Mixing Hot and Cold with Sound

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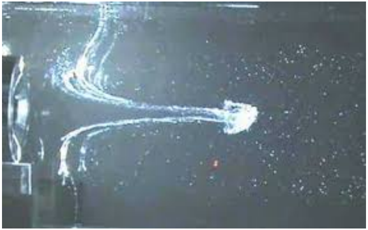
IPAM Workshop: Transport and Mixing in Complex/Turbulent Flows
January 13th, 2021

GPC and GM gratefully acknowledge support from the Woods Hole Summer Program in Geophysical Fluid Dynamics.

Wave-Driven Mean Flows



Steady streaming refers to time-averaged Eulerian flows that arise in periodically forced fluid systems, leading to: water-wave induced sediment transport; winds in equatorial stratosphere; and even circulation of cerebro-spinal fluid in spinal canal!

Acoustic streaming is a technologically important form of steady streaming in which oscillatory driving is provided by high-frequency sound waves

- Eckart streaming: depends on viscous dissipation of waves in body of fluid
- Rayleigh streaming: driven in Stokes boundary layers adjacent to solid walls

Role of Fluctuating Vorticity

Decomposing into time-mean $\overline{(\cdot)}$ and a fluctuation $(\cdot)'$, it can be shown under fairly general circumstances[1] that:

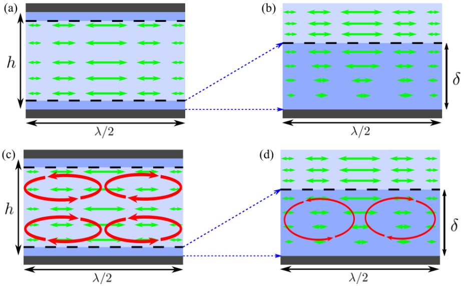
- If fluctuating flow is **irrotational** ($\omega' \equiv \mathbf{0}$):

$$\boxed{\bar{\rho}\,\overline{\mathbf{u}' \cdot \nabla \mathbf{u}'} = \nabla \left(\frac{\bar{\rho}}{2} \overline{|\mathbf{u}'^2|} \right)}$$

- Induces (or modifies) mean pressure gradient, **not** streaming

[1] Eckart streaming is an exception!

Rayleigh Streaming (Lord Rayleigh, 1884)



(i) Small-amplitude **standing** acoustic waves: $\epsilon \equiv u_*/(\omega\lambda) \ll 1$

(ii) Thin geometry $h/\lambda \ll 1$

(iii) Reynolds stress divergence in **viscous** boundary layer (BL) drives mean flow

(iv) Mean flow finite at edge of BL:

$$\boxed{\frac{\bar{u}}{u_*}\Big|_{y\to 0^+} \sim -(3\epsilon/4)\overline{u'\partial_x u'} = O(\epsilon)}$$

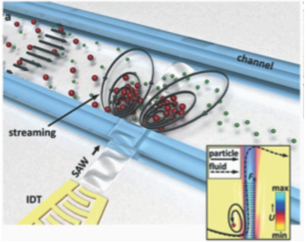
Attributes of Rayleigh streaming:

- **Intensity:** streaming flow outside BL is $O(\epsilon)$ relative to oscillating flow, implying strictly **one-way coupling** (i.e. waves can be computed as if streaming absent)
- **Pattern:** counter-rotating cells, stacked in the wall-normal direction, and at edge of BL, streaming is directed toward velocity nodes of standing acoustic wave

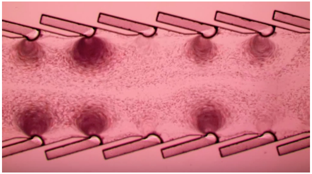
Technological Applications of Acoustic Streaming

- Microfluidic pumping, mixing, particle capture
- Disruption of dendrites in lithium batteries
- Activated irrigation in medical procedures
- Micro-*g* heat transfer
- Control of high-intensity discharge (HID) lamps

