Small scale formations in the incompressible porous media equation

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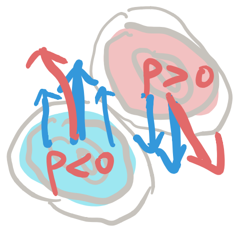
Incompressible Porous Media (IPM) equation

- $\rho(x,t)$: density of incompressible fluid in porous media.
- $\boldsymbol{u}(x,t)$: velocity field of fluid.

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0 \\ \nabla \cdot \boldsymbol{u} = 0 \end{cases} \quad \text{in } \mathbb{R}^2 \times [0,T) \ \text{ or } \ \mathbb{T}^2 \times [0,T).$$

- Darcy's law for flow in porous medium:



pressure

viscosity

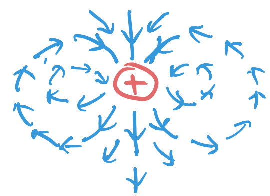
permeability

$$\frac{\mu}{\kappa}\boldsymbol{u} = -\nabla p - \begin{pmatrix} 0 \\ g\rho \end{pmatrix}.$$

acceleration due to gravity

- In $\mathbb{R}^2$ or $\mathbb{T}^2$, setting $\mu = \kappa = g = 1$, the Biot-Savart law becomes

$$\boldsymbol{u} = \partial_{x_1} \nabla^{\perp} (-\Delta)^{-1} \rho.$$



Comparison with other active scalar equations

In all 3 equations below, the scalar $\rho(x,t)$ is transported by an incompressible $\boldsymbol{u}(x,t)$:

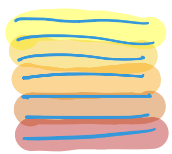
$$\partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = 0.$$

	2D Euler	SQG	2D IPM
Biot-Savart law	$\nabla^\perp(-\Delta)^{-1}\rho$	$\nabla^\perp(-\Delta)^{-1/2}\rho$	$\partial_{x_1}\nabla^\perp(-\Delta)^{-1}\rho$
Fourier symbol	$i\boldsymbol{k}^\perp\lvert\boldsymbol{k}\rvert^{-2}$	$i\boldsymbol{k}^\perp\lvert\boldsymbol{k}\rvert^{-1}$	$-k_1\boldsymbol{k}^\perp\lvert\boldsymbol{k}\rvert^{-2}$
Local well-posed?	Yes (in H^s)	Yes (in H^s)	Yes (in H^s)
Global well-posed?	Yes	Unknown	Unknown

On well-posedness of IPM

- Córdoba–Gancedo–Orive '07: Local well-posedness in H^s, and various blow-up criteria (in terms of the geometry structure of the level sets and $\int_0^T \|R\rho\|_{L^\infty} dt$).
- Friedlander–Gancedo–Sun–Vicol '11: The more singular equation $\boldsymbol{u} = \partial_{x_1} \nabla^\perp (-\Delta)^{-1+\beta} \rho$ is ill-posed for $0 < \beta \leq 1$ in H^s, but patch solution is locally well-posed for $0 < \beta < 1/2$.

