Phase-space approach to bosonic mean field dynamics

Francis Nier,
IRMAR, Univ. Rennes 1
After joint works with Z. Ammari

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Outline

N-body bosonic problem

Let $\mathcal{Z} = L^2(X, m; \mathbb{C})$ be the one particle space.
e.g. $(X, m) \subset$ measured add. group. $\mathcal{Z}$ separable.
$X \subset \mathbb{R}^d$, $X \subset \mathbb{Z}^d$, $X = \mathbb{Z}^d/(K\mathbb{Z})^d$.
The bosonic N-particle space is $L^2_s(X^N, m^{\otimes N}; \mathbb{C}) = \bigvee^N \mathcal{Z}$.
$(A, D(A))$ is a s.a. operator on $\mathcal{Z}$
$V(x) = V(-x)$ is a real function on X.

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N-body quantum dynamics

$$\begin{cases} i\partial_t \Psi = \sum_{i=1}^N A_i \Psi + \frac{1}{N} \sum_{i<j} V(x_i - x_j)\Psi \\ \Psi(t=0) = \Psi_0 \in \bigvee^N \mathcal{Z} \end{cases}$$

with $A_i = \mathrm{Id}_{\mathcal{Z}} \otimes \cdots \otimes \underbrace{A}_{i} \otimes \cdots \otimes \mathrm{Id}_{\mathcal{Z}}$.

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Mean field limit $N \to \infty$

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$$\begin{cases} i\varepsilon\partial_t\Psi = \varepsilon\sum_{i=1}^N A_i\Psi + \varepsilon^2\sum_{i<j} V(x_i - x_j)\Psi \\ \Psi(t=0) = \Psi_0 \in \bigvee^{\frac{1}{\varepsilon}} \mathcal{Z} \end{cases}$$

with $A_i = \mathrm{Id}_{\mathcal{Z}} \otimes \cdots \otimes \underbrace{A}_{i} \otimes \cdots \otimes \mathrm{Id}_{\mathcal{Z}}$.

Mean field limit $N \to \infty$ $\varepsilon \to 0$.
Divide by N and set $\varepsilon = \frac{1}{N}$.

Fock space formulation

Francis Nier, IRMAR, Univ. Rennes 1
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Rewriting in the bosonic Fock space $\Gamma(\mathcal{Z}) = \oplus_{n=0}^{\infty} \bigvee^n \mathcal{Z}$.

$$\begin{cases} i\varepsilon\partial_t \Psi^\varepsilon = H^\varepsilon \Psi^\varepsilon \\ \Psi^\varepsilon(t=0) = \Psi_0^\varepsilon \in \Gamma(\mathcal{Z}), \end{cases}$$

where

$$H^\varepsilon = \int_{X^2} A(x,y) a^*(x) a(y) dm(x) dm(y) + \frac{1}{2}\int_{X^2} V(x-y) a^*(x) a^*(y) a(x) a(y) \; dm(x) dm(y)$$

is the Wick quantization of the energy

$$\mathcal{E}(z,\bar{z}) = \int_{X^2} \bar{z}(x) A(x,y) z(y) dm(x) dm(y) + \frac{1}{2}\int_{X^2} V(x-y) |z(x)|^2 |z(y)|^2 \; dm(x) dm(y),$$

with the ε-dependent CCR's:

$$[a(x), a(y)] = [a^*(x), a^*(y)] = 0 \quad , \quad [a(x), a^*(y)] = \varepsilon\delta(x-y).$$

Mean field=semiclassical

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The relation between

$$i\varepsilon\partial_t\Psi^\varepsilon = H^\varepsilon\Psi^\varepsilon$$

and the hamiltonian dynamics in the phase-space $(\mathcal{Z}, \mathrm{Im}\,\langle\ ,\ \rangle_{\mathcal{Z}})$

$$i\partial_t z = \partial_{\overline{z}}\mathcal{E}$$

in the limit is actually a semiclassical problem.

$$\varepsilon \to 0 \quad , \quad [a(x), a^*(y)] = \varepsilon\delta(x-y) \quad , \quad H^\varepsilon = \mathcal{E}^{Wick}\,.$$

Perfectly known when $X = \{1, \dots, K\} = \mathbb{Z}/(K\mathbb{Z})$, $\mathcal{Z} = \mathbb{C}^K \sim \mathbb{R}^{2K}$, $\Gamma(\mathcal{Z}) \sim L^2(\mathbb{R}^K, dx; \mathbb{C})$, (semiclassical pseudo-differential calculus, Fourier integral operators, Egorov's theorem, higher order expansions are possible).