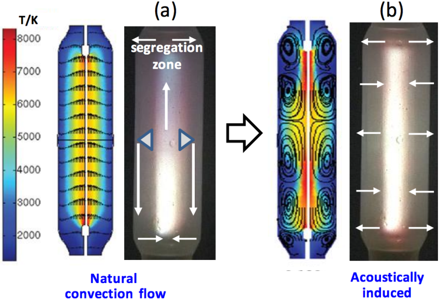
Eckart Streaming



Rayleigh Streaming



Inapplicability of Classical Rayleigh Theory: Motivating Application–HID Lamps

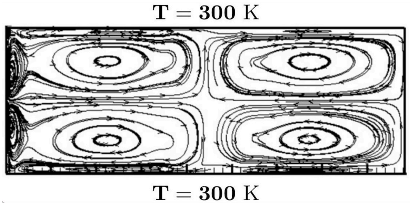


Dreeben & Chini (2011)[2] investigated acoustic streaming in high-intensity discharge (HID) lamps, showing that classical Rayleigh streaming theory *not* relevant. . .

HID Streaming	Rayleigh Theory
0.5 m/s	0.005 m/s
Away from velocity nodes	Toward velocity nodes

[2] Dreeben, T. D. & Chini, G. P. 2011. 2D streaming flows in high-intensity discharge lamps. *Phys. Fluids* **23**, 056101.

Profound Effect of Background Temperature Gradient

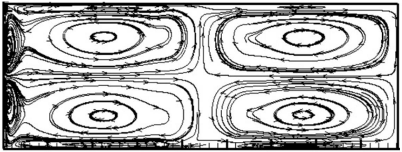


Lin & Farouk (2008)[3] performed DNS of full compressible Navier–Stokes equations to investigate impact of acoustic streaming on heat transfer in channels

[3] Lin, Y. & Farouk, B. 2008. Heat transfer in a rectangular chamber with differentially heated horizontal walls: Effects of a vibrating sidewall. *Int. J. Heat Mass Transfer* **51**, 3179–3189.

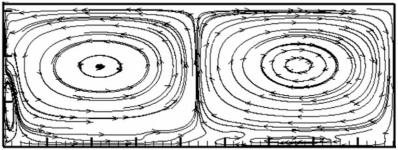
Profound Effect of Background Temperature Gradient

$\mathbf{T} = \mathbf{300}$ K



$\mathbf{T} = \mathbf{300}$ K

$\mathbf{T} = \mathbf{320}$ K



$\mathbf{T} = \mathbf{300}$ K

Lin & Farouk (2008)[4] performed DNS of full compressible Navier–Stokes equations to investigate impact of acoustic streaming on heat transfer in channels

Their simulations confirm that character of Rayleigh streaming – both intensity and pattern – is fundamentally altered by imposed (mean) temperature gradient

[4] Lin, Y. & Farouk, B. 2008. Heat transfer in a rectangular chamber with differentially heated horizontal walls: Effects of a vibrating sidewall. *Int. J. Heat Mass Transfer* **51**, 3179–3189.

Baroclinic Acoustic Streaming

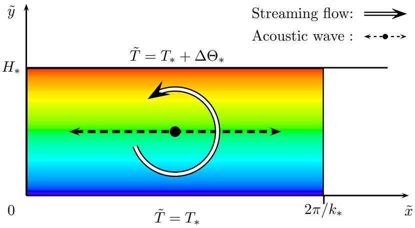
Fluctuating vorticity produced **baroclinically** rather than by viscous torques!

$$\boxed{\bar{\rho}\partial_t \mathbf{u}' \approx -\nabla p' \quad \Rightarrow \quad \partial_t \left(\nabla \times \mathbf{u}'\right) \approx \frac{\nabla \bar{\rho} \times \nabla p'}{\bar{\rho}^2}}$$

Chini, Malecha & Dreeben (2014)[5] demonstrated the potential for *strong* baroclinically-driven acoustic streaming

[5] Chini, G. P., Malecha, Z. & Dreeben, T. D. 2014. Large-amplitude acoustic streaming. *J. Fluid Mech.* **744**, 329–351.