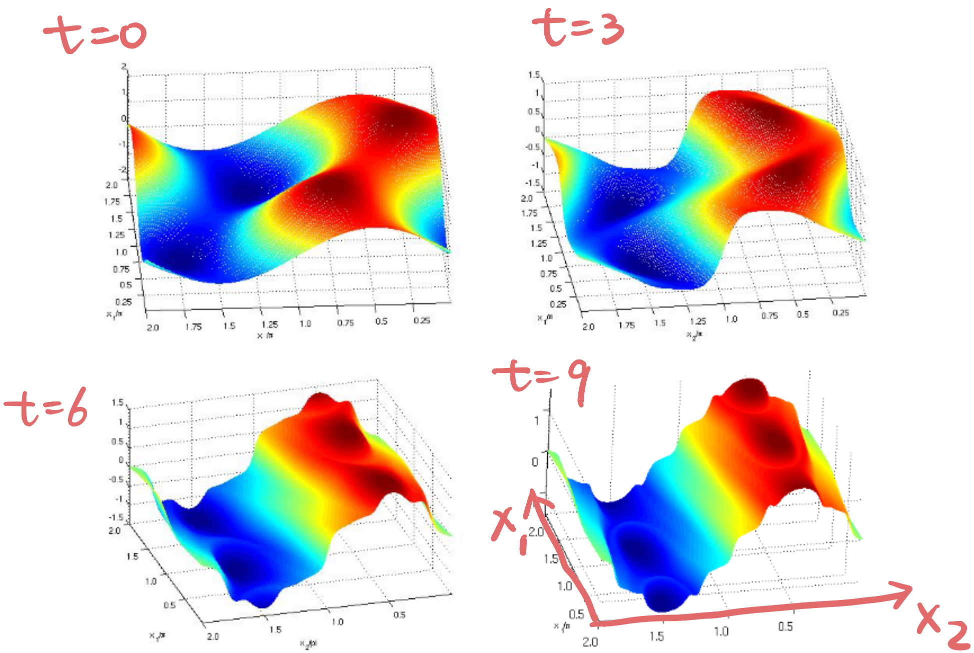
$\mathbb{R}^2$



- Elgindi '17: The stratified steady state $\rho(x_1, x_2) = -x_2$ is asymptotically stable in H^s for large s.
- Castro–Córdoba–Lear '19: In a bounded strip $\mathbb{T} \times [-l, l]$, smooth initial data near the stratified solution $-x_2$ remain globally regular.

Numerics on IPM: small scale formation?

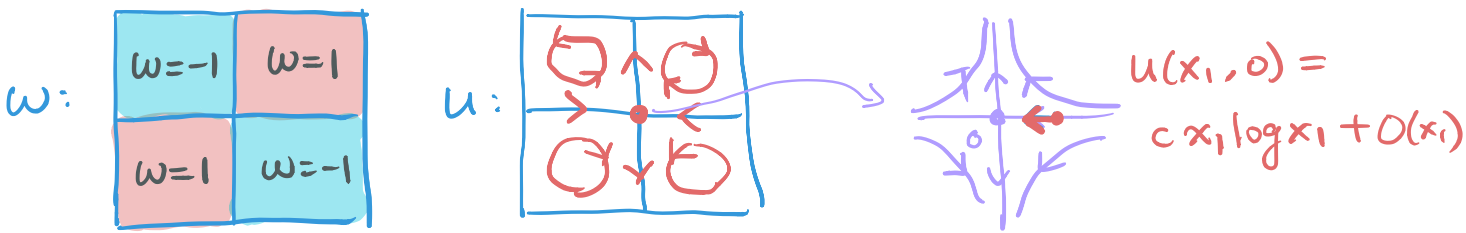
- Numerical simulations by Córdoba–Gancedo–Orive '07 suggest that $\|\nabla\rho\|_{L^\infty}$ is growing as $t \to \infty$, although no evidence for finite-time blow-up.



- Goal: Assuming a global-in-time solution ρ in $\mathbb{T}^2 \times [0,\infty)$, want to rigorously prove the growth of $\nabla\rho$ as $t \to \infty$.

Small scale formation of 2D Euler and SQG

- The following results shows that growth of $\|\nabla\omega\|_{L^\infty}$ can indeed happen for 2D Euler:
  - Yudovich '74: some infinite growth of $\|\nabla\omega\|_{L^\infty}$.
  - Nadirashvili '91: linear growth of $\|\nabla\omega\|_{L^\infty}$ in an annulus.
  - Bahouri-Chemin '94: steady state with "hyperbolic flow" in $\mathbb{T}^2$:



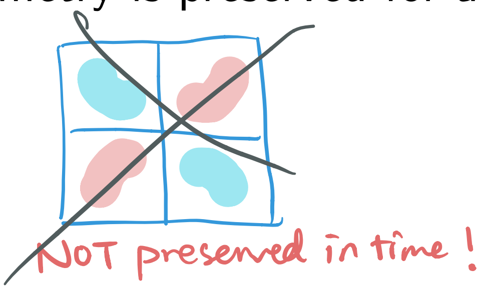
  - Denisov '09, '15: superlinear growth of $\|\nabla\omega\|_{L^\infty}$ (for all t), and exponential growth (for arbitrarily long time) in $\mathbb{T}^2$.
  - Kiselev-Šverák '14: double-exponential growth in a disk.
  - Zlatoš '15: exponential growth in $\mathbb{T}^2$.

