# Bounds on mean vertical heat transport in convection driven by internal heating


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IPAM Worskshop

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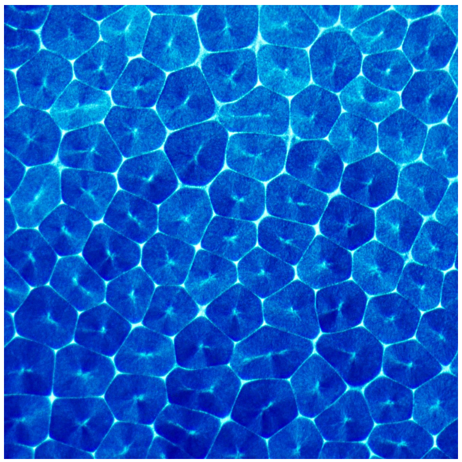
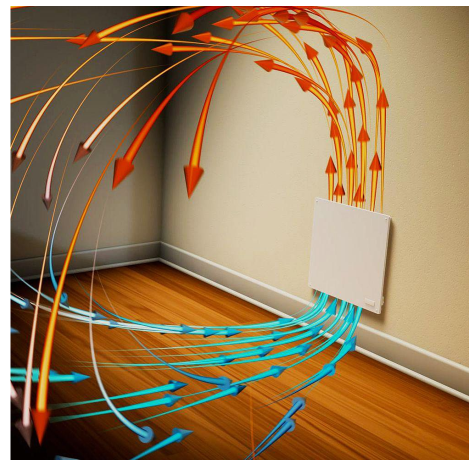
11 January 2021

# Convection is important!

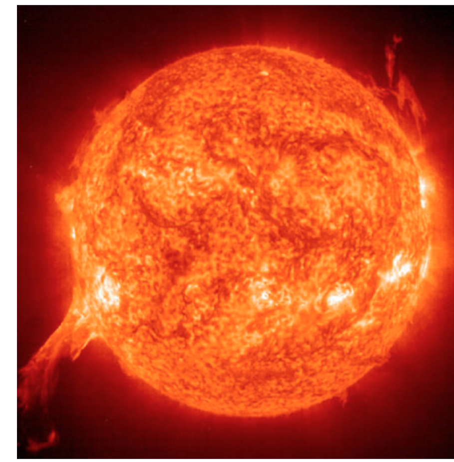
Drying paint



Central heating

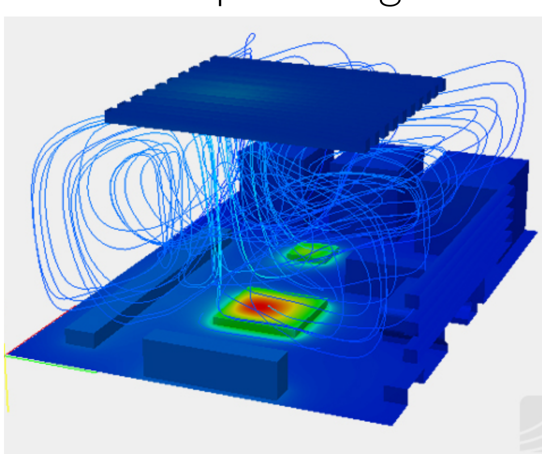


Stars



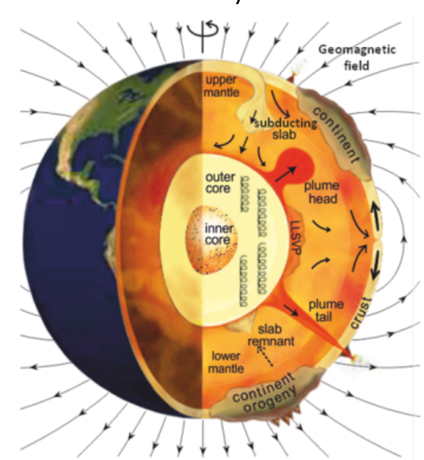
*scale*

Chip cooling



Soup

Mantle dynamics



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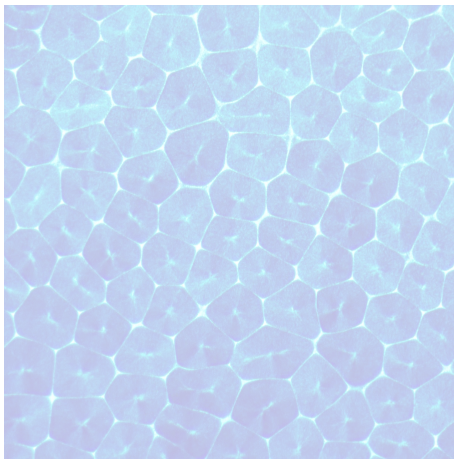

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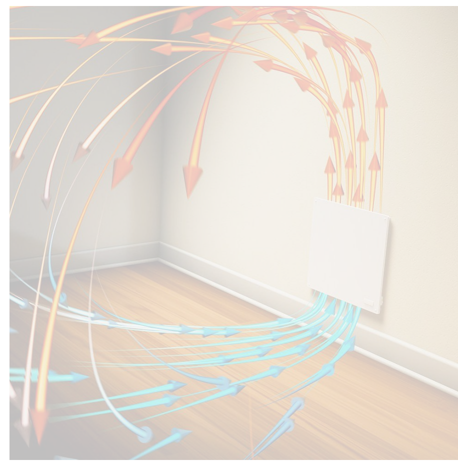

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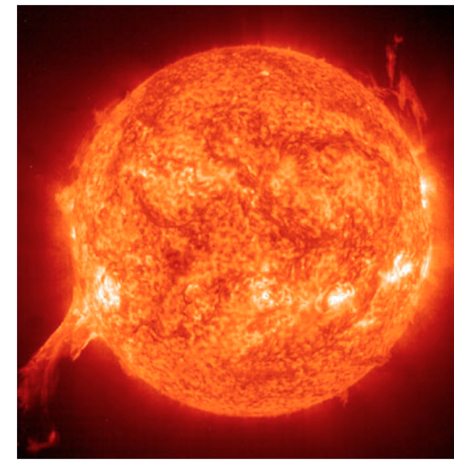

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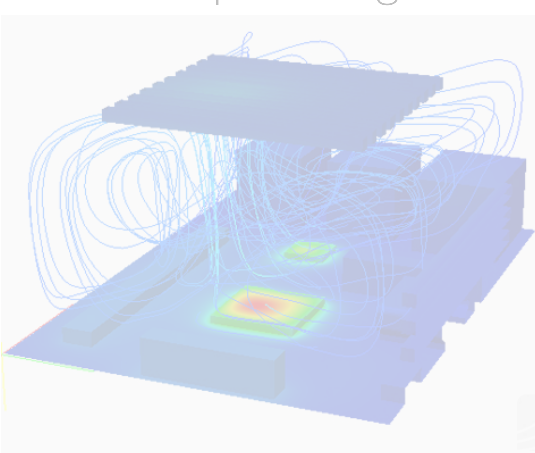