Problem Formulation



BCs: $u = V = \Theta = 0$

$T = T_B(Y) + \Theta(x, Y, t)$

where background temperature:

$T_B(Y) = 1 + \Gamma Y$

Note: Buoyancy forces **neglected**

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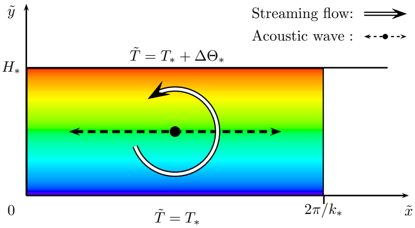

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Compressible NS Equations

$$
\begin{aligned}
\rho\Big[\partial_t u + (\mathbf{u}\cdot\nabla)\, u\Big] &= -\frac{1}{\gamma}\partial_x \pi + \frac{1}{Re_w}\Big[\nabla^2 u + \frac{1}{3}\partial_x(\nabla\cdot\mathbf{u})\Big] \\
\rho\Big[\partial_t V + (\mathbf{u}\cdot\nabla)\, V\Big] &= -\frac{1}{\delta^2\gamma}\partial_Y \pi + \frac{1}{Re_w}\Big[\nabla^2 V + \frac{1}{3\delta^2}\partial_Y(\nabla\cdot\mathbf{u})\Big] \\
\partial_t\rho + \partial_x(\rho u) + \partial_Y(\rho V) &= 0 \\
\partial_t\Theta + (\mathbf{u}\cdot\Theta) + V\frac{dT_B}{dY} &= (1-\gamma)(T_B+\Theta)(\nabla\cdot\mathbf{u}) + \frac{\gamma}{\rho Pr\, Re_w}\nabla^2\Theta \\
1 + \pi &= \rho\,(T_B + \Theta)
\end{aligned}
$$

where $\mathbf{u} = (u, \delta V)$ and $\nabla(\,\cdot\,) = [\partial_x(\cdot), \frac{1}{\delta}\partial_Y(\cdot)]$

Dimensionless Parameters, Distinguished Limit & Multiple Scale (WKBJ) Analysis

Parameter	Definition	Scaling
Wave Amplitude ϵ	$k_* u_* / \omega_* = u_* / a_*$	$\epsilon \ll 1$
Aspect Ratio δ	$k_* H_*$	$\delta \equiv \epsilon^{1/2} h$
Temperature Gradient Γ	$\Delta\Theta_* / T_*$	$\Gamma = O(1)$
Reynolds Number Re_w	$a_* / k_* \nu_*$	$Re_w \equiv \epsilon^{-2} R$
Prandtl Number Pr	ν_* / κ_*	$Pr = O(1)$
Specific Heat Ratio γ	c_p / c_v	$\gamma = O(1)$

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WKBJ analysis to capture fast and slow evolution, using phase variable $\phi(t) \equiv \Phi(T)/\epsilon$ and slow time scale $T \equiv \epsilon t$, where the slowly-varying frequency $\omega = -\partial_T \Phi$:

$$\boxed{f(x, Y, t; \epsilon) \quad = \quad \bar{f}(x, Y, T; \epsilon) + f'(x, Y, \phi; T, \epsilon)}$$

Asymptotic expansions for various fields:

$$\begin{aligned} [u, V, \pi] \quad &\sim \quad \epsilon[\bar{u}_1 + u_1', \bar{V}_1 + V_1', \pi_1'] + \epsilon^2[\bar{u}_2 + u_2', \bar{V}_2 + V_2', \bar{\pi}_2 + \pi_2'] + \ldots \\ [\Theta, \rho] \quad &\sim \quad \left[\bar{\Theta}_0, \bar{\rho}_0\right] + \epsilon\left[\bar{\Theta}_1 + \Theta_1', \bar{\rho}_1 + \rho_1'\right] + \ldots \end{aligned}$$

