The Brownian transport map

Yair Shenfeld

MIT

Joint work with Dan Mikulincer

Poincaré inequalities

Poincaré inequalities

A measure μ satisfies a Poincaré inequality with constant C_μ if, for any test function f,

$$\mathrm{Var}_\mu(f) \leq C_\mu \mathbb{E}_\mu \left[|\nabla f|^2 \right].$$

Poincaré inequalities

A measure μ satisfies a Poincaré inequality with constant C_μ if, for any test function f,

$$\mathrm{Var}_\mu(f) \leq C_\mu \mathbb{E}_\mu \left[|\nabla f|^2 \right].$$

Example: When $\mu = \gamma_d =$ standard Gaussian measure on $\mathbb{R}^d$ it satisfies a Poincaré inequality with the optimal constant $C_{\gamma_d} = 1$.

Poincaré inequalities for other measures

Proving Poincaré inequality for γ_d is not too difficult because the Gaussian is special (e.g., explicit formulas).

Poincaré inequalities for other measures

Proving Poincaré inequality for γ_d is not too difficult because the Gaussian is special (e.g., explicit formulas).

How to show Poincaré inequalities for $\mu \neq \gamma_d$?

Poincaré inequalities for other measures

Proving Poincaré inequality for γ_d is not too difficult because the Gaussian is special (e.g., explicit formulas).

How to show Poincaré inequalities for $\mu \neq \gamma_d$?

Expect Poincaré inequality when:

Poincaré inequalities for other measures

Proving Poincaré inequality for γ_d is not too difficult because the Gaussian is special (e.g., explicit formulas).

How to show Poincaré inequalities for $\mu \neq \gamma_d$?

Expect Poincaré inequality when:

- μ is strongly log-concave.

Poincaré inequalities for other measures

Proving Poincaré inequality for γ_d is not too difficult because the Gaussian is special (e.g., explicit formulas).

How to show Poincaré inequalities for $\mu \neq \gamma_d$?

Expect Poincaré inequality when:

- μ is strongly log-concave.
- μ is log-concave and has bounded support.

Poincaré inequalities for other measures

Proving Poincaré inequality for γ_d is not too difficult because the Gaussian is special (e.g., explicit formulas).

How to show Poincaré inequalities for $\mu \neq \gamma_d$?

Expect Poincaré inequality when:

- μ is strongly log-concave.
- μ is log-concave and has bounded support.
- μ is a mixture of Gaussians.

Poincaré inequalities for other measures

Proving Poincaré inequality for γ_d is not too difficult because the Gaussian is special (e.g., explicit formulas).

How to show Poincaré inequalities for $\mu \neq \gamma_d$?

Expect Poincaré inequality when:

- μ is strongly log-concave.
- μ is log-concave and has bounded support.
- μ is a mixture of Gaussians.
- μ is a bounded perturbation of γ_d.

Poincaré inequalities for other measures

Proving Poincaré inequality for γ_d is not too difficult because the Gaussian is special (e.g., explicit formulas).

How to show Poincaré inequalities for $\mu \neq \gamma_d$?

Expect Poincaré inequality when:

- μ is strongly log-concave.
- μ is log-concave and has bounded support.
- μ is a mixture of Gaussians.
- μ is a bounded perturbation of γ_d.
- ...

Transportation of Poincaré inequalities

Transportation of Poincaré inequalities

Suppose there exist an L-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ which transports γ_d to μ.

Transportation of Poincaré inequalities

Suppose there exist an L-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ which transports γ_d to μ.

Then μ satisfies a Poincaré inequality with constant L^2:

Transportation of Poincaré inequalities

Suppose there exist an L-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ which transports γ_d to μ.