Take $a^*(k) = -h\partial_{x_k} + x_k$, $a(k) = h\partial_{x_k} + x_k$ $(k \in X\,,\, x_k \in \mathbb{R})$ with

$$(\frac{1}{N}) \sim \varepsilon = 2h.$$

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Weyl observables

For $f \in \mathcal{Z}$, $W(f) = e^{i\Phi(f)}$.

$$\phi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(f)) = \int_X f(x)a^*(x)\ dm(x) + \int_X \overline{f}(x)a(x)\ dm(x)\,,$$

$$W(f_1) \circ W(f_2) = e^{-i\frac{\varepsilon}{2}\,\mathrm{Im}\,\langle f_1, f_2\rangle} W(f_1 + f_2)\,.$$

Separation of variables: $\Gamma(\mathcal{Z}) = \Gamma(\mathfrak{p}\mathcal{Z}) \otimes \Gamma((\mathfrak{p}\mathcal{Z})^{\perp})$ and $b^{Weyl} = b^{Weyl} \otimes \mathrm{Id}_{\Gamma((\mathfrak{p}\mathcal{Z})^{\perp})}$.

Can be extended to the Weyl-Hörmander symbol class $\cup_{\mathfrak{p}} \cup_n \mathcal{S}((1 + |\mathfrak{p}z|^2)^{n/2}, d|\mathfrak{p}z|^2)$ which contains cylindrical polynomial functions and forms an algebra for the Moyal product with

$$b_1^{Weyl} \circ b_2^{Weyl} = (b_1 b_2)^{Weyl} + \varepsilon R(b_1, b_2, \varepsilon)\,.$$

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Let $\mathfrak{p}$ be a finite rank orthogonal projection. When $b \in \mathcal{S}_{\text{cyl}}(\mathcal{Z})$ based on $\mathfrak{p}\mathcal{Z}$, i.e. $b(z) = b(\mathfrak{p}z)$ with $b \in \mathcal{S}(\mathfrak{p}\mathcal{Z})$,

$$b^{Weyl} = \int_{\mathfrak{p}\mathcal{Z}} \mathcal{F}b(\xi) W(\sqrt{2}\pi\xi)\ L_{\mathfrak{p}\mathcal{Z}}(d\xi)\,,$$

with $$\mathcal{F}b(\xi) = \int_{\mathfrak{p}\mathcal{Z}} e^{2i\pi\, \mathrm{Re}\, \langle \xi, z\rangle} b(z)\ L_{\mathfrak{p}\mathcal{Z}}(dz)\,.$$

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Monomials: $\tilde{b} \in \mathcal{L}(\bigvee^{p} \mathcal{Z}; \bigvee^{q} \mathcal{Z})$,

$$[\tilde{b}u](x_1, \ldots, x_q) = \int_{X^p} \tilde{b}(x,y)u(y_1,\ldots,y_p)\ dm^{\otimes p}(y)\,,$$

$$b(z) = \langle z^{\otimes q}\,,\ \tilde{b}z^{\otimes p}\rangle\,,$$

$$b^{Wick} = \int_{X^{p+q}} \tilde{b}(x,y)a^*(x_1)\ldots a^*(x_q)a(y_1)\ldots a(y_p)\ dm^{\otimes q}(x)dm^{\otimes p}(y)\,.$$

With the product $\sharp^{Wick}$, $b_1^{Wick} \circ b_2^{Wick} = (b_1 \sharp^{Wick} b_2)^{Wick}$, $\mathcal{P}(\mathcal{Z})$ is an algebra and

$$b_1 \sharp^{Wick} b_2 = \sum_{k=0}^{\min\{p_1,q_2\}} \frac{\varepsilon^k}{k!} \partial_z^k b_1 . \partial_{\bar{z}}^k b_2\,.$$

