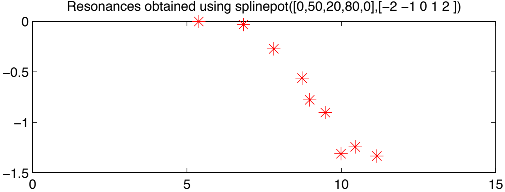
$$C(f,g)(t) = \sum_{\operatorname{Im}\lambda_j > -A} e^{-i\lambda_j t} \int_{\mathbb{R}} f u_j \, dx \int_{\mathbb{R}} g u_j \, dx + \mathcal{O}(e^{-At}),$$

λ_j's are the poles of $G(\lambda, x, y)$.

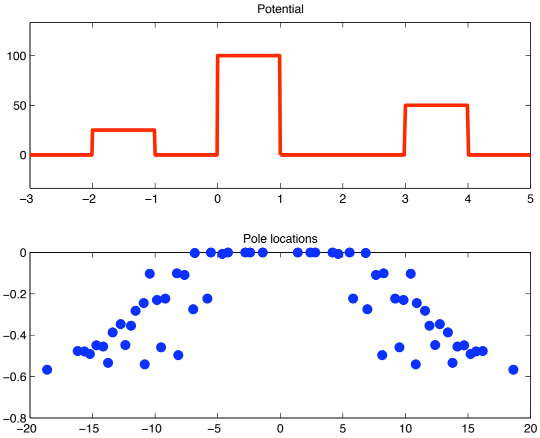
$$C(f,g)(t) = \sum_{\mathrm{Im}\,\lambda_j > -A} e^{-i\lambda_j t} \int_{\mathbb{R}} f u_j \, dx \int_{\mathbb{R}} g u_j \, dx + \mathcal{O}(e^{-At}),$$

$$\widehat{C(f,g)}(-\lambda) = \sum_{\mathrm{Im}\,\lambda_j > -A} \frac{c_j}{\lambda_j - \lambda} + \mathcal{O}\left(\frac{1}{A}\right)$$





Quantum Resonances describe the resonating waves:



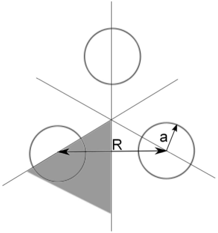
Computed using `squarepot.m`

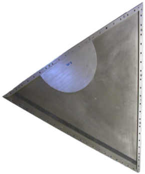
`http://www.cims.nyu.edu/~dbindel/resonant1d/`

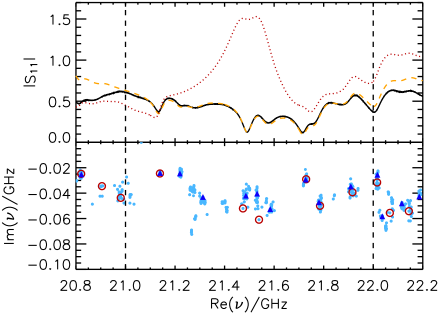
Here is how they sound:

```
time = linspace(0,500,5000);
sound(real(exp(-i*z*time)))
```

A real experimental example







Potzuweit-Weich-Barkhofen-Kuhl-Stöckmann-Z '12

Resonances for three discs:

Barkhofen-Kuhl-Weich '13

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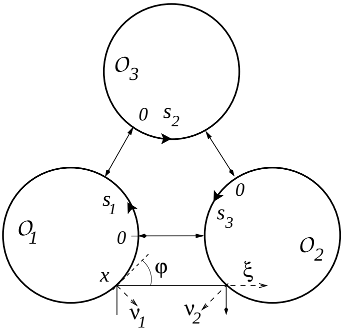
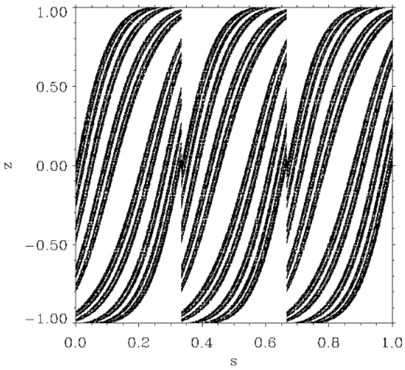
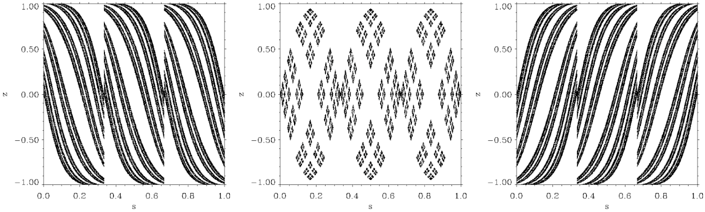