Leading-Order, Quasilinear (QL) Wave/Mean-Flow System

Streaming Flow

$$
\begin{aligned}
\bar{\rho}_0 \left(\partial_T \bar{u}_1 + \bar{u}_1 \partial_x \bar{u}_1 + \bar{V}_1 \partial_Y \bar{u}_1\right) + \frac{1}{\gamma}\partial_x \bar{\pi}_2 &= -\partial_x \left(\bar{\rho}_0 \overline{u_1'^2}\right) - \partial_Y \left(\bar{\rho}_0 \overline{u_1' V_1'}\right) + \frac{1}{Rh^2}\partial_Y^2 \bar{u}_1 \\
\partial_Y \bar{\pi}_2 &= 0 \\
\partial_x \bar{u}_1 + \partial_Y \bar{V}_1 &= \frac{1}{PrRh^2}\partial_Y^2 \Theta \\
\partial_T \bar{\Theta}_0 + \bar{u}_1 \partial_x \bar{\Theta}_0 + \bar{V}_1 \left(\partial_Y \bar{\Theta}_0 + \frac{dT_B}{dY}\right) &= (1-\gamma)(T_B + \bar{\Theta}_0)(\partial_x \bar{u}_1 + \partial_Y \bar{V}_1) \\
&\quad + \left(\frac{\gamma}{PrRh^2}\right)\frac{1}{\bar{\rho}_0}\partial_Y^2 \bar{\Theta}_0
\end{aligned}
$$

$$
\bar{\rho}_0 \left(T_B + \bar{\Theta}_0\right) = 1
$$

Leading-Order, Quasilinear (QL) Wave/Mean-Flow System

Streaming Flow

$$\begin{aligned}
\bar{\rho}_0\left(\partial_T \bar{u}_1 + \bar{u}_1\partial_x \bar{u}_1 + \bar{V}_1\partial_Y \bar{u}_1\right) + \frac{1}{\gamma}\partial_x \bar{\pi}_2 &= -\partial_x\left(\bar{\rho}_0\overline{u_1'^2}\right) - \partial_Y\left(\bar{\rho}_0\overline{u_1' V_1'}\right) + \frac{1}{Rh^2}\partial_Y^2 \bar{u}_1 \\
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\bar{\rho}_0\left(T_B + \bar{\Theta}_0\right) = 1 \qquad & \qquad + \left(\frac{\gamma}{PrRh^2}\right)\frac{1}{\bar{\rho}_0}\partial_Y^2 \bar{\Theta}_0
\end{aligned}$$

Acoustic Waves

$$\begin{aligned}
\omega_0\bar{\rho}_0\partial_\phi u_1' &= -\frac{1}{\gamma}\partial_x \pi_1' \\
\partial_Y \pi_1' &= 0 \\
\omega_0\partial_\phi \rho_1' + \partial_x\left(\bar{\rho}_0 u_1'\right) + \partial_Y\left(\bar{\rho}_0 V_1'\right) &= 0 \\
\omega_0\partial_\phi \Theta_1' + u_1'\partial_x \bar{\Theta}_0 + V_1'\left(\partial_Y \bar{\Theta}_0 + \frac{dT_B}{dY}\right) &= (1-\gamma)(T_B + \bar{\Theta}_0)(\partial_x u_1' + \partial_Y V_1') \\
\pi_1' &= \rho_1'(T_B + \bar{\Theta}_0) + \bar{\rho}_0\Theta_1'
\end{aligned}$$

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\bar{\rho}_0\left(\partial_T\bar{u}_1+\bar{u}_1\partial_x\bar{u}_1+\bar{V}_1\partial_Y\bar{u}_1\right)+\frac{1}{\gamma}\partial_x\bar{\pi}_2 &= -\partial_x\left(\bar{\rho}_0\overline{u_1'^2}\right)-\partial_Y\left(\bar{\rho}_0\overline{u_1'V_1'}\right)+\frac{1}{Rh^2}\partial_Y^2\bar{u}_1\\
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&\quad+\left(\frac{\gamma}{PrRh^2}\right)\frac{1}{\bar{\rho}_0}\partial_Y^2\bar{\Theta}_0
\end{aligned}$$

$$\bar{\rho}_0\left(T_B+\bar{\Theta}_0\right)=1$$

Acoustic Waves

$$\partial_x\left[\left(\frac{1}{\bar{\rho}_0}\right)\partial_x\pi_1'\right]+\partial_Y\left[\left(\frac{1}{\bar{\rho}_0}\right)\partial_Y\pi_1'\right] = \omega_0^2\,\partial_\phi^2\pi_1'$$

NOTE: Stokes layers are **passive**.

