Entropy and C^0 stability of hypersurfaces

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Outline

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Isoperimetric inequality and Bonnesen’s inequality

Classical Isoperimetric inequality

The isoperimetric inequality in the plane states that if $\Omega \subset \mathbb{R}^2$ is a (reasonable) domain, then

$$4\pi A(\Omega) \leq L(\partial\Omega)^2.$$

with equality only when $\Omega = B_r(p)$ is a disk. Here $A(\Omega)$ is the area of Ω and $L(\partial\Omega)$ the length of its boundary.

This generalizes to higher dimensions as

$$c_n \mathrm{Vol}(\Omega) \leq \left(\mathcal{H}^n(\partial\Omega)\right)^{\frac{n+1}{n}}$$

where $\Omega \subset \mathbb{R}^{n+1}$, c_n is an appropriate dimensional constant and one has equality only on balls.

Bonnesen's Inequality

Given a sharp inequality it is natural to study its stability.

A classical example is Bonnesen's inequality:

If $\Omega \subset \mathbb{R}^2$ is a domain, then

$$\pi^2(R_{out} - R_{in})^2 \leq L(\partial\Omega)^2 - 4\pi A(\Omega).$$

Here,

- R_{out} is the radius of the smallest disk containing Ω;
- R_{in} is the radius of the largest disk contained in Ω.

When the RHS, the *isoperimetric defect*, is small then Ω is close, as a set, to a disk and this holds in a quantitative fashion.

Osserman surveyed a number of such results which he called "Bonnesen-style" inequalities.

What about higher dimensions?

It’s not hard to see that a direct analog of Bonnesen’s inequality can’t hold in higher dimensions.

Indeed, a ball with a long “spike” will have arbitrarily small isoperimetric defect while not being close to a ball as a set.

Instead, one can study other notions of stability.

E.G., work of Fuglede, Hall and Fusco-Maggi-Pratelli.

Colding-Minicozzi Entropy

Colding-Minicozzi Entropy

Inspired by Huisken's monotonicity formula for mean curvature flow, Colding-Minicozzi introduced a notion of entropy associated to a hypersurface $\Sigma \subset \mathbb{R}^{n+1}$.

One starts with the Gaussian surface area of $\Sigma \subset \mathbb{R}^{n+1}$:

$$F[\Sigma] = (4\pi)^{-n/2} \int_{\Sigma} e^{-\frac{|\mathbf{x}|^2}{4}} \, d\mathcal{H}^n.$$

The (Colding-Minicozzi) entropy of Σ is then:

$$\lambda[\Sigma] = \sup_{\mathbf{y} \in \mathbb{R}^{n+1}, \rho > 0} F[\rho\Sigma + \mathbf{y}].$$

That is, $\lambda[\Sigma]$ is the Gaussian surface area of all translations and dilations of Σ.

NB: This functional shares certain formal similarities with the Willmore energy.

Basic properties of CM entropy

We list here some basic properties of this functional:

- λ is invariant under rigid motions and dilations;
- $\lambda[\Sigma] \geq 1$ for any Σ with equality for a hyperplane;
- If $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is the unit sphere, then a computation of Stone yields:

$$\lambda[\mathbb{S}^1] \approx 1.52 > \lambda[\mathbb{S}^2] \approx 1.47 > \lambda[\mathbb{S}^3] > \cdots \downarrow \sqrt{2},$$

  I.E., the entropy of $\mathbb{S}^n$ is decreases as the dimension increases;
- If $\Sigma \times \mathbb{R}^k \subset \mathbb{R}^{n+k+1}$ is the cylinder over $\Sigma \subset \mathbb{R}^{n+1}$, then $\lambda[\Sigma] = \lambda[\Sigma \times \mathbb{R}^k]$.

Some not so basic properties

Some more sophisticated properties:

- $\lambda[\Sigma_t]$ is non-increasing along a mean curvature flow $\{\Sigma_t\}$;
- If Σ is a self-shrinker – i.e., $\mathbf{H}_\Sigma = -\frac{\mathbf{x}^\perp}{2}$, then $\lambda[\Sigma] = F[\Sigma]$;
- If Σ is a non-flat self-shrinker, then $\lambda[\Sigma] > 1$. In fact, there is a dimensional constant $\epsilon_n > 1$ so $\lambda[\Sigma] \geq \epsilon_n$.
- If Σ is closed (compact and w/o boundary), then $\lambda[\Sigma] \geq \epsilon_n$.

A natural question is to determine the sharp choice of ϵ_n.

As entropy is supposed to measure complexity, it is natural to guess the minimum is achieved on round spheres.

NB: Hyperplanes are now known to be the only hypersurfaces with entropy one (due to L. Chen).

Conjecture of Colding-Ilmanen-Minicozzi-White

Colding-Ilmanen-Minicozzi-White conjectured:

Conjecture (CIMW)

Suppose $\Sigma \subset \mathbb{R}^{n+1}$ *is a closed hypersurface, then*

$$\lambda[\Sigma] \geq \lambda[\mathbb{S}^n]$$

with equality only if $\Sigma = \rho\mathbb{S}^n + \mathbf{y}$.