Barkhofen-Kuhl-Weich '13

The outgoing set



incoming set trapped set outgoing set

Poon-Campos-Ott-Grebogi '96

$$C(f,g)(t) = \sum_{\operatorname{Im}\lambda_j > -A} e^{-i\lambda_j t} \int_{\mathbb{R}} f u_j \, dx \int_{\mathbb{R}} g u_j \, dx + \mathcal{O}(e^{-At}),$$

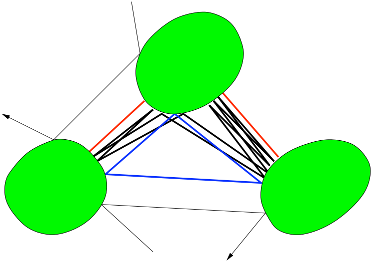
In general to have a finite expansion of correlations with exponentially decaying remainders we need to know that

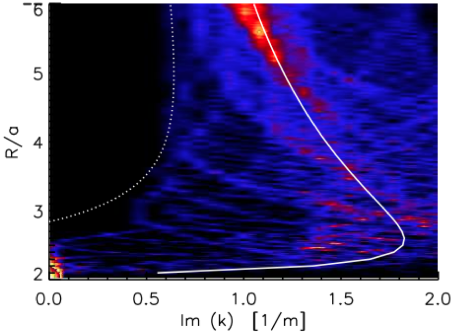
the number of poles of $G(\lambda)$ is finite in a strip $\operatorname{Im}\lambda > -\gamma$.

and to have some effective bounds in $G(\lambda)$.

Hence exponetial decay of correlations is closely related to *resonance free strips*.

How do we determine that gap at the high frequency limit when the dynamics is hyperbolic?

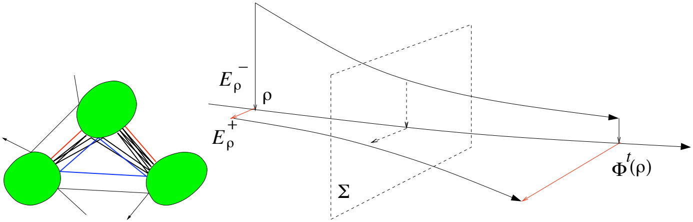




(b)3 Disk Simulation

Gaspard-Rice '89, Lu-Sridhar-Z '03, Potzuweit et al '12

Ikawa '88, Burq '93, Nonnenmacher-Z'09, Naud '04,'12, Petkov-Stoyanov'11



We define the topological pressure associated to the unstable Jacobian:

$$J_t^+(\rho) = \det\left(d\Phi^t_{|E_\rho^+}\right)$$

$$\mathcal{P}_E(s) = \lim_{T\to\infty} \frac{1}{T} \log \sum_{T-1<T_\gamma<T} J^+(\gamma)^{-s} \, ,$$

where γ are closed orbits with period T_γ.

Ikawa '88, Nonnemacher-Z '09, Petkov-Stoyanov '11:
If $\mathcal{P}_E(1/2) < 0$ then no resonances with $\mathrm{Im}\,\lambda > P_E(1/2)$.
(at high energies)

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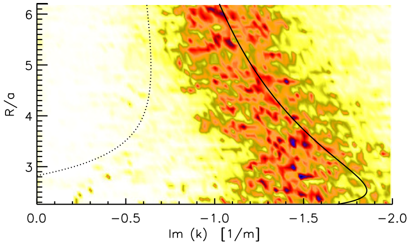
The decay of correlations is closely related to *resonance free strips*.