Elimination of Fast Acoustic Wave Dynamics. I. Mode Structure

Quasilinear (QL) structure of wave equations enables linear WKBJ *ansatz*, i.e.

$$\boxed{f_1'(x, Y, \phi; T) \quad = \quad A(T)\left[\hat{f}_1(x, Y; \bar{\rho}_0)e^{i\phi} + c.c.\right]}$$

2D **non-separable** partial differential eigenvalue problem for $\hat{f}_1$ can be simplified:

$$\hat{u}_1(x, Y; \bar{\rho}_0) = -\frac{g'(x; \bar{\rho}_0)}{\omega_0^2 \bar{\rho}_0(x, Y, T)} \qquad \hat{v}_1(x, Y; \bar{\rho}_0) = Yg(x; \bar{\rho}_0) + \partial_x\left(\frac{g'(x; \bar{\rho}_0)}{\omega_0^2}\int_0^Y \frac{\mathrm{d}Y}{\bar{\rho}_0}\right)$$

where $g(x)$ and $\omega_0 = \partial_T \Phi$ satisfy the **1D** e-value problem

$$\boxed{g''(x) + \frac{\alpha'(x)}{\alpha(x)}g'(x) + \frac{\omega_0^2}{\alpha(x)}g(x) = 0, \text{ where } \alpha(x) = \int_0^1 \frac{\mathrm{d}Y}{\bar{\rho}_0} \text{ and } g'(0) = g'(2\pi) = 0}$$

with normalization condition $\int_0^{2\pi} g^2(x; \bar{\rho}_0)\mathrm{d}x = 1$.

Elimination of Fast Acoustic Wave Dynamics. II. Modal Amplitude

Novel **amplitude equation** obtained by extending analysis of waves to higher order:

$$\boxed{\begin{aligned}\frac{2}{A\omega_0^{-1}}\frac{\mathrm{d}(A\omega_0^{-1})}{\mathrm{d}T} &= -\frac{i\omega_0}{Pe_s h^2}\int \mathrm{d}x\mathrm{d}Y\, g\partial_Y^2\hat{\Theta}_1 + \frac{1}{2\omega_0^2}\int \mathrm{d}x\mathrm{d}Y\, g'^2\left(\bar{u}_1\partial_x\bar{\rho}_0^{-1} + \bar{v}_1\partial_Y\bar{\rho}_0^{-1}\right)\\ &+ \int \mathrm{d}x\mathrm{d}y(\partial_x\bar{u}_1 + \partial_y\bar{v}_1)\left[(\gamma+1)g^2 + 2i\omega_0\hat{\Theta}_1 g\bar{\rho}_0 + \frac{g'^2}{\omega_0^2\bar{\rho}_0}\left(Pr - \frac{1}{2}\right)\right]\end{aligned}}$$

- May be interpreted as **slow-time energy balance** for acoustic wave, since

$$\overline{E}_1 \equiv \frac{1}{2}\int_0^{2\pi}\mathrm{d}x\int_0^1 \mathrm{d}Y\,\bar{\rho}_0\overline{u_1'^2} = \left(\frac{A(T)}{2\omega_0}\right)^2 \quad\Rightarrow\quad \frac{2}{A\omega_0^{-1}}\frac{\mathrm{d}(A\omega_0^{-1})}{\mathrm{d}T} = \frac{1}{\overline{E}_1}\frac{\mathrm{d}\overline{E}_1}{\mathrm{d}T}$$

- Derivation requires proper accounting of variation of eigenfunctions with slow time owing to their functional dependence on $\bar{\rho}_0(x, Y, T)$

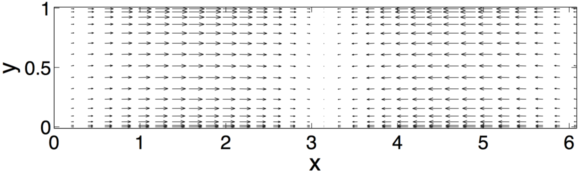
Virtue of approach is that computations can be performed strictly on slow time scale of streaming flow by time-advancing mean + amplitude equations and, each slow-time step, solving eigenvalue problem for wave frequency and mode structure.