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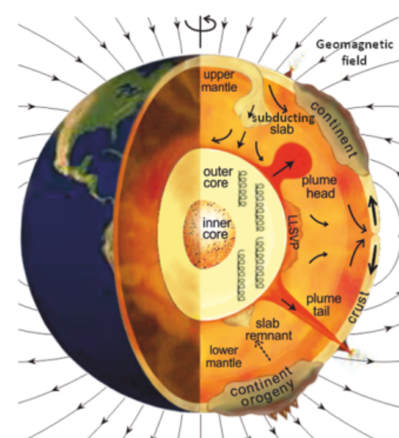
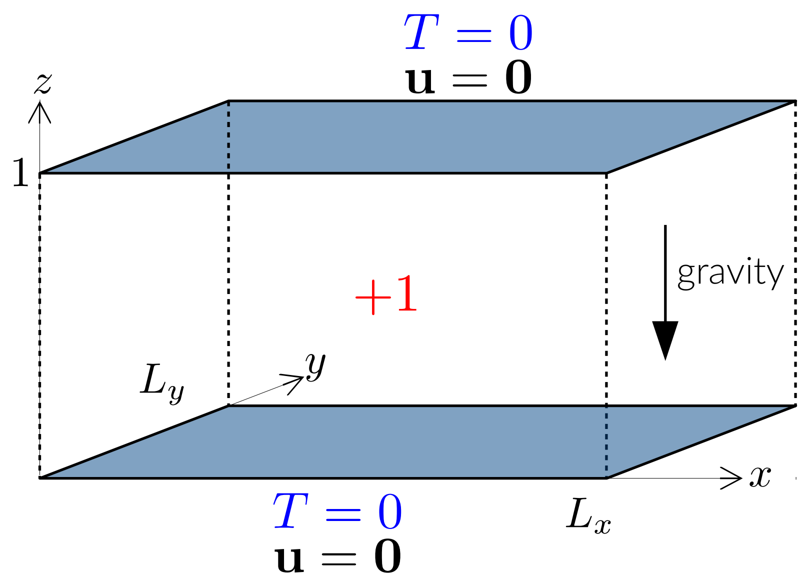


# An idealized nondimensional model




$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = Pr(\nabla^2 \mathbf{u} + RT\hat{z})$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \nabla^2 T + 1$$

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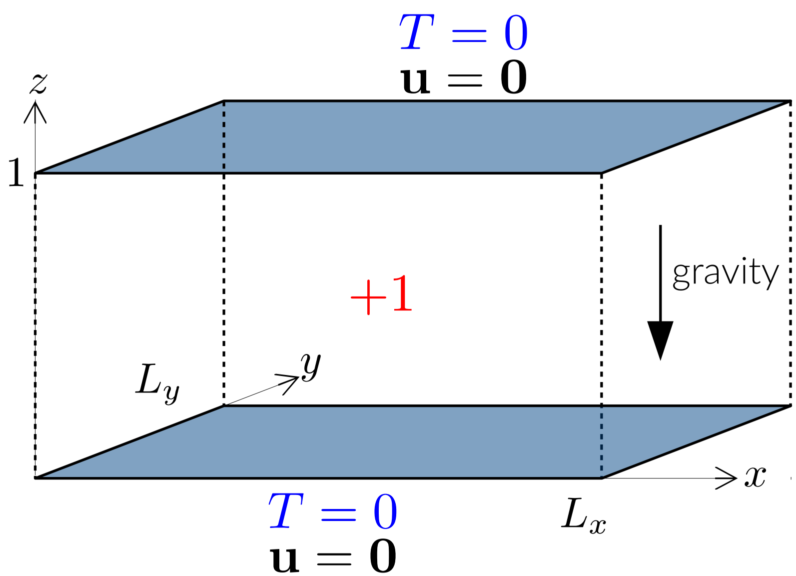


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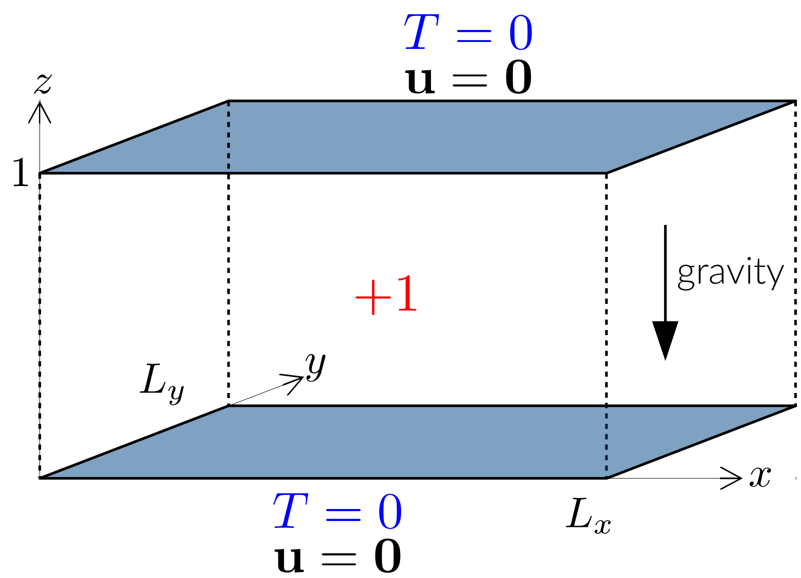


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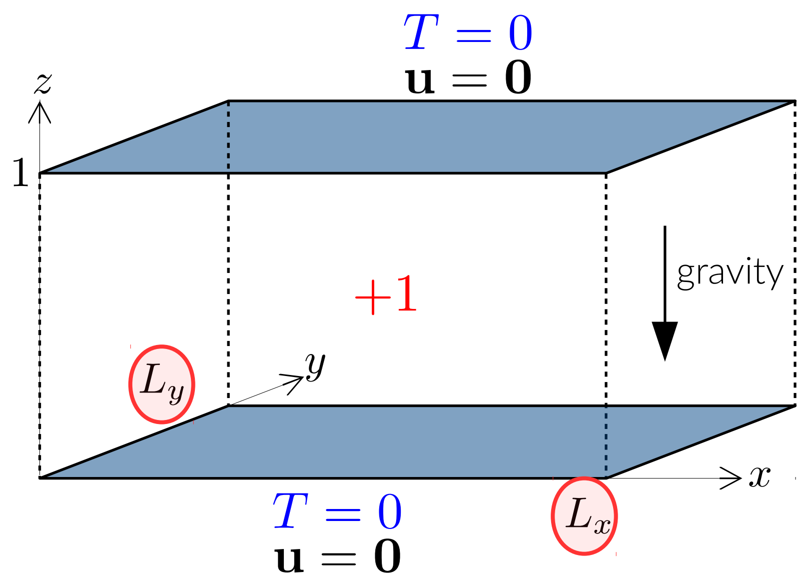