Potzuweit-Weich-Barkhofen-Kuhl-Stöckmann-Z, PRL '13

Lu-Sridhar-Z '03: concentration of decay rates at $P(1)/2$, PRL '03

A very different setting: chaotic dynamics (more precisely contact Anosov flows)

$$\varphi_t : X \to X, \qquad X \text{ compact}$$

$$U(t) : C^\infty(X) \to C^\infty(X), \quad U(t)f \stackrel{\text{def}}{=} \varphi_t^* f.$$

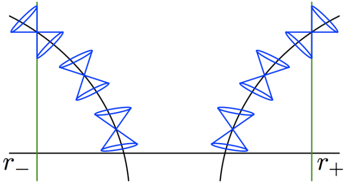
Dolgopyat '98, Liverani '04 , Tsujii '11:

$$\int_X U(t)fg\,dx = \int_X f\,dx \int_X g\,dx + O(e^{-\gamma t}), \quad t \to \infty.$$

A completely different setting:

Wave equation for Kerr-de Sitter black holes

$$X = (r_-, r_+) \times \mathbb{S}^2, \quad U(t) = \text{ wave propagator } .$$



Bony-Häfner'08, Melrose-Sá Barreto-Vasy'10, Dyatlov'11, Vasy'12:

$$\int_X U(t) fg \, dx = \int_X f \, dx \int_X g \, dx + O(e^{-\gamma t}), \quad t \to \infty.$$

$$\int_X U(t) fg \, dx = \int_X f \, dx \int_X g \, dx + O(e^{-\gamma t}), \quad t \to \infty.$$

$$X \text{ a compact manifold }, \quad U(t) = \varphi_t^*$$

$$X = (r_-, r_+) \times \mathbb{S}^{n-1}, \quad U(t) = \text{ wave group }.$$

What do the two problems have in common?

1) Normally hyperbolic trapped set

2) A strip with finitely many resonances

3) A resonance at 0 (this is quite special)

At this point, except for one hyperbolic orbit, I do not know of an experimental example of this set-up.

Trapped set:

$$K_E = \{\rho \in p^{-1}(E) \subset T^*X : \exp tH_p(x,\xi) \not\to \infty,\ t \to \pm\infty\}.$$

The flow is normally hyperbolic at $K = \bigcup_{|E-E_0|<\delta} K_E$:

$$\exists\, \lambda_0 > 0, \quad K \ni \rho \mapsto E_\rho^\pm \subset T_\rho(T^*X), \quad (\exp tH_p)_* E_\rho^\pm = E^\pm_{\exp tH_p(\rho)},$$

$$T_\rho K \cap E_\rho^\pm = E_\rho^+ \cap E_\rho^- = \{0\}, \quad \dim E_\rho^\pm = d_t,$$

$$T_\rho(T^*X) = T_\rho K + E_\rho^+ + E_\rho^-,$$

$$\|(\exp(\mp tH_p))_* v\|_{\exp(\mp tH_p)(\rho)} \le Ce^{-\lambda_0 t}\|v\|_\rho, \quad v \in E_\rho^\pm.$$

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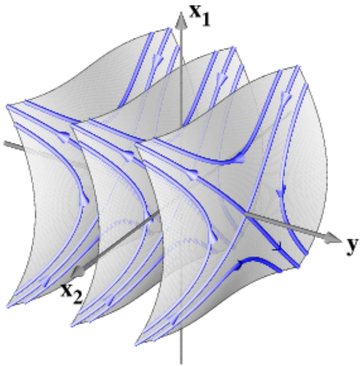
$$\|(\exp(\mp t H_p))_* v\|_{\exp(\mp t H_p)(\rho)} \le C e^{-\lambda_0 t}\|v\|_\rho, \qquad v \in E_\rho^\pm.$$



Resonance gap for normally hyperbolic trapping:

Nonnenmacher-Z '13: If the trapped set is normally hyperbolic then there are no resonances with $\mathrm{Im}\,\lambda > -\gamma$

($+$ bounds on the Green function which allow consequences for correlations).

$$\gamma < \liminf_{t\to\infty} \frac{1}{t} \inf_{\rho\in K} \log\det d(\exp tH_p)|_{E_\rho^+}.$$

Requires fine analysis near the trapped set then "glued to infinity" using

1) Euclidean infinity: complex scaling of Aguilar-Combes, '71 Balslev-Combes '71, Simon '72, Helffer-Sjöstrand, '85...