Weak Streaming: 1-Way Coupling

Parameters corresponding to Lin & Farouk (2008) DNS...

$$\epsilon = 10^{-2},\ \gamma = 1.4,\ Pr = 0.71,\ h = 2.3,\ \Gamma = 0.2,\ R = 5.7,\ A \approx 6$$

Acoustic Wave Velocity Field



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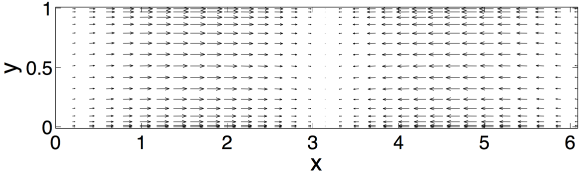
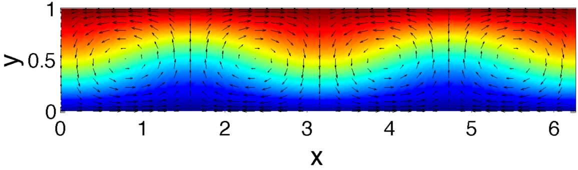


Streaming Velocity and Temperature Fields



Weak Streaming: Quantitative Comparison w/Lin & Farouk (2008)

No fitting parameters. . .

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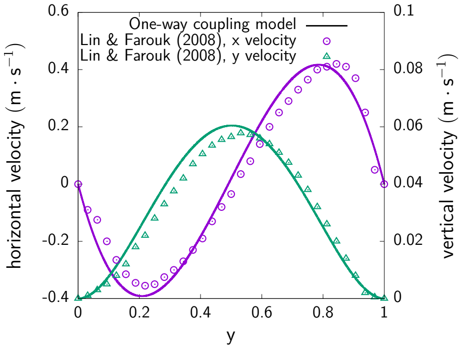

DNS of full (2D) compressible NS equations by Lin & Farouk (2008):

- Included additional physics (e.g. temperature-dependent diffusivities)
- **Computationally-expensive. . .** computed only 1 eddy-turnover time!

Finite-Amplitude Streaming: 2-Way Coupling

Slow-flow system solved using *Dedalus* code with constant power $\mathcal{P}$ injection:

$$\boxed{\frac{1}{\overline{\overline{E}}_1}\frac{\mathrm{d}\overline{E}_1}{\mathrm{d}T} \quad = \quad \ldots + \frac{\mathcal{P}}{\overline{\overline{E}}_1}}$$

Using parameters:

$\epsilon = 10^{-2}$, $\gamma = 1.4$, $Pr = 1$,
$h = 4$, $\Gamma = 1$, $R = 4$
$\mathcal{P} = 0.004$

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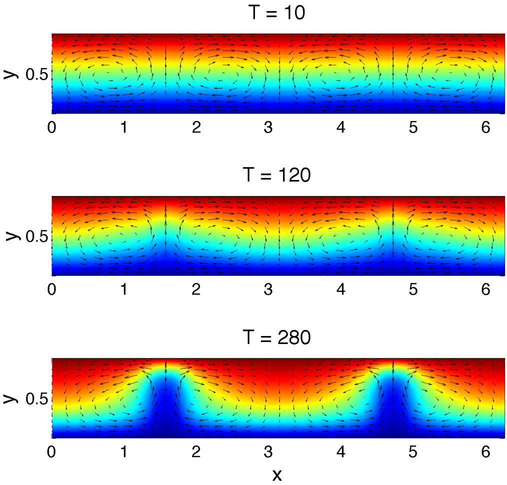

Summary: Rayleigh vs. Baroclinic Acoustic Streaming

Property	Rayleigh Streaming	Baroclinic Streaming
Relative magnitude	$\bar{u}/u_* = O(\epsilon)$	$\bar{u}/u_* = O(1)$
Pattern	Stacked cells	Cells span channel
Wave/mean-flow coupling	1-way	2-way
Heat transport	[6] $Nu - 1 \propto \epsilon^2 A^4 R^2 h^2$	$Nu - 1 \propto A^4 R^4 h^8$

[6] Vainshtein, P. Fichman, M. & Gutfinger, G. 1995 Acoustic streaming enhancement of heat transfer between two parallel plates. *Int. J. Heat. Mass Transfer.* **38**, 1893–1899.

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However, prefactors in $(Nu - 1)$–Ra relationship are rather small (at least in one-way coupling limit for which scaling result easily derived)...

Increasing Nu by increasing aspect-ratio δ *is* feasible for baroclinic acoustic streaming.