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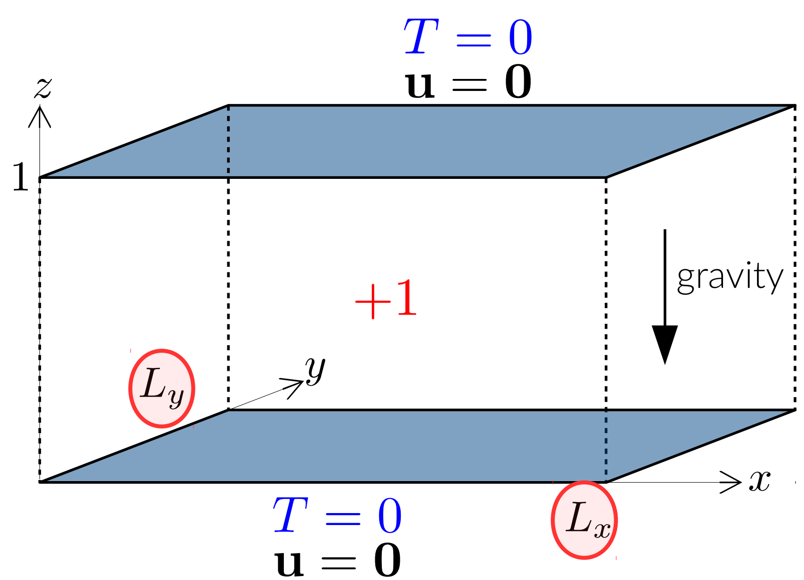


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Mean vertical heat flux:

$$\langle wT\rangle = \limsup_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau ⨍_\Omega wT\,\mathrm{d}\mathbf{x}\mathrm{d}t$$

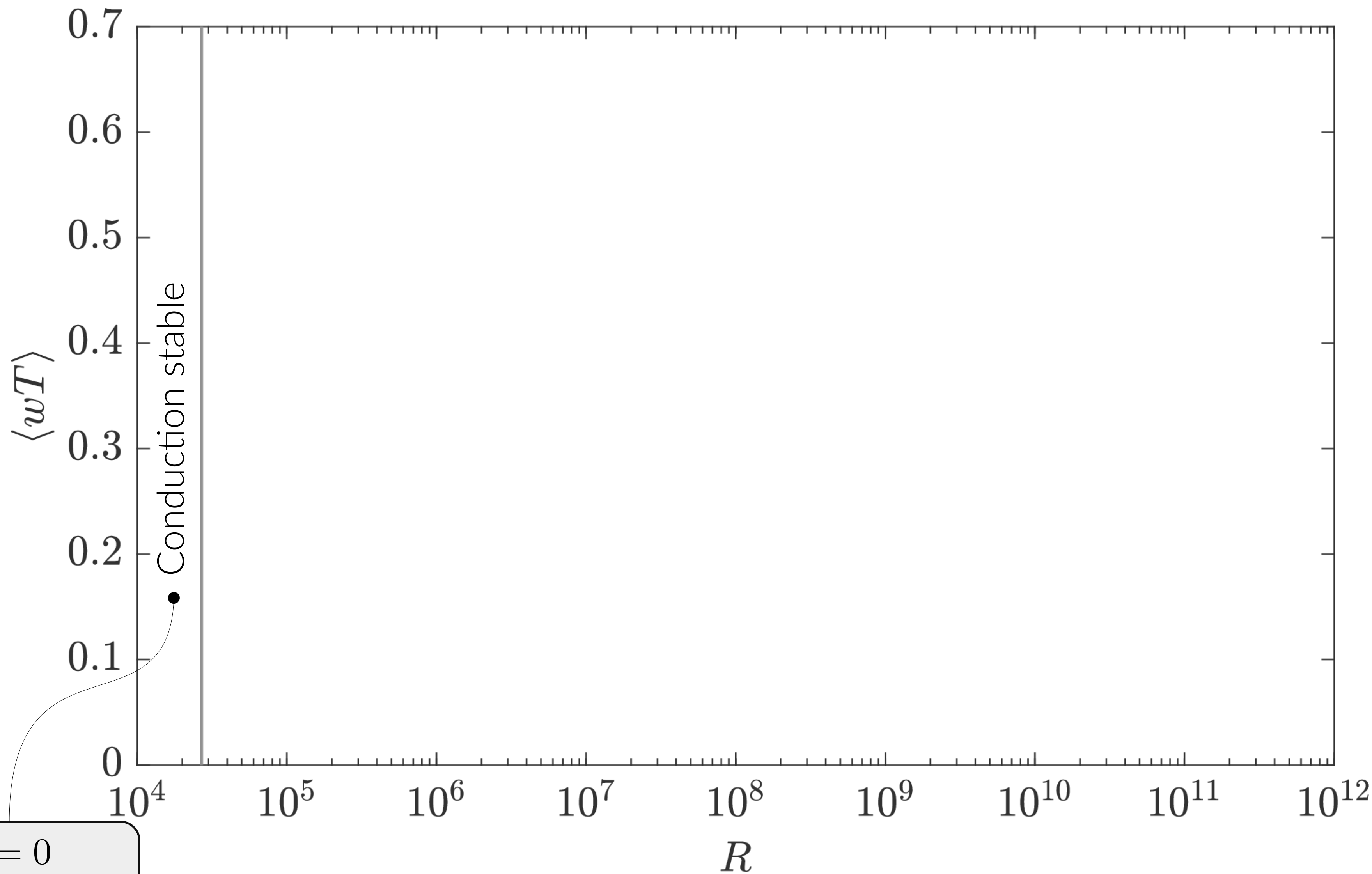
Heat flux through
top and bottom boundaries:

$$\mathcal{F}_T = -\overline{\partial_z T}|_{z=1} = \frac{1}{2} + \langle wT\rangle$$

$$\mathcal{F}_B = \overline{\partial_z T}|_{z=0} = \frac{1}{2} - \langle wT\rangle$$