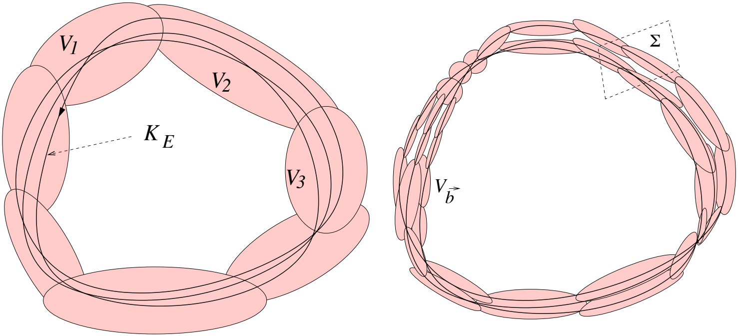
2) de Sitter black holes, hyperbolic infinities: Vasy '13, Dyatlov '11

3) dynamical systems: Faure-Sjöstrand '11

Conclusions

- Exponential decay of correlations comes from the high-energy (semiclassical) resonance gap; in other words from a lower bound on decay rates.
- Resonance gap is determined by the dynamical structure of the trapped set the set of points in phase space which do not escape to infinity.
- For hyperbolic (fractal) trapped sets the gap is measured using topological pressure; it occurs when the set is filamentary.
- For normally hyperbolic (smooth) trapped sets the gap is estimated using transversal Lyapunov exponents
- It is essential that the gap estimates are quantitative; that is they come with estimates on the Green function.

An alternative definition of the topological pressure



To define the pressure of Φ^t on K_E, one starts from an open cover $(V_b)_{b\in B}$ of K_E. This cover is then refined T times through the flow, producing sets

$$V_{\vec{b}} = V_{b_0} \cap \Phi^{-1} V_{b_1} \cap \Phi^{-2} V_{b_2} \cap \cdots \cap \Phi^{-T+1} V_{b_{T-1}}, \qquad \vec{b} \in B^T.$$

Keep the $V_{\vec{b}}$ intersecting K_E.

We weigh each $V_{\vec{b}}$ using the coarse-grained unstable Jacobian:

$$w_T(V_{\vec{b}}) \stackrel{\text{def}}{=} \sup_{\rho \in V_{\vec{b}} \cap K_E} \left(J_T^+(\rho)\right)^{-1/2} \sim \mathrm{e}^{-T(d-1)\bar{\lambda}/2},$$

where $\bar{\lambda}$ is an "average" stretching exponent for initial points in $V_{\vec{b}}$.

One then considers the partition function

$$\mathcal{Z}_T \stackrel{\text{def}}{=} \inf\{\sum_{\vec{b} \in \mathcal{B}_T} w_T(V_{\vec{b}}) \;:\; \mathcal{B}_T \subset B^T,\; K_E \subset \bigcup_{\vec{b} \in \mathcal{B}_T} V_{\vec{b}}\}.$$

The pressure $\mathcal{P}_E(1/2)$ is finally given by

$$\mathcal{P}_E(1/2) = \lim_{\mathrm{diam}(V_b) \to 0} \lim_{T \to \infty} \frac{1}{T} \log \mathcal{Z}_T.$$

Yet another definition (perhaps not rigorous but computationally very successful):

$$P(s) = \frac{1}{i} \text{ first zero of } \; \zeta_s(\lambda)$$

$$\zeta_s(\lambda) = \prod_{\gamma^\#} \left(1 - \frac{e^{i\lambda T_\gamma^\#}}{J_+(\gamma^\#)^s} \right),$$

$$J_+(\gamma^\#) = \exp \left(\int_0^{T_\gamma^\#} \log \det d\Phi_t(\rho)|_{E^u(\rho)} \right).$$

$P(0) =$ topological entropy, $P(1) = -$ classical escape rate,

$P(\delta) = 0$, where $\dim K = 2\delta + 1$, $K \subset S^*(\mathbb{R}^2 \setminus \mathcal{O})$, the trapped set.