[6] Vainshtein, P. Fichman, M. & Gutfinger, G. 1995 Acoustic streaming enhancement of heat transfer between two parallel plates. *Int. J. Heat. Mass Transfer.* **38**, 1893–1899.

Ongoing Work: Impact of Aspect Ratio, BCs, Control Parameters on Heat Transfer

- What is **aspect-ratio dependence** of heat transport?
  For finite aspect-ratio [$\delta = O(1)$] case, must repeatedly solve non-separable 2D partial differential eigenvalue problem for wave field:

$$\boxed{\partial_x \left[\overline{\rho}_0^{-1} \partial_x \hat{\pi}_1\right] + \delta^{-2} \partial_y \left[\overline{\rho}_0^{-1} \partial_y \hat{\pi}_1\right] = -\,\omega_0^2\, \hat{\pi}_1, \quad \text{where: } \overline{\rho}_0 = \overline{\rho}_0(x, y, T)}$$

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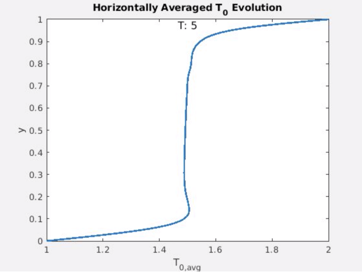
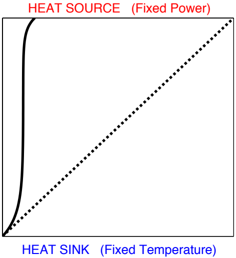
$$\partial_x \left[\overline{\rho}_0^{-1} \partial_x \hat{\pi}_1 \right] + \delta^{-2} \partial_y \left[\overline{\rho}_0^{-1} \partial_y \hat{\pi}_1 \right] = -\omega_0^2 \, \hat{\pi}_1, \quad \text{where: } \overline{\rho}_0 = \overline{\rho}_0(x, y, T)$$

- Will streaming intensity (and, hence, cooling) be reduced when **interior** temperature/density field is **well mixed**? What if at least one boundary satisfies a **fixed heat flux** rather than fixed temperature condition?





Finite Aspect-Ratio Baroclinic Acoustic Streaming: Turbulent Transport

Relaxing small aspect-ratio constraint: $\delta \equiv k_* H_* = \epsilon^{1/2} h \;\Rightarrow\; \delta \equiv k_* H_* = 1$
Other Parameters: $\epsilon = 10^{-2}$, $\gamma = 1.4$, $Pr = 0.7$, $\Gamma = 1$, $R = 5$, $A = 5$

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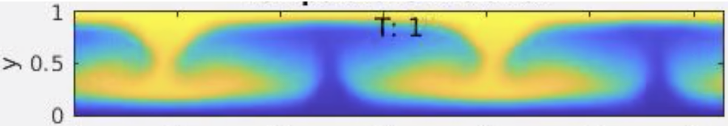

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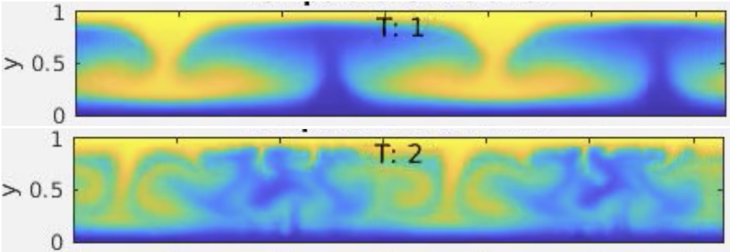

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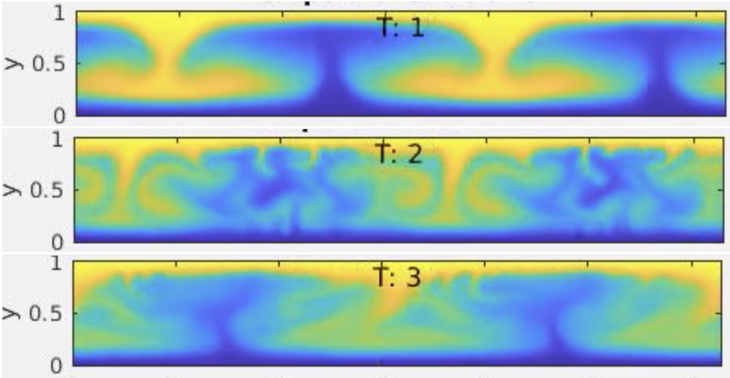