**Q:** Variation with heating rate (*R*)?

# What do we know?




$$\mathbf{u} = 0$$

$$T = \frac{1}{2} z(1 - z)$$

$$\langle wT \rangle = 0$$

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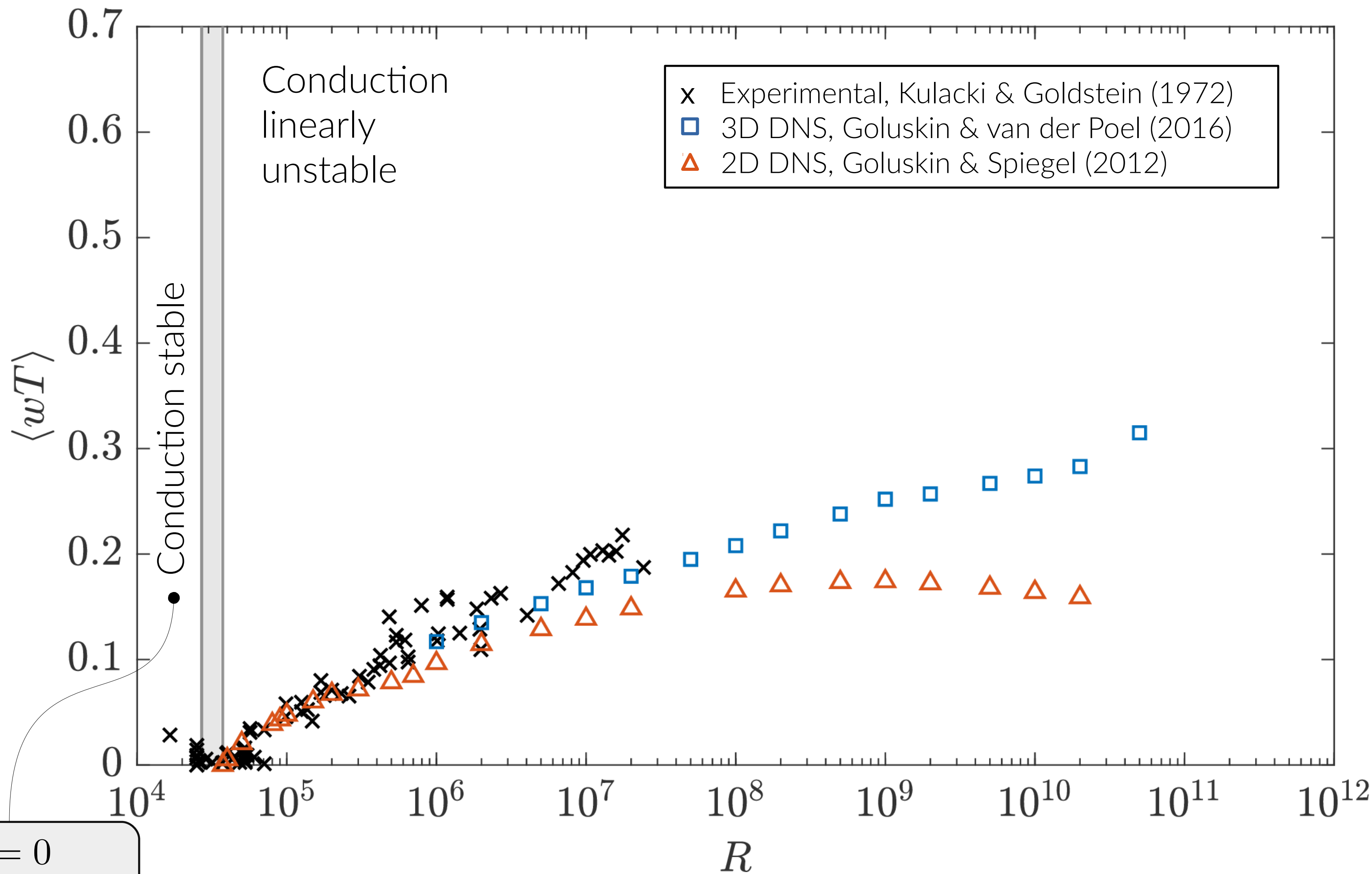


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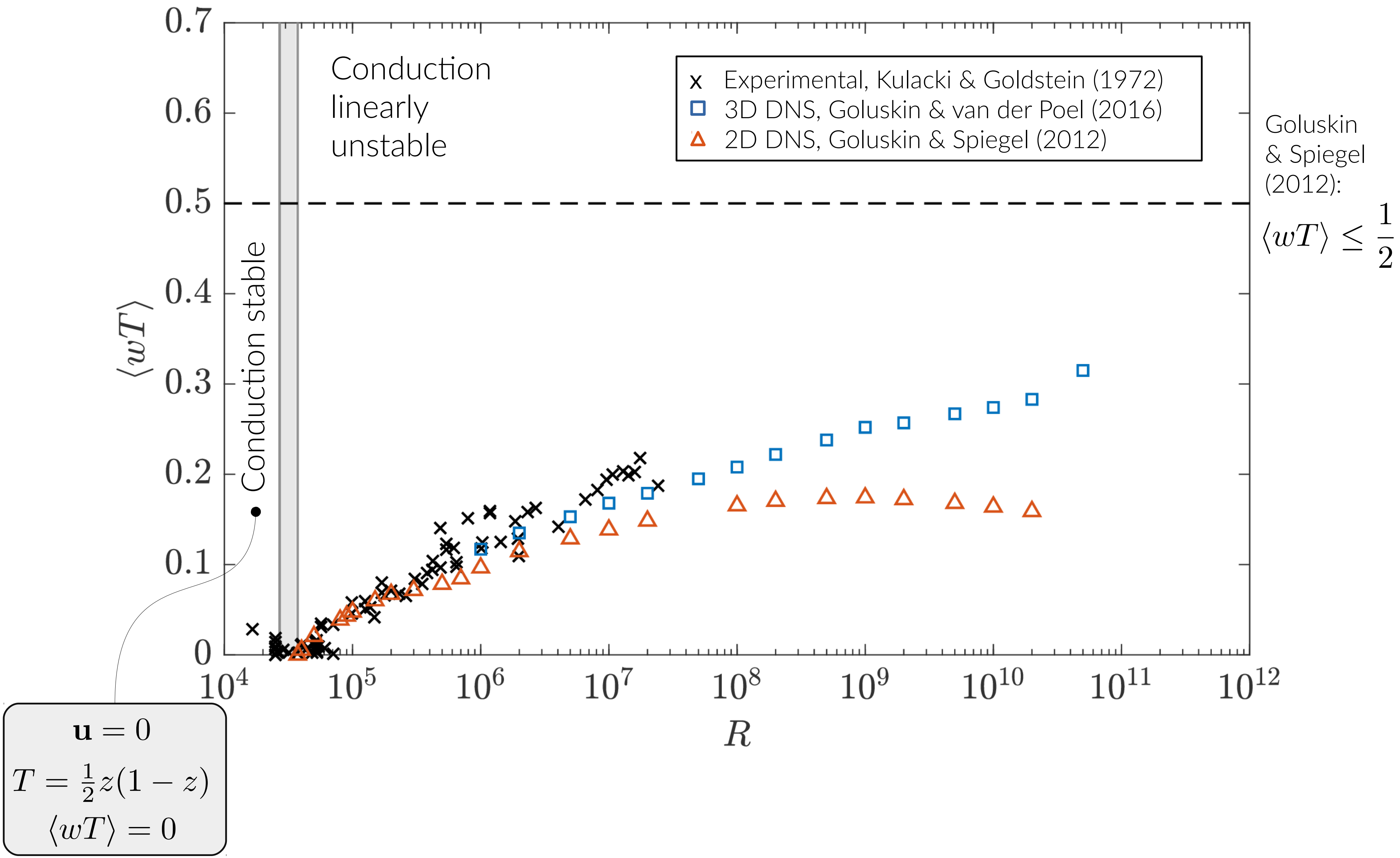