In particular

$$\varepsilon^{-1}\left[b_1^{Wick}\,,\ b_2^{Wick}\right] = \partial_z b_1 . \partial_{\bar{z}} b_2 - \partial_z b_2 . \partial_{\bar{z}} b_1 + \varepsilon R(b_1, b_2, \varepsilon)\,.$$

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Main difficulty

There is a natural quantization of cylindrical symbols (Weyl quantization) and of rather general polynomials (Wick quantization). Although these classes are algebra for the operator product (Moyal or $\sharp^{Wick}$ product), they are not preserved by the action of a nonlinear symplectic flow.

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Definition

Consider the quantity $\mathrm{Tr}\ [A\varrho^\varepsilon]$ which extends the $\langle\psi^\varepsilon\,,\,A\psi^\varepsilon\rangle$ when ϱ^ε is a general normal state (parametrized by ε).

Admissible sets $\mathcal{E}$ for the parameter ε are bounded subsets of $(0,+\infty)$ such that $0\in\overline{\mathcal{E}}$.

Definition: Let $(\varrho^\varepsilon)_{\varepsilon\in\mathcal{E}}$ be a family or normal states in $\Gamma(\mathcal{Z})$, with $\mathcal{Z}$ separable and $\mathcal{E}$ admissible. The set of Wigner measures of $(\varrho^\varepsilon)_{\varepsilon\in\mathcal{E}}$ $\mathcal{M}(\varrho^\varepsilon,\varepsilon\in\mathcal{E})$ is the set of Borel probability measures μ on $\mathcal{Z}$ such that there exists an admissible subset $\mathcal{E}_\mu\in\mathcal{E}$ for which

$$\forall\xi\in\mathcal{Z}\,,\qquad \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in\mathcal{E}_\mu}}\ \mathrm{Tr}\ \left[W(\sqrt{2}\pi\xi)\varrho^\varepsilon\right]=\int_{\mathcal{Z}}e^{2i\pi\,\mathrm{Re}\,\langle\xi\,,\,z\rangle}\,d\mu(z)\,.$$

Theorem (Ammari N. Ann.IHP 08): With $\mathbf{N}=(|z|^2)^{Wick}=\varepsilon\mathbf{N}_{\varepsilon=1}$, the simple condition

$$\exists\delta>0\,,\forall\varepsilon\in\mathcal{E}\,,\quad \mathrm{Tr}\ \left[\mathbf{N}^\delta\varrho^\varepsilon\right]\leq C_\delta\,,$$

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$$\forall\xi\in\mathcal{Z}\,,\qquad \lim_{\substack{\varepsilon\to 0\\ \varepsilon\in\mathcal{E}_\mu}}\ \mathrm{Tr}\,\left[W(\sqrt{2}\pi\xi)\varrho^\varepsilon\right]=\int_{\mathcal{Z}}e^{2i\pi\,\mathrm{Re}\,\langle\xi\,,\,z\rangle}\,d\mu(z)\,.$$

Theorem (Ammari N. Ann.IHP 08): With $\mathbf{N}=(|z|^2)^{Wick}=\varepsilon\mathbf{N}_{\varepsilon=1}$, the simple condition

$$\exists\delta>0\,,\forall\varepsilon\in\mathcal{E}\,,\quad \mathrm{Tr}\,\left[\mathbf{N}^\delta\varrho^\varepsilon\right]\le C_\delta\,,$$

ensures $\mathcal{M}(\varrho^\varepsilon,\varepsilon\in\mathcal{E})\neq\emptyset$.

Definition

Consider the quantity $\mathrm{Tr}\,[A\varrho^\varepsilon]$ which extends the $\langle\psi^\varepsilon\,,\,A\psi^\varepsilon\rangle$ when ϱ^ε is a general normal state (parametrized by ε).
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Simple properties

$\mathcal{M}(\varrho^\varepsilon, \varepsilon \in \mathcal{E}) = \{\mu\}$ is characterized by

$$\forall b \in \mathcal{S}_{cyl}(\mathcal{Z}), \qquad \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathcal{E}}} \mathrm{Tr}\left[b^{Weyl} \varrho^\varepsilon\right] = \int_{\mathcal{Z}} b(z)\, d\mu(z).$$

Moreover $\int_{\mathcal{Z}} |z|^{2\delta}\, d\mu(z) \le C_\delta$.