Then μ satisfies a Poincaré inequality with constant L^2:

$$\begin{aligned} &\mathrm{Var}_\mu(f) = \mathrm{Var}_{\gamma_d}(f \circ T) \leq \mathbb{E}_{\gamma_d}\left[|\nabla(f \circ T)|^2\right] \\ &\leq \mathbb{E}_{\gamma_d}\left[|DT|^2|\nabla f(T)|^2\right] \leq L^2\mathbb{E}_{\gamma_d}\left[|\nabla f(T)|^2\right] = L^2\mathbb{E}_\mu\left[|\nabla f|^2\right]. \end{aligned}$$

Transportation of functional inequalities

The existence of an L-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ which transports γ_d to μ implies numerous other functional inequalities for μ:

Transportation of functional inequalities

The existence of an L-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ which transports γ_d to μ implies numerous other functional inequalities for μ:

- Isoperimetric inequalities.

Transportation of functional inequalities

The existence of an L-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ which transports γ_d to μ implies numerous other functional inequalities for μ:

- Isoperimetric inequalities.
- Dimension-free Φ-Sobolev inequalities (generalize both Poincaré and log-Sobolev inequalities).

Transportation of functional inequalities

The existence of an L-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ which transports γ_d to μ implies numerous other functional inequalities for μ:

- Isoperimetric inequalities.
- Dimension-free Φ-Sobolev inequalities (generalize both Poincaré and log-Sobolev inequalities).
- Comparisons of high-order eigenvalues of weighted Laplacian.

Transportation of functional inequalities

The existence of an L-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ which transports γ_d to μ implies numerous other functional inequalities for μ:

- Isoperimetric inequalities.
- Dimension-free Φ-Sobolev inequalities (generalize both Poincaré and log-Sobolev inequalities).
- Comparisons of high-order eigenvalues of weighted Laplacian.
- q- Poincaré inequalities.

Transportation of functional inequalities

The existence of an L-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ which transports γ_d to μ implies numerous other functional inequalities for μ:

- Isoperimetric inequalities.
- Dimension-free Φ-Sobolev inequalities (generalize both Poincaré and log-Sobolev inequalities).
- Comparisons of high-order eigenvalues of weighted Laplacian.
- q- Poincaré inequalities.
- ...

Optimal transport

Optimal transport

For a transport map $T : \mathbb{R}^d \to \mathbb{R}^d$ to be Lipschitz we expect it to have some special structure.

Optimal transport

For a transport map $T : \mathbb{R}^d \to \mathbb{R}^d$ to be Lipschitz we expect it to have some special structure.

The map T_{ot} is an *optimal transport map* if it a minimizer of

$$\min_{\{S \text{ transport map of } \gamma_d \text{ to } \mu\}} \int_{\mathbb{R}^d} |x - S(x)|^2 d\gamma_d.$$

Lipschitz properties of optimal transport

Lipschitz properties of optimal transport

- Suppose μ is κ-log-concave with $\kappa > 0$.

Lipschitz properties of optimal transport

- Suppose μ is κ-log-concave with $\kappa > 0$. Then $C_\mu = \frac{1}{\kappa}$ so we expect T_{ot} to be $\frac{1}{\sqrt{\kappa}}$-Lipschitz.

Lipschitz properties of optimal transport

- Suppose μ is κ-log-concave with $\kappa > 0$. Then $C_\mu = \frac{1}{\kappa}$ so we expect T_{ot} to be $\frac{1}{\sqrt{\kappa}}$-Lipschitz. True by Caffarelli's contraction theorem.

Lipschitz properties of optimal transport

- Suppose μ is κ-log-concave with $\kappa > 0$.Then $C_\mu = \frac{1}{\kappa}$ so we expect T_{ot} to be $\frac{1}{\sqrt{\kappa}}$-Lipschitz. True by Caffarelli's contraction theorem.
- Suppose μ is log-concave ($\kappa = 0$) with $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$.