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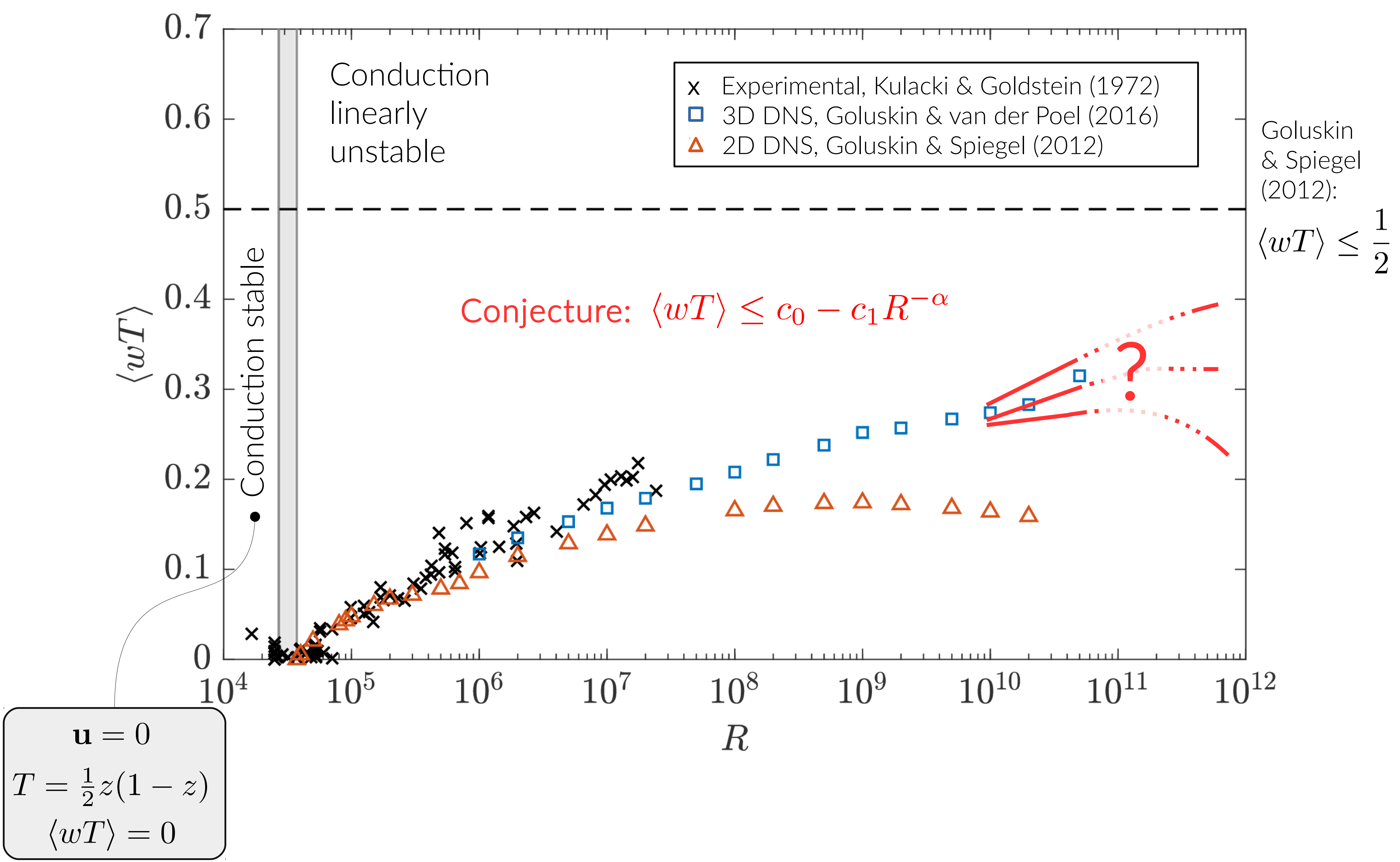

# Upper bounds on heat transfer

- Infinite-time averages of time derivatives of bounded functionals vanish!

$$\limsup_{\tau\to\infty} \underbrace{\frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{V}\{\mathbf{u}(\cdot,t),T(\cdot,t)\}\,\mathrm{d}t}_{=\,\frac{\mathcal{V}\{\mathbf{u}(\cdot,\tau),T(\cdot,\tau)\}-\mathcal{V}\{\mathbf{u}(\cdot,0),T(\cdot,0)\}}{\tau}} = 0$$

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$$= \limsup_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \fint_\Omega wT + \frac{\delta\mathcal{V}}{\delta\mathbf{u}} \cdot \partial_t \mathbf{u} + \frac{\delta\mathcal{V}}{\delta T} \cdot \partial_t T \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$

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**If $\mathcal{S}\{\mathbf{u},T\}\geq 0$ for all $\mathbf{u}(\mathbf{x})$ and $T(\mathbf{x})$ consistent with physical constraints, then $\langle wT\rangle \leq U$**

# Reinterpreting the background method

$$\mathcal{V}\{\mathbf{u},T\} = \int_\Omega \frac{a}{2PrR}|\mathbf{u}|^2 + \frac{b}{2}|T|^2 - [1 - z + \psi(z)]\,T\,\mathrm{d}\mathbf{x}$$

- After some algebra:

$$\mathcal{S}\{\mathbf{u},T\} = U - \frac{1}{2} + \psi(1)\int_{z=1} \partial_z T\,\mathrm{d}x\mathrm{d}y - [\psi(0) - 1]\int_{z=0} \partial_z T\,\mathrm{d}x\mathrm{d}y$$
$$+ \int_\Omega \frac{a}{R}|\nabla\mathbf{u}|^2 + b|\nabla T|^2 - (a - \psi')wT + (bz - \psi' - 1)\partial_z T + \psi\,\mathrm{d}\mathbf{x}$$

- A **linear** optimization problem:

$$\langle wT\rangle \leq \inf_{U,a,b,\psi(z)} \quad U$$
$$\text{s.t.} \quad \mathcal{S}\{\mathbf{u},T\} \geq 0 \quad \forall \mathbf{u},T : \begin{cases} \text{BCs} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

# Reinterpreting the background method

balance parameters

$$\mathcal{V}\{\mathbf{u},T\} = \int_\Omega \frac{a}{2PrR}|\mathbf{u}|^2 + \frac{b}{2}|T|^2 - \underbrace{[1 - z + \psi(z)]}_{\text{(rescaled) background temperature field}} T \, \mathrm{d}\mathbf{x}$$