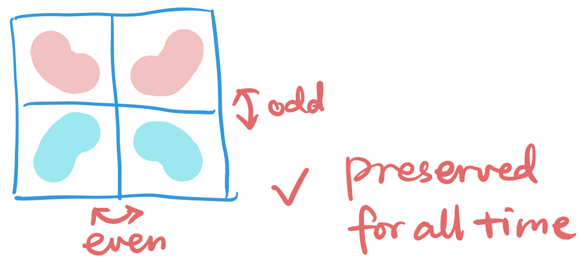
- For SQG equation, Kiselev–He '19 proved exponential growth of $\|\nabla\rho\|_{L^\infty}$, assuming the solution remain regular for all times.

Difficulties with extending to IPM

The hyperbolic flow scenario could not be adapted to IPM for the following reason:

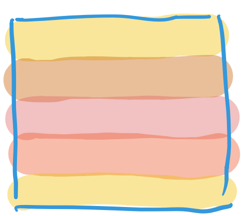
- For IPM on $\mathbb{T}^2$, there is no odd-odd symmetry; instead, odd-even symmetry is preserved for all time.





- There are no nontrivial steady states – all steady states of IPM are stratified (i.e. constant in x_1), where $\boldsymbol{u} \equiv 0$.

Only steady states:



$\Longrightarrow u \equiv 0$

$\partial_{x_1} \nabla^{\perp} (-\Delta)^{-1} \rho$

Small scale formation of IPM

We take a completely different approach and prove the following result:

Theorem (Kiselev–Y., forthcoming)

There exists smooth initial data $\rho_0 \in C^\infty(\mathbb{T}^2)$ leading to a solution $\rho(x,t)$ such that, provided it remains smooth for all time, satisfies

$$\int_0^\infty \|\partial_{x_1}\rho(\cdot,t)\|_{\dot H^\beta}^{-\frac{2}{2\beta+1}}\,dt \le C(\beta,\rho_0) < \infty \quad \text{for all } \beta > -\frac{1}{2}.$$

In particular, it implies that for all $\beta > -\frac{1}{2}$,

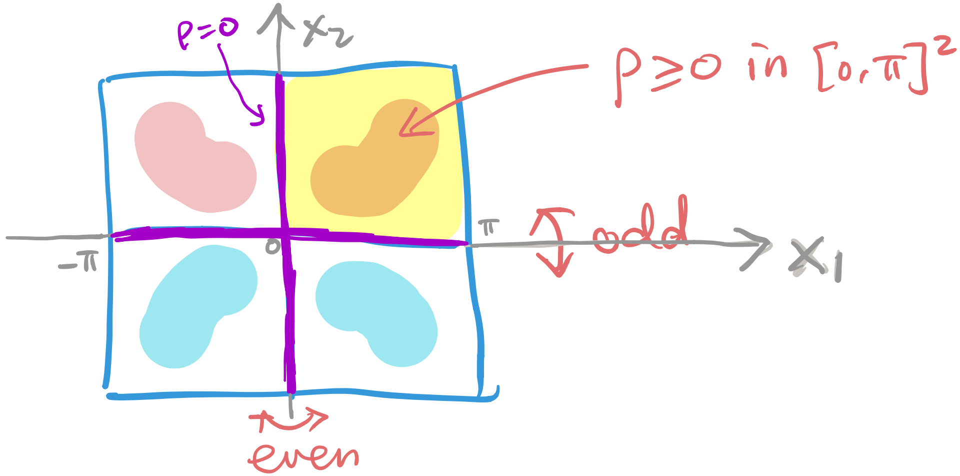
$$\limsup_{t\to\infty} \frac{\|\rho(\cdot,t)\|_{\dot H^{\beta+1}}}{t^{\frac{2\beta+1}{2}}} \ge \limsup_{t\to\infty} \frac{\|\partial_{x_1}\rho(\cdot,t)\|_{\dot H^\beta}}{t^{\frac{2\beta+1}{2}}} = \infty,$$

meaning that ρ has infinite-in-time growth in $\dot H^\beta$ for $\beta > 1/2$.