Examples

Coherent states: $\Psi^\varepsilon = E(z_0) = W(\frac{\sqrt{2}}{i\varepsilon} z_0)|\Omega\rangle$,
$\varrho^\varepsilon = |E(z_0)\rangle\langle E(z_0)|$.

$$\mathcal{M}(|E(z_0)\rangle\langle E(z_0)|, \varepsilon \in \mathcal{E}) = \{\delta_{z_0}\}.$$

Hermite states: $\Psi^\varepsilon = z_0^{\otimes N}$ with $|z_0| = 1$ and $N = \left[\frac{1}{\varepsilon}\right]$.

$$\mathcal{M}(|z_0^N\rangle\langle z_0^{\otimes N}|, \varepsilon \in \mathcal{E}) = \left\{\delta_{z_0}^{S^1}\right\}$$

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An example with mean field correlations

$$\text{Take} \qquad \Psi^{\varepsilon} = \frac{1}{\sqrt{\varepsilon^{N_1+N_2} N_1! N_2!}} [a^*(\phi_1)]^{N_1} [a^*(\phi_2)]^{N_2} |\Omega\rangle \,,$$

$$\text{with} \qquad \lim_{\varepsilon \to 0} \varepsilon N_1 = \lim_{\varepsilon \to 0} \varepsilon N_2 = \frac{1}{2} \,,$$

$$\text{and} \qquad \phi_1 \perp \phi_2 \quad , \quad |\phi_1| = |\phi_2| = 1 \,.$$

$$\begin{array}{lllllll}
\Gamma(\mathcal{Z}) = & \Gamma(\mathbb{C}\phi_1) & \otimes & \Gamma(\mathbb{C}\phi_2) & \otimes & \Gamma(\{\phi_1, \phi_2\}^{\perp}) \\
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μ carried by a torus. Other writing:

$$\psi_0 = \frac{1}{\sqrt{2}}(\phi_1 + \phi_2) \quad , \quad \psi_{\frac{\pi}{2}} = \frac{i}{\sqrt{2}}(\phi_1 - \phi_2) \,,$$

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Francis Nier, IRMAR, Univ. Rennes 1
After joint works with Z. Ammari

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Wigner measures and Wick observables

Assume

$$\forall k \in \mathbb{N}\,, \exists C_k > 0\,, \forall \varepsilon \in \mathcal{E} \quad \mathrm{Tr}\,\left[\varrho^{\varepsilon}\mathbf{N}^k\right] \leq C_k$$

and $\mathcal{M}(\varrho^{\varepsilon}\,, \varepsilon \in \mathcal{E}) = \{\mu\}$.

Then

$$\forall b \in \mathcal{P}^{\infty}(\mathcal{Z})\,, \qquad \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathcal{E}}} \mathrm{Tr}\,\left[b^{Wick}\varrho^{\varepsilon}\right] = \int_{\mathcal{Z}} b(z)\; d\mu(z)\,.$$

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But it is not true for general $b \in \mathcal{P}(\mathcal{Z})$ without additional assumptions.
Example: $(e_n)_{n \in \mathbb{N}^*}$ Hilbert basis of $\mathcal{Z}$.
Take $\varrho^\varepsilon = |E(e_{[\varepsilon^{-1}]})\rangle\langle E(e_{[\varepsilon^{-1}]})|$. Then $\mathcal{M}(\varrho^\varepsilon, \varepsilon \in \mathcal{E}) = \{\delta_0\}$ and

$$0 = \int_{\mathcal{Z}} |z|^2 \delta_0(z) \neq 1 = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathcal{E}}} \mathrm{Tr}\, [\varrho^\varepsilon \mathbf{N}] = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathcal{E}}} \mathrm{Tr}\, \left[\varrho^\varepsilon (|z|^2)^{Wick}\right].$$