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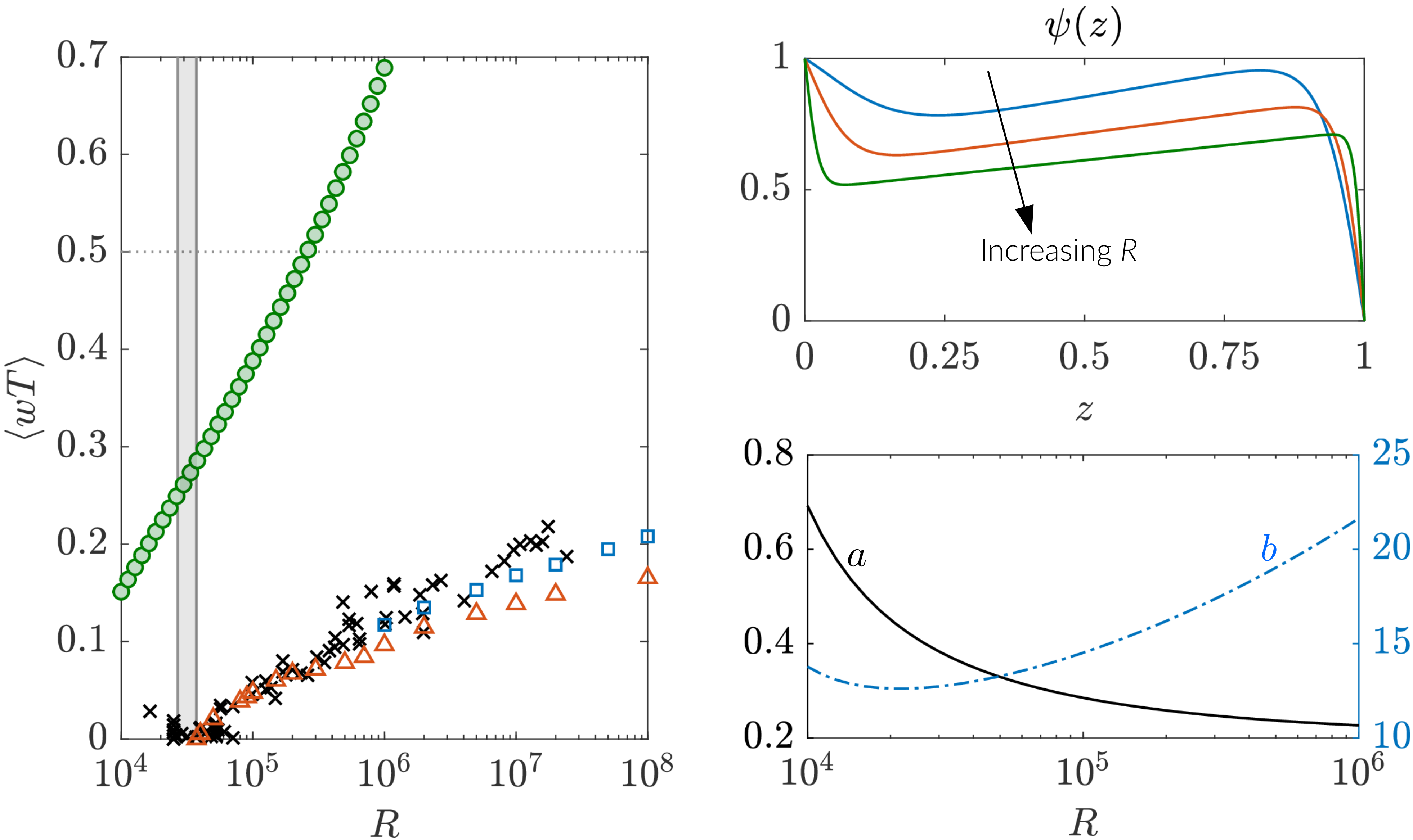
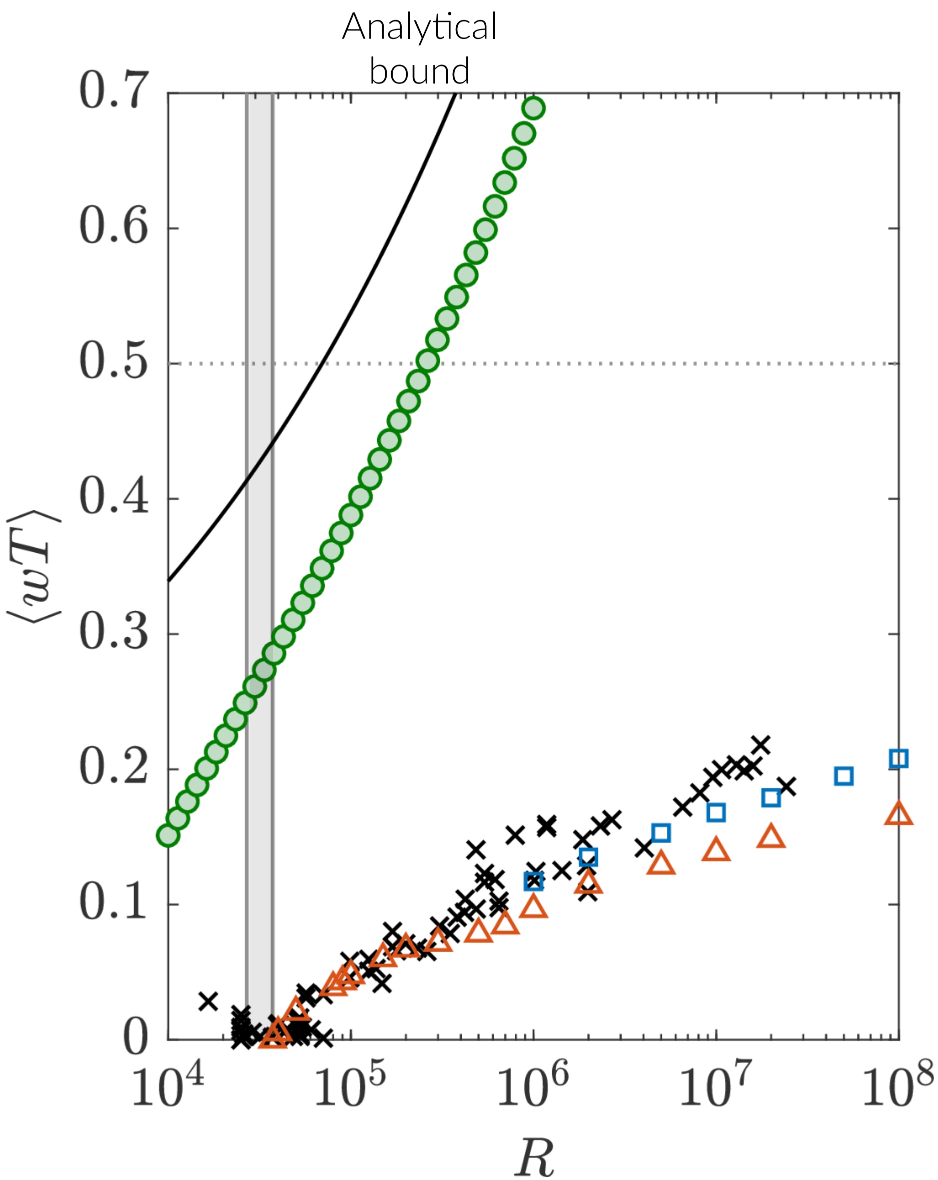
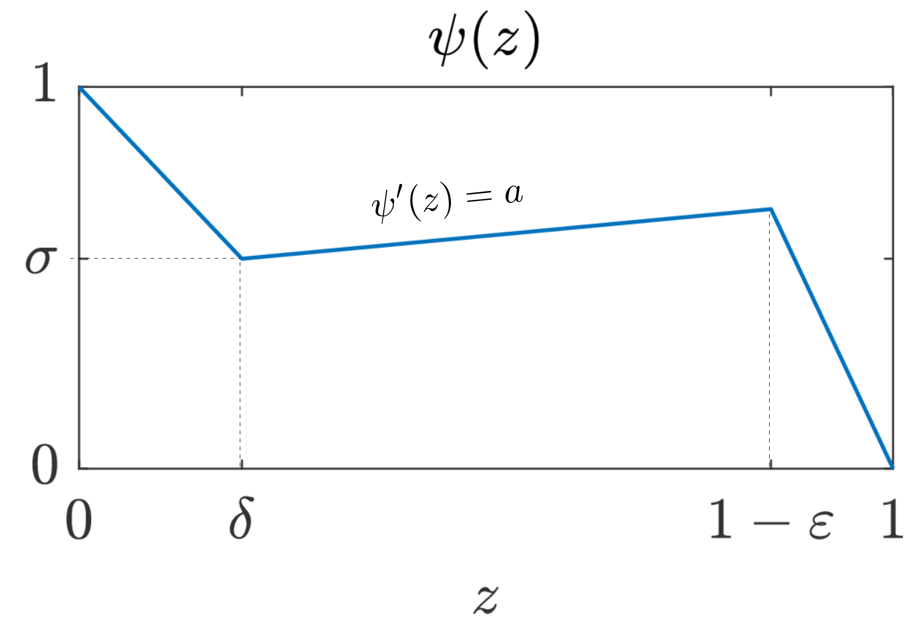
$$\text{s.t.} \qquad \inf_{\substack{\mathbf{u},T: \\ \text{BCs} \\ \nabla\cdot\mathbf{u}=0}} \mathcal{S}\{\mathbf{u},T\} \ge 0$$