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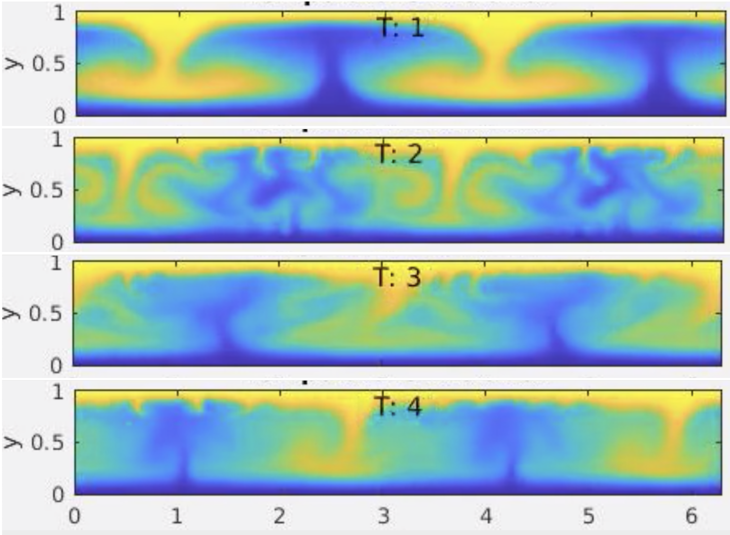
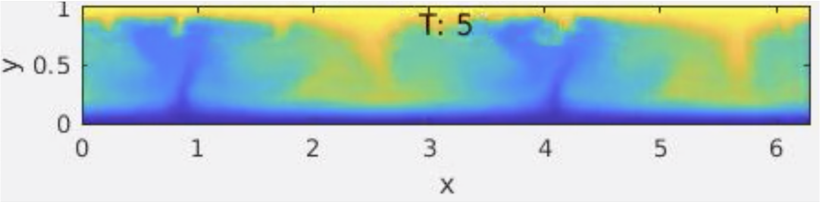


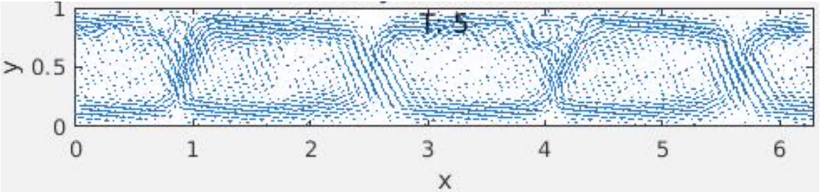
Finite Aspect-Ratio Baroclinic Acoustic Streaming: Turbulent Transport

Streaming Temperature Field: $T_B + \bar{\Theta}_0$ **(**$Nu \simeq 8$**)**



Wave forcing concentrates in streaming thermal boundary layers!

Streaming Velocity Field: **(**$\bar{u}_0, \bar{V}_0$**)**



Summary Points and Open Questions

Baroclinic acoustic streaming flows[7] [8]:

- Arise in presence of ambient/imposed **transverse temperature/density gradients**
- Characterized by velocities that are **comparable** to acoustic wave motions
- Generically involve **two-way coupling** between waves and mean flows

Unless temperature differences very small, **cannot** simply couple Rayleigh streaming predictions to appropriate (e.g. heat) transport equation, as in Richardson (1967), Davidson (1973), Vainshtein *et al.* (1995)...

Derived a **novel amplitude equation** that properly incorporates slow changes in eigenfunctions owing to mean-field evolution [cf. Karlsen *et al.* (2016, 2017)]

Open **mathematical** questions:

(i) **well-posedness** of reduced system?

(ii) rigorous **bounds** on heat transport as function of control parameters?

(iii) **efficient algorithms** for 2D eigenvalue problem?

[7] Michel, G. & Chini, G. P. Strong wave–mean-flow coupling in baroclinic acoustic streaming. 2019. *J. Fluid Mech.* **858**, 536–564.

[8] Nama, N. Mixing hot and cold with sound. 2019. *J. Fluid Mech.* Focus on Fluids **866**, 1–4.