Lipschitz properties of optimal transport

- Suppose μ is κ-log-concave with $\kappa > 0$. Then $C_\mu = \frac{1}{\kappa}$ so we expect T_{ot} to be $\frac{1}{\sqrt{\kappa}}$-Lipschitz. True by Caffarelli's contraction theorem.
- Suppose μ is log-concave ($\kappa = 0$) with $\operatorname{diam}(\operatorname{supp}(\mu)) \leq R$. Then $C_\mu = O(R^2)$ so we expect T_{ot} to be $O(R)$-Lipschitz.

Lipschitz properties of optimal transport

- Suppose μ is κ-log-concave with $\kappa > 0$. Then $C_\mu = \frac{1}{\kappa}$ so we expect T_{ot} to be $\frac{1}{\sqrt{\kappa}}$-Lipschitz. True by Caffarelli's contraction theorem.
- Suppose μ is log-concave ($\kappa = 0$) with $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$. Then $C_\mu = O(R^2)$ so we expect T_{ot} to be $O(R)$-Lipschitz. Open problem.

Lipschitz properties of optimal transport

- Suppose μ is κ-log-concave with $\kappa > 0$. Then $C_\mu = \frac{1}{\kappa}$ so we expect T_{ot} to be $\frac{1}{\sqrt{\kappa}}$-Lipschitz. True by Caffarelli's contraction theorem.
- Suppose μ is log-concave ($\kappa = 0$) with $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$. Then $C_\mu = O(R^2)$ so we expect T_{ot} to be $O(R)$-Lipschitz. Open problem.
- Suppose μ is a mixture of Gaussians such that $\mathrm{diam}(\mathrm{supp}(\text{mixing measure})) \leq R$.

Lipschitz properties of optimal transport

- Suppose μ is κ-log-concave with $\kappa > 0$.Then $C_\mu = \frac{1}{\kappa}$ so we expect T_{ot} to be $\frac{1}{\sqrt{\kappa}}$-Lipschitz. True by Caffarelli's contraction theorem.
- Suppose μ is log-concave ($\kappa = 0$) with $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$. Then $C_\mu = O(R^2)$ so we expect T_{ot} to be $O(R)$-Lipschitz. Open problem.
- Suppose μ is a mixture of Gaussians such that $\mathrm{diam}(\mathrm{supp}(\text{mixing measure})) \leq R$. Then $C_\mu = O(e^{R^2})$ (Bardet, Gozlan, Malrieu and Zitt) so we expect T_{ot} to be $O(e^{\frac{R^2}{2}})$-Lipschitz.

Lipschitz properties of optimal transport

- Suppose μ is κ-log-concave with $\kappa > 0$. Then $C_\mu = \frac{1}{\kappa}$ so we expect T_{ot} to be $\frac{1}{\sqrt{\kappa}}$-Lipschitz. True by Caffarelli's contraction theorem.
- Suppose μ is log-concave ($\kappa = 0$) with $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$. Then $C_\mu = O(R^2)$ so we expect T_{ot} to be $O(R)$-Lipschitz. Open problem.
- Suppose μ is a mixture of Gaussians such that $\mathrm{diam}(\mathrm{supp}(\text{mixing measure})) \leq R$. Then $C_\mu = O(e^{R^2})$ (Bardet, Gozlan, Malrieu and Zitt) so we expect T_{ot} to be $O(e^{\frac{R^2}{2}})$-Lipschitz. Open problem.

Transportation from Wiener space

Transportation from Wiener space

Let $\Omega = C([0,1]; \mathbb{R}^d)$ be the Wiener space and γ_∞ be the Wiener measure, and let μ be a measure on $\mathbb{R}^d$.

Transportation from Wiener space

Let $\Omega = C([0,1];\mathbb{R}^d)$ be the Wiener space and γ_∞ be the Wiener measure, and let μ be a measure on $\mathbb{R}^d$. Let $T = (T_t)_{t\in[0,1]} : \Omega \to \Omega$ be a map such that $T_1 \sim \mu$ and T is L-Lipschitz in the sense that $|DT_1| \leq L$ (Malliavin derivative).