Condition (PI) and consequences

Assuming $\mathcal{M}(\varrho^\varepsilon, \varepsilon \in \mathcal{E}) = \{\mu\}$, the condition (PI) says

$$\forall k \in \mathbb{N}, \qquad \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathcal{E}}} \mathrm{Tr}\left[\varrho^\varepsilon \mathbf{N}^k\right] = \int_{\mathcal{Z}} |z|^{2k} \, d\mu(z).$$

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It is equivalent (Ammari-N. J.M.P.A 11) to

$$\forall b \in \mathcal{P}(\mathcal{Z}), \quad \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathcal{E}}} \mathrm{Tr}\left[\varrho^\varepsilon b^{Wick}\right] = \int_{\mathcal{Z}} b(z) \, d\mu(z) .$$

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Define the reduced density matrices (see BBGKY hierarchy) by

$$\forall \tilde{b} \in \mathcal{L}(\bigvee^p \mathcal{Z}), \quad \mathrm{Tr}\left[\gamma_p^\varepsilon \tilde{b}\right] = \frac{\mathrm{Tr}\left[\varrho^\varepsilon b^{Wick}\right]}{\mathrm{Tr}\left[\varrho^\varepsilon (|z|^{2p})^{Wick}\right]}$$

The condition (PI) implies (Ammari-N. J.M.P.A 11)

$$\lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathcal{E}}} \left\|\gamma_p^\varepsilon - \frac{\int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}|\, d\mu(z)}{\int_{\mathcal{Z}} |z|^{2p}\, d\mu(z)}\right\|_{\mathcal{L}^1(\bigvee^p \mathcal{Z})} = 0\,.$$

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Review of the framework

$\mathcal{Z} = L^2(X, dm; \mathbb{C})$ separable. (e.g. $(X, +, m) \subset$ meas. add. group)
$(A, D(A))$ self-adjoint on $\mathcal{Z}$, $\quad V = V(x) = V(-x)$.
$[a(x), a^*(y)] = \varepsilon\delta(x - y)$.

$$H^\varepsilon = d\Gamma(A) + \frac{1}{2}\int_{X\times X} V(x-y)a^*(x)a^*(y)a(x)a(y)\ dm(x)dm(y)$$

$$H^\varepsilon = \mathcal{E}(z)^{Wick}$$

$$\mathcal{E}(z) = \langle z\,, Az\rangle + \frac{1}{2}\int_{X\times X} V(x-y)|z(x)|^2|z(y)|^2\ dm(x)dm(y)\,.$$

$H^\varepsilon, D(H^\varepsilon)$ s.a. in $\Gamma(\mathcal{Z})$.

$$\varrho^\varepsilon(t) = e^{-i\frac{t}{\varepsilon}H^\varepsilon}\varrho^\varepsilon e^{i\frac{t}{\varepsilon}H^\varepsilon}\,.$$

$$\Phi(t) \text{ flow of } i\partial_t z = \partial_{\overline{z}}\mathcal{E}(z)\,.$$

Result for $V \in L^\infty$.

See (Ammari-N. J.Math.Phys. 09, JMPA 11)
Assume $V \in L^\infty(X, dm)$, $\mathcal{M}(\varrho^\varepsilon, \varepsilon \in (0, \varepsilon_0)) = \{\mu_0\}$ and

$$\forall n \in \mathbb{N}, \quad \lim_{\varepsilon \to 0} \mathrm{Tr}\,[\varrho^\varepsilon \mathbf{N}^n] = \int_{\mathcal{Z}} |z|^{2n}\, d\mu_0(z).$$

Then for all $t \in \mathbb{R}$, the following result hold

$$\mathcal{M}(\varrho^\varepsilon(t), \varepsilon \in (0, \varepsilon_0)) = \{\mu(t) = \Phi(t)_* \mu_0\},$$

$$\forall b \in \mathcal{P}(\mathcal{Z}), \quad \lim_{\varepsilon \to 0} \mathrm{Tr}\,\left[\varrho^\varepsilon(t) b^{Wick}\right] = \int_{\mathcal{Z}} b(z)\, d\mu_t(z),$$

$$\lim_{\varepsilon \to 0} \gamma_p^\varepsilon(t) = \frac{\int_{\mathcal{Z}} |z^{\otimes p}\rangle\langle z^{\otimes p}|\, d\mu_t(z)}{\int_{\mathcal{Z}} |z|^{2p}\, d\mu_0(z)} = \frac{\int_{\mathcal{Z}} |z_t^{\otimes p}\rangle\langle z_t^{\otimes p}|\, d\mu_0(z)}{\int_{\mathcal{Z}} |z|^{2p}\, d\mu_0(z)}.$$