# Computational results



# Analytical results



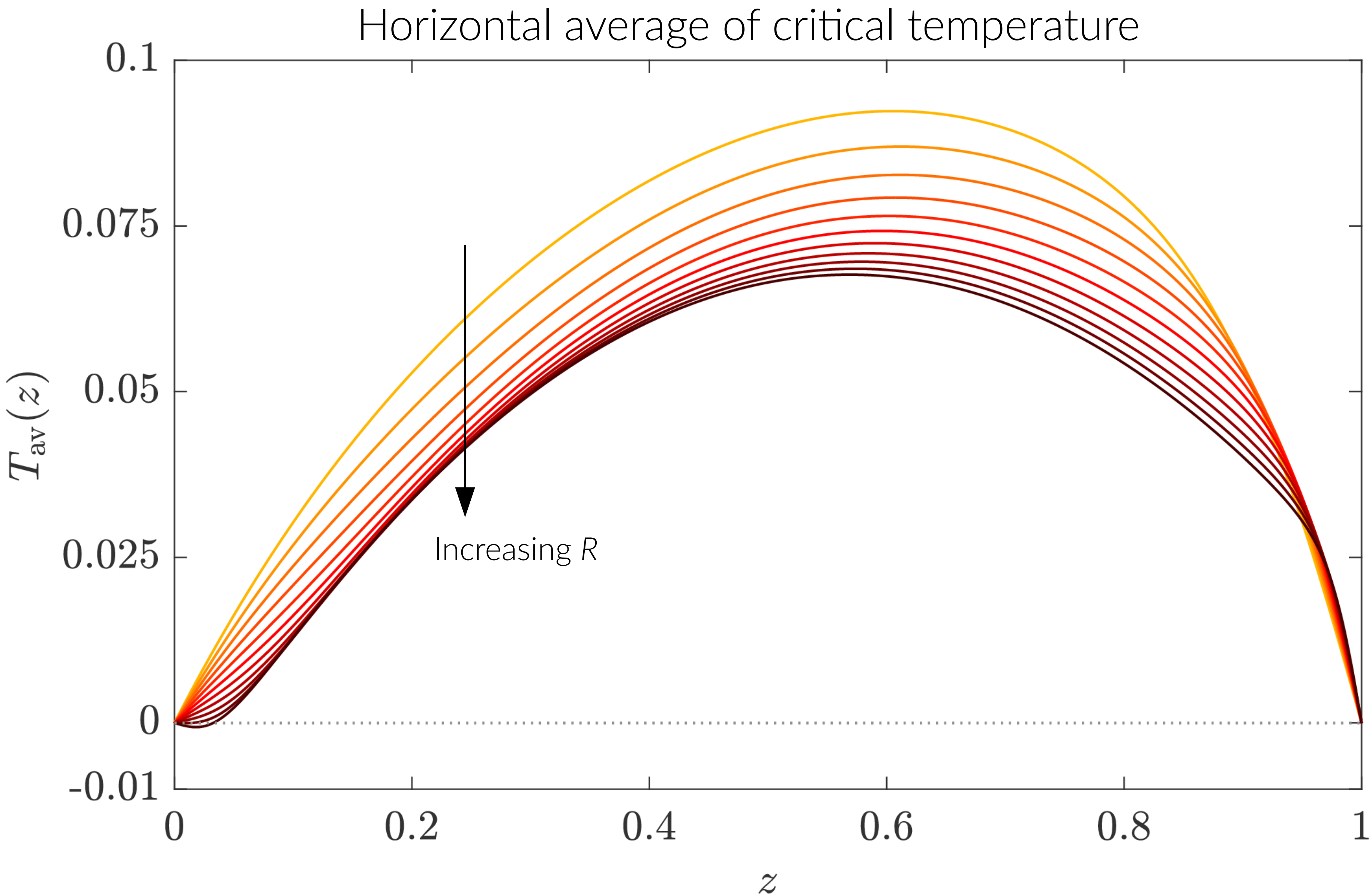




$$\langle wT \rangle \leq \tfrac{1}{16}[8(3\sqrt{2}-4)^2]^{\frac{1}{5}} R^{\frac{1}{5}}$$