Sketch of the proof: problem set-up

- Set up of initial data:

ρ_0 smooth,
even in x_1,
odd in x_2



(Note that the even-odd symmetry is preserved for all times.)

- Main tool: monotonicity of the potential energy

$$E(t) := \int_{\mathbb{T}^2} \rho(x,t) x_2 \, dx.$$

- $\|\rho(t)\|_{L^\infty}$ invariant in time $\Longrightarrow$ as long as we have a smooth solution,

$$|E(t)| \leq 4\pi^3 \|\rho_0\|_{L^\infty} \text{ for all } t.$$

Monotonicity of the potential energy

- A quick computation gives (using ρ is periodic in x_1 and odd in x_2)

$$E'(t) = \int_{\mathbb{T}^2} -\nabla \cdot (\rho \boldsymbol{u})\, x_2 \, dx = \int_{\mathbb{T}^2} \rho u_2 \, dx.$$

- Plug in $\boldsymbol{u} = \partial_{x_1} \nabla^\perp (-\Delta)^{-1} \rho$:

$$E'(t) = \int_{\mathbb{T}^2} \rho \, \partial^2_{x_1 x_1} (-\Delta)^{-1} \rho \, dx = - \underbrace{\|\partial_{x_1} \rho\|^2_{\dot{H}^{-1}}}_{=:\delta(t)}.$$

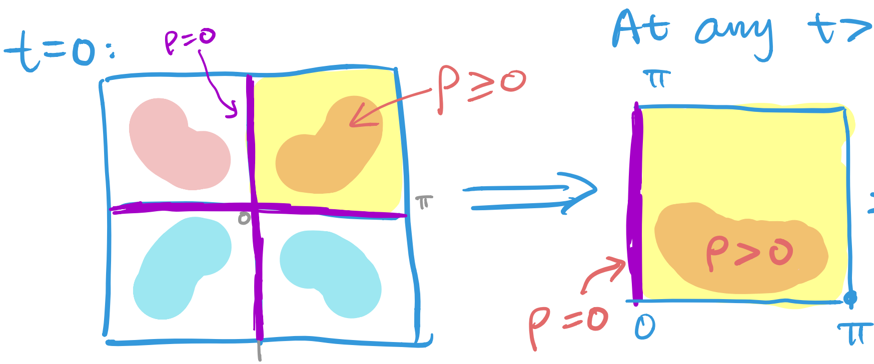
- The uniform bound of E implies that $\delta(t)$ is integrable in time:

$$|E(t)| \le 4\pi^3 \|\rho_0\|_{L^\infty} \Longrightarrow \int_0^\infty \delta(t) dt < C.$$

- Key observation: "$\delta \ll 1 \quad \Longrightarrow \quad \|\partial_{x_1} \rho\|_{\dot{H}^s} \gg 1$".

Relating $\delta(t)$ with $\|\partial_{x_1}\rho\|_{\dot{H}^s}$

- Recall: $\delta(t) := \|\partial_{x_1}\rho\|^2_{\dot{H}^{-1}}$. Known: $\int_0^\infty \delta(t)dt < C$.
- Goal: "$\delta \ll 1 \quad \Rightarrow \quad \|\partial_{x_1}\rho\|_{\dot{H}^s} \gg 1$".
- Observation for the "bubble" solution:



$$\Rightarrow \int_0^\pi\int_0^\pi |\partial_{x_1}\rho|\,dx > c > 0, \ \forall t$$

$$\Rightarrow \|\partial_{x_1}\rho\|_{L^2} > c$$

$$\Rightarrow \|\partial_{x_1}\rho\|_{\dot{H}^1} \gtrsim \delta^{-\frac{1}{2}}$$

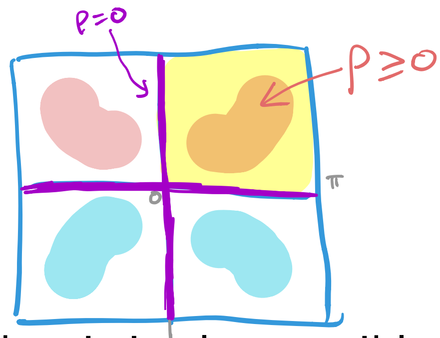
- A slightly more sophisticated argument on the Fourier side yields

$$\|\partial_{x_1}\rho\|_{\dot{H}^\beta} \gtrsim \delta^{-(\beta+\frac{1}{2})} \qquad \text{for all } \beta > -1/2,$$