Transportation from Wiener space

Let $\Omega = C([0,1];\mathbb{R}^d)$ be the Wiener space and γ_∞ be the Wiener measure, and let μ be a measure on $\mathbb{R}^d$. Let $T = (T_t)_{t\in[0,1]} : \Omega \to \Omega$ be a map such that $T_1 \sim \mu$ and T is L-Lipschitz in the sense that $|DT_1| \leq L$ (Malliavin derivative).

Poincaré inequality for γ_∞: For any test function $f : \Omega \to \mathbb{R}$,

$$\mathrm{Var}_{\gamma_\infty}(f) \leq \mathbb{E}_{\gamma_\infty}[|Df|^2].$$

Transportation from Wiener space

Let $\Omega = C([0,1];\mathbb{R}^d)$ be the Wiener space and γ_∞ be the Wiener measure, and let μ be a measure on $\mathbb{R}^d$. Let $T = (T_t)_{t\in[0,1]} : \Omega \to \Omega$ be a map such that $T_1 \sim \mu$ and T is L-Lipschitz in the sense that $|DT_1| \le L$ (Malliavin derivative).

Poincaré inequality for γ_∞: For any test function $f : \Omega \to \mathbb{R}$,

$$\mathrm{Var}_{\gamma_\infty}(f) \le \mathbb{E}_{\gamma_\infty}[|Df|^2].$$

Poincaré inequality for μ: Let $f : \mathbb{R}^d \to \mathbb{R}$ be a test function,

$$\mathrm{Var}_\mu(f) = \mathrm{Var}_{\gamma_\infty}(f \circ T_1) \le \mathbb{E}_{\gamma_\infty}\left[|D(f \circ T_1)|^2\right]$$
$$\le \mathbb{E}_{\gamma_\infty}\left[|DT_1|^2|\nabla f(T_1)|^2\right] \le L^2\mathbb{E}_{\gamma_\infty}\left[|\nabla f(T_1)|^2\right] = L^2\mathbb{E}_\mu\left[|\nabla f|^2\right].$$

Transport maps on Wiener space

Transport maps on Wiener space

Optimal transport maps on Wiener space exist (Feyel and Üstünel) but in our settings they reduce to optimal transport on $\mathbb{R}^d$ so no gain.

Transport maps on Wiener space

Optimal transport maps on Wiener space exist (Feyel and Üstünel) but in our settings they reduce to optimal transport on $\mathbb{R}^d$ so no gain.

Causal optimal transport transport measures in an optimal way while respecting causality (filtration).

The Brownian transport map

Given a measure μ on $\mathbb{R}^d$ consider the problem

$$\min \mathbb{E}_{\gamma_\infty}\left[\int_0^1 |u_t|^2 dt\right] \text{ over all **adapted** } (u_t)_{\in[0,1]} \text{ s.t. } \omega_1 + \int_0^1 u_t dt \sim \mu,$$

and let u^* be the optimizer.

The Brownian transport map

Given a measure μ on $\mathbb{R}^d$ consider the problem

$$\min \mathbb{E}_{\gamma_\infty}\left[\int_0^1 |u_t|^2 dt\right] \text{ over all } \textbf{adapted } (u_t)_{\in[0,1]} \text{ s.t. } \omega_1 + \int_0^1 u_t dt \sim \mu,$$

and let u^* be the optimizer.

The optimal map $X : \Omega \to \Omega$, given by $X(\omega)_t = \omega_t + \int_0^t u_s^* ds$, is the **Brownian transport map** between γ_∞ and μ.

The Brownian transport map

Given a measure μ on $\mathbb{R}^d$ consider the problem

$$\min \mathbb{E}_{\gamma_\infty}\left[\int_0^1 |u_t|^2 dt\right] \text{ over all } \textbf{adapted } (u_t)_{\in[0,1]} \text{ s.t. } \omega_1 + \int_0^1 u_t dt \sim \mu,$$

and let u^* be the optimizer.