Singular case ($\pm$ Coulombic case $d=3$)

See (Ammari-N. online Ann. Sc. Norm. di Pisa).
Take $X=\mathbb{R}^d$ and $A=-\Delta$ and assume $V(1-\Delta)^{-\frac{1}{2}}$ bounded and $(1-\Delta)^{-\frac{1}{2}}V(1-\Delta)^{-\frac{1}{2}}$ compact. Under the sole condition

$$\exists\delta>0\,,\ \forall\varepsilon\in(0,\varepsilon_0)\,,\ \mathrm{Tr}\ \left[(d\Gamma(1-\Delta))^{\delta}\varrho^{\varepsilon}\right]\leq C_{\delta}$$

and $\mathcal{M}(\varrho^{\varepsilon},\,\varepsilon\in(0,\varepsilon_0))=\{\mu_0\}$, the following results hold for all $t\in\mathbb{R}$:

- $\mathcal{M}(\varrho^{\varepsilon}(t),\varepsilon\in(0,\varepsilon_0))=\{\mu(t)\}$ where $\mu(t)$ is a Borel probability measure on $\mathcal{Z}_1=\mathcal{H}^1(\mathbb{R}^d;\mathbb{C})$ and $\mu(t)=\Phi(t)_*\mu_0$ (Φ well-defined on $\mathcal{Z}_1$).
- When $(\varrho^{\varepsilon})_{\varepsilon\in(0,\varepsilon_0)}$ satisfies the condition (PI) then $(\varrho^{\varepsilon}(t))_{\varepsilon\in(0,\varepsilon_0)}$ and

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$$\lim_{\varepsilon}\gamma_p^{\varepsilon}(t)=\frac{\int_{\mathcal{Z}}|z^{\otimes p}\rangle\langle z^{\otimes p}|\ d\mu_t(z)}{\int_{\mathcal{Z}}|z|^{2p}\ d\mu_0(z)}=\frac{\int_{\mathcal{Z}}|z_t^{\otimes p}\rangle\langle z_t^{\otimes p}|\ d\mu_0(z)}{\int_{\mathcal{Z}}|z|^{2p}\ d\mu_0(z)}\,.$$

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Take $X = \mathbb{R}^d$ and $A = -\Delta$ and assume $V(1-\Delta)^{-\frac{1}{2}}$ bounded and $(1-\Delta)^{-\frac{1}{2}}V(1-\Delta)^{-\frac{1}{2}}$ compact. Under the sole condition

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Evolution of mean field correlations

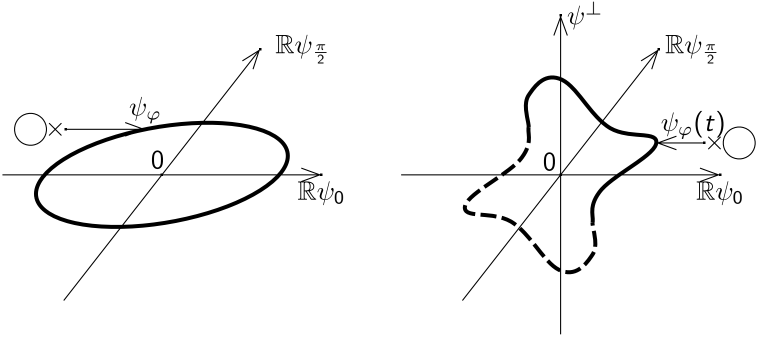

Fig.1: Evolution of the measure initially carried by a torus in $\mathbb{C}\psi_0 \oplus \mathbb{C}\psi_{\frac{\pi}{2}}$.

The complex gauge parameter $e^{i\theta}$ is represented by the small circle.