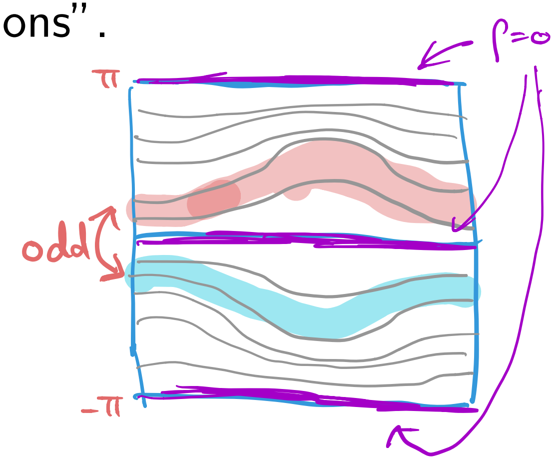
and combining it with $\int_0^\infty \delta(t)dt < C$ gives us the desired estimate.

From bubble to layers

- The above argument strongly depends on the “bubble” structure.

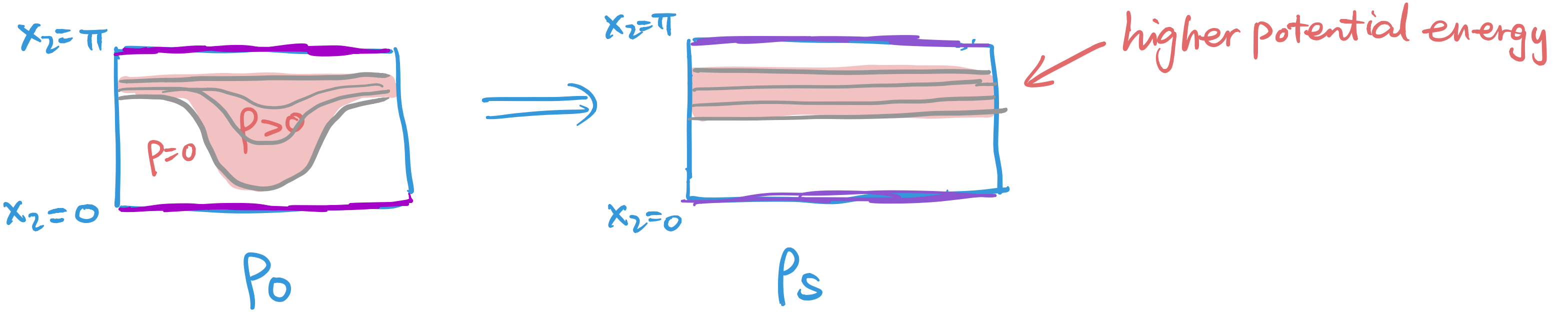


- Next we will show that it is also possible to have norm growth in “layered solutions”.



Growth for layered solution

- Assume $\rho_0 \in \mathbb{T}^2$ is an odd-in-x_2 "layered" initial data. There exists a *unique* stratified steady state $\rho_s(x) = g(x_2)$ that is odd-in-x_2 and reachable from ρ_0 by a measure-preserving diffeomorphism.
- Suppose $E[\rho_0] - E[\rho_s] = -c < 0$. Then $E[\rho(t)] < E[\rho_s] - c$ for all t, i.e. $\rho(t)$ can never get very close to ρ_s.



- So level sets of $\rho(t)$ can never be too close to horizontal lines - a geometric argument gives $\int_{\mathbb{T}^2} |\partial_{x_1} \rho(x,t)| dx > c(\rho_0) > 0$ for all t.
- Using that $\|\partial_{x_1}\rho\|^2_{\dot{H}^{-1}}$ is integrable in time, this leads to infinite-in-time growth of $\|\partial_{x_1}\rho\|^2_{\dot{H}^1}$ and higher norms.

Thank you for your attention!