The optimal map $X : \Omega \to \Omega$, given by $X(\omega)_t = \omega_t + \int_0^t u_s^* ds$, is the **Brownian transport map** between γ_∞ and μ.

Optimal transport solves the problem

$$\min \mathbb{E}_{\gamma_\infty}\left[\int_0^1 |u_t|^2 dt\right] \text{ over all } (u_t)_{\in[0,1]} \text{ s.t. } \omega_1 + \int_0^1 u_t dt \sim \mu.$$

Lipschitz properties of the Brownian transport map

Theorem (Mikulincer, S.)

The Brownian transport map $X : \Omega \to \Omega$*, with* $X_1 \sim \mu$*, satisfies:*

1. *If* μ *is* κ*-log-concave* ($\kappa > 0$),

$$|DX_1| \leq \frac{1}{\sqrt{\kappa}}.$$

2. *If* μ *is log-concave with* $\mathrm{diam}(\mathrm{supp}(\mu)) \leq R$,

$$|DX_1| \leq O(R).$$

3. *If* $\mu = \gamma_d \star \nu$ *and* $\mathrm{diam}(\mathrm{supp}(\nu)) \leq R$,

$$|DX_1| \leq O(e^{R^2}).$$

The Kannan–Lovász–Simonovits conjecture

The Kannan–Lovász–Simonovits conjecture

All isotropic log-concave ($\kappa = 0$) measures μ satisfy Poincaré inequality with dimension-free constant C_{kls}: for any test function f,

$$\mathrm{Var}_{\mu}(f) \leq C_{\mathsf{kls}} \mathbb{E}_{\mu}\left[|\nabla f|^2\right].$$

The Kannan–Lovász–Simonovits conjecture

All isotropic log-concave ($\kappa = 0$) measures μ satisfy Poincaré inequality with dimension-free constant C_{kls}: for any test function f,

$$\mathrm{Var}_{\mu}(f) \leq C_{\mathsf{kls}} \mathbb{E}_{\mu} \left[|\nabla f|^2 \right].$$

Transportation approach: Given isotropic log-concave μ find a transport map T pushing γ_d to μ which is C_{kls}-Lipschitz.

The Kannan–Lovász–Simonovits conjecture

All isotropic log-concave ($\kappa = 0$) measures μ satisfy Poincaré inequality with dimension-free constant C_{kls}: for any test function f,

$$\mathrm{Var}_{\mu}(f) \leq C_{\mathsf{kls}} \mathbb{E}_{\mu} \left[|\nabla f|^2 \right].$$

Transportation approach: Given isotropic log-concave μ find a transport map T pushing γ_d to μ which is C_{kls}-Lipschitz. Not possible because such a map would imply properties which are false for some isotropic log-concave measures.

The Kannan–Lovász–Simonovits conjecture

All isotropic log-concave ($\kappa = 0$) measures μ satisfy Poincaré inequality with dimension-free constant C_{kls}: for any test function f,

$$\mathrm{Var}_{\mu}(f) \leq C_{\mathsf{kls}} \mathbb{E}_{\mu} \left[|\nabla f|^2 \right].$$

Transportation approach: Given isotropic log-concave μ find a transport map T pushing γ_d to μ which is C_{kls}-Lipschitz. Not possible because such a map would imply properties which are false for some isotropic log-concave measures. However, E. Milman showed that KLS would follow as soon as T is C_{kls}-Lipschitz on average: $\mathbb{E}_{\gamma_d}[|DT|^2] \leq C_{\mathsf{kls}}$.

KLS and Brownian transport map

KLS and Brownian transport map

Current best bound on KLS (Chen): $C_{\mathsf{kls}} \leq d^{o(1)}$.

KLS and Brownian transport map

Current best bound on KLS (Chen): $C_{\mathrm{kls}} \leq d^{o(1)}$.

Theorem (Mikulincer, S.)