- Properties of the mean curvature flow imply the conjecture when $n = 2$ or Σ convex (or even mean convex);
- CIMW showed it for self-shrinkers;
- B.-L. Wang showed it in full generality when $2 \leq n \leq 6$;
- Zhu extend this to $n \geq 7$.

Entropy and C^0 estimates

Bonnesen-like inequality

There are several natural ways to study the stability of the CIMW conjecture.

One of these is inspired by the classical Bonnesen inequality:

Theorem (B.-L. Wang)

Given $\epsilon > 0$, there exists a $\delta > 0$ so that if $\Sigma \subset \mathbb{R}^3$ is closed and $\lambda[\Sigma] \leq \lambda[\mathbb{S}^2] + \delta$, then

$$1 \leq \frac{R_{out}}{R_{in}} < 1 + \epsilon.$$

Where R_{out} and R_{in} are as in the Bonnesen inequality.

NB: In our paper we stated this result in terms of Hausdorff distance to an appropriate $\mathbb{S}^2$.

Idea of proof

The idea of the proof is that the CM entropy looks at all scales and so if there is a small “spike,” then it contributes a non-trivial amount of entropy as the spike looks cylindrical.

To make this precise we use the fact that we were able to completely classify a low-entropy self-shrinkers in $\mathbb{R}^3$. They must be either:

$$2\mathbb{S}^2, \sqrt{2}\mathbb{S}^1 \times \mathbb{R}, \text{ or } \mathbb{R}^2,$$

the shrinking sphere, the shrinking round cylinder or the static plane (which is not a singularity model).

As $\lambda[\mathbb{S}^1 \times \mathbb{R}] > \lambda[\mathbb{S}^2]$, a low entropy closed surface must flow under mean curvature smoothly until it disappears in a round point. Moreover, the speed is bounded in a scale invariant way that controls the change of in and out radius.

Quantitative Bonnesen-like inequality

In fact, using a more careful analysis we were able to show the following quantitative version of the stability result:

Theorem (B.-L. Wang)

There is a constant $K \geq 1$, so that if $\Sigma \subset \mathbb{R}^3$ is closed, then there is a $\rho >$ and $\mathbf{x} \in \mathbb{R}^3$ so:

$$\rho^{-1}\mathrm{dist}_H(\Sigma, \rho\mathbb{S}^2 + \mathbf{x}) \leq K(\lambda[\Sigma] - \lambda[\mathbb{S}^2])^{1/8}.$$

Here dist_H is the Hausdorff distance between closed sets.

REM: The exponent $1/8$ is likely not sharp.

Extension to higher dimension

The proof given by L. Wang and myself depended on having a complete classification of self-shrinkers and this is not known in higher dimensions.

Nevertheless, my student S. Wang was able to extend the stability result to all dimensions:

Theorem (S. Wang)

Given $\epsilon > 0$, there is a $\delta > 0$ so that if $\Sigma \subset \mathbb{R}^{n+1}$ is closed and has $\lambda[\Sigma] \leq \lambda[\mathbb{S}^n] + \delta$, then

$$1 \leq \frac{R_{out}}{R_{in}} < 1 + \epsilon.$$

Where here R_{out} is the radius of the smallest ball containing Σ and R_{in} is the radius of the largest ball inside of Σ.

Thinness estimate

We end by observing an (unpublished) uniform lower bound on how thin a low entropy closed surface can be relative to it's diameter. The proof of this follows same lines as the stability result.

For $\Sigma \subset \mathbb{R}^{n+1}$, let the *thinness* of Σ, $Th(\Sigma)$, be the width of the smallest slab containing Σ.

Theorem (B.-L. Wang)

Let $\Sigma \subset \mathbb{R}^3$ be closed. For every $\epsilon > 0$, there is a $C(\epsilon) > \frac{1}{2}$ so that if $\lambda[\Sigma] \leq \lambda[\mathbb{S}^1 \times \mathbb{R}] - \epsilon$, then

$$\frac{1}{2} \leq \frac{R_{out}}{Th(\Sigma)} \leq C(\epsilon).$$

Higher dimensions

The analogous result in higher dimensions is harder as S. Wang's most general argument only works for hypersurfaces whose entropy is very close to that of the round sphere.

We do have the following:

Theorem (S. Wang, B.-S. Wang, B.-L. Wang)

Let $\Sigma \subset \mathbb{R}^4$ be closed. For every $\epsilon > 0$, there is a $C(\epsilon) > \frac{1}{2}$ so that if $\lambda[\Sigma] \leq \lambda[\mathbb{S}^2 \times \mathbb{R}] - \epsilon$, then

$$\frac{1}{2} \leq \frac{R_{out}}{Th(\Sigma)} \leq C(\epsilon).$$

This is likely true in all dimensions.