$$\leq \frac{1}{2} \text{ if } R \leq 69\,572$$

# What is missing?

# Physically realistic T should be positive!



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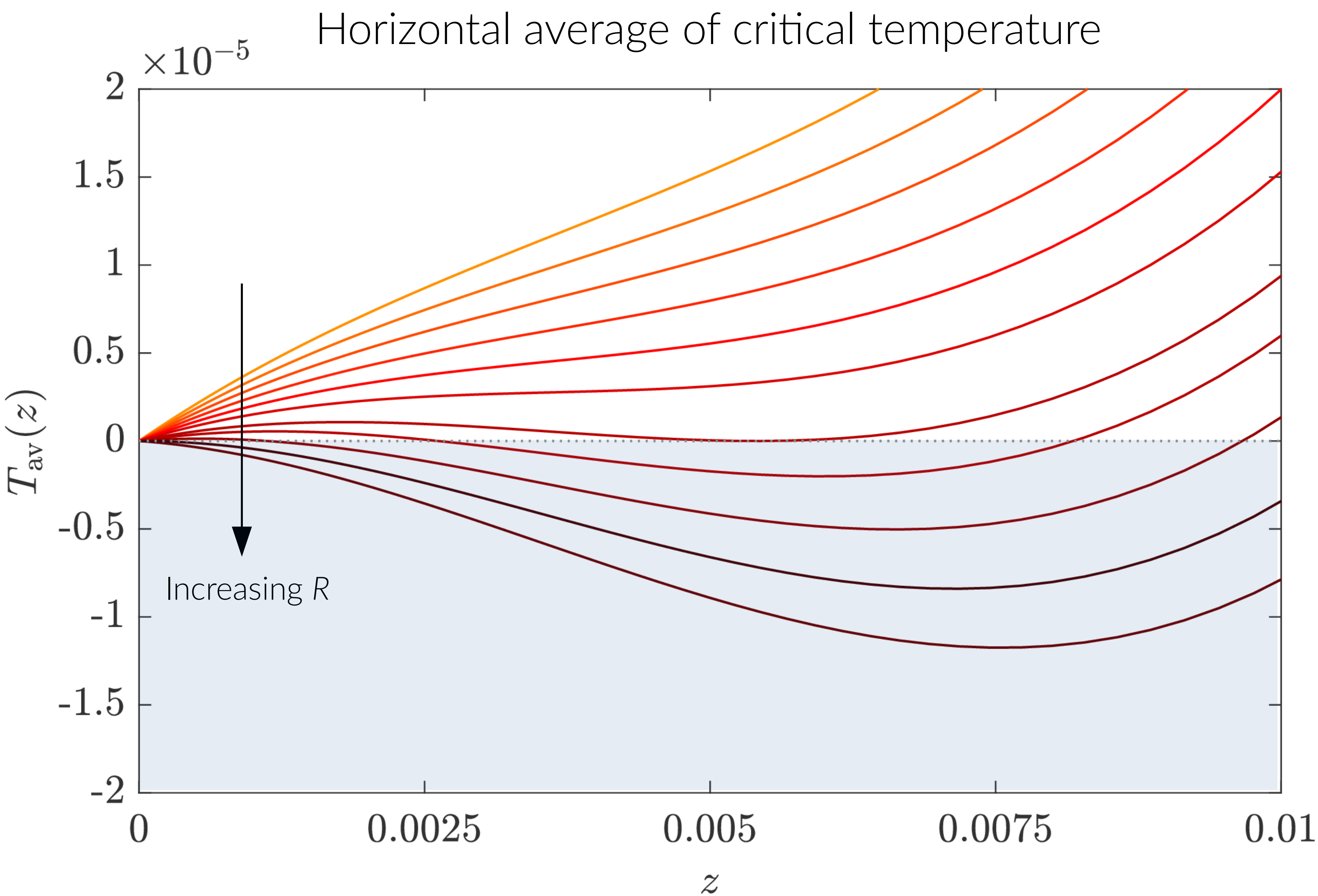

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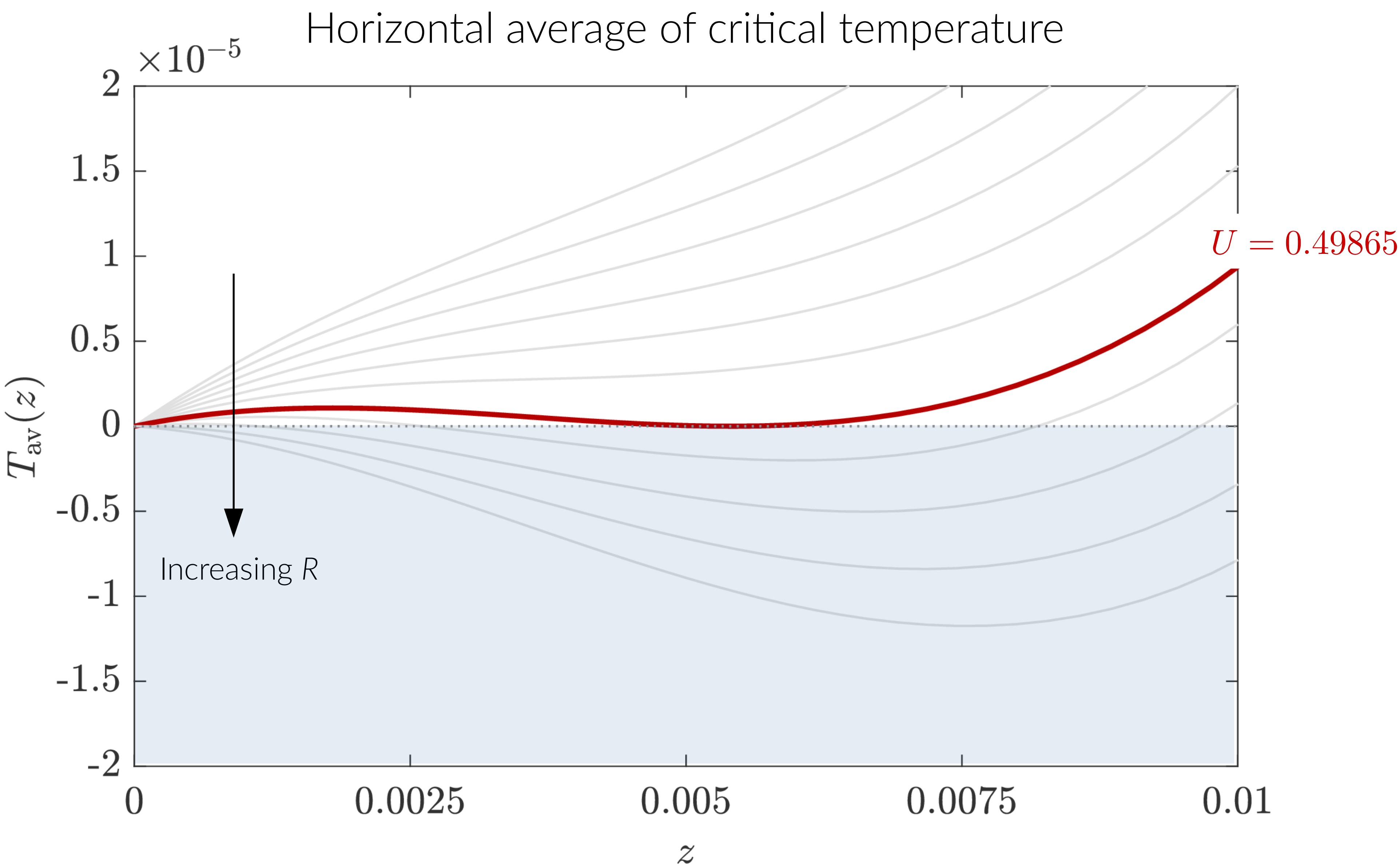

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