Let μ be an isotropic log-concave measure and let $X_1 : \Omega \to \mathbb{R}^d$ be the Brownian transport map between γ_∞ to μ. Then,

$$\mathbb{E}_{\gamma_\infty}\left[|DX_1|^2\right] \leq d^{o(1)}.$$

The Föllmer process

Let μ be a measure on $\mathbb{R}^d$ and let (P_t) be the heat semigroup. The solution to the stochastic differential equation

$$dX_t = \nabla \log P_{1-t}\left(\frac{d\mu}{d\gamma_d}\right)(X_t)dt + d\omega_t, \quad X_0 = 0,$$

is the **Brownian transport map**.

The Föllmer process

Let μ be a measure on $\mathbb{R}^d$ and let (P_t) be the heat semigroup. The solution to the stochastic differential equation

$$dX_t = \nabla \log P_{1-t}\left(\frac{d\mu}{d\gamma_d}\right)(X_t)dt + d\omega_t, \quad X_0 = 0,$$

is the **Brownian transport map**.

History: Motivation for this *entropy-minimizing* process goes back to Schrödinger (30's) and was formulated by Föllmer (80').

The Föllmer process

Let μ be a measure on $\mathbb{R}^d$ and let (P_t) be the heat semigroup. The solution to the stochastic differential equation

$$dX_t = \nabla \log P_{1-t}\left(\frac{d\mu}{d\gamma_d}\right)(X_t)dt + d\omega_t, \quad X_0 = 0,$$

is the **Brownian transport map**.

History: Motivation for this *entropy-minimizing* process goes back to Schrödinger (30's) and was formulated by Föllmer (80'). Lassalle (2013) showed that process solves *causal optimal transport*.

The Föllmer process

Let μ be a measure on $\mathbb{R}^d$ and let (P_t) be the heat semigroup. The solution to the stochastic differential equation

$$dX_t = \nabla \log P_{1-t}\left(\frac{d\mu}{d\gamma_d}\right)(X_t)dt + d\omega_t, \quad X_0 = 0,$$

is the **Brownian transport map**.

History: Motivation for this *entropy-minimizing* process goes back to Schrödinger (30's) and was formulated by Föllmer (80'). Lassalle (2013) showed that process solves *causal optimal transport*. Lehec (2010) showed how to use process in context of functional inequalities.

Final remarks

- We in fact showed stronger Lipschitz properties for the Brownian transport map taking into account semi-log-concavity.

Final remarks

- We in fact showed stronger Lipschitz properties for the Brownian transport map taking into account semi-log-concavity.
- In the log-concave case the Lipschitz-on-average property of the Brownian transport map has applications to Stein kernels and central limit theorems.

Final remarks

- We in fact showed stronger Lipschitz properties for the Brownian transport map taking into account semi-log-concavity.
- In the log-concave case the Lipschitz-on-average property of the Brownian transport map has applications to Stein kernels and central limit theorems.
- On Wiener space there are stronger Lipschitz notions for which we showed that in some cases Brownian transport map cannot be Lipschitz while optimal transport can.

Final remarks

- We in fact showed stronger Lipschitz properties for the Brownian transport map taking into account semi-log-concavity.
- In the log-concave case the Lipschitz-on-average property of the Brownian transport map has applications to Stein kernels and central limit theorems.
- On Wiener space there are stronger Lipschitz notions for which we showed that in some cases Brownian transport map cannot be Lipschitz while optimal transport can.
- Many remaining questions...

Final remarks

- We in fact showed stronger Lipschitz properties for the Brownian transport map taking into account semi-log-concavity.
- In the log-concave case the Lipschitz-on-average property of the Brownian transport map has applications to Stein kernels and central limit theorems.
- On Wiener space there are stronger Lipschitz notions for which we showed that in some cases Brownian transport map cannot be Lipschitz while optimal transport can.
- Many remaining questions...
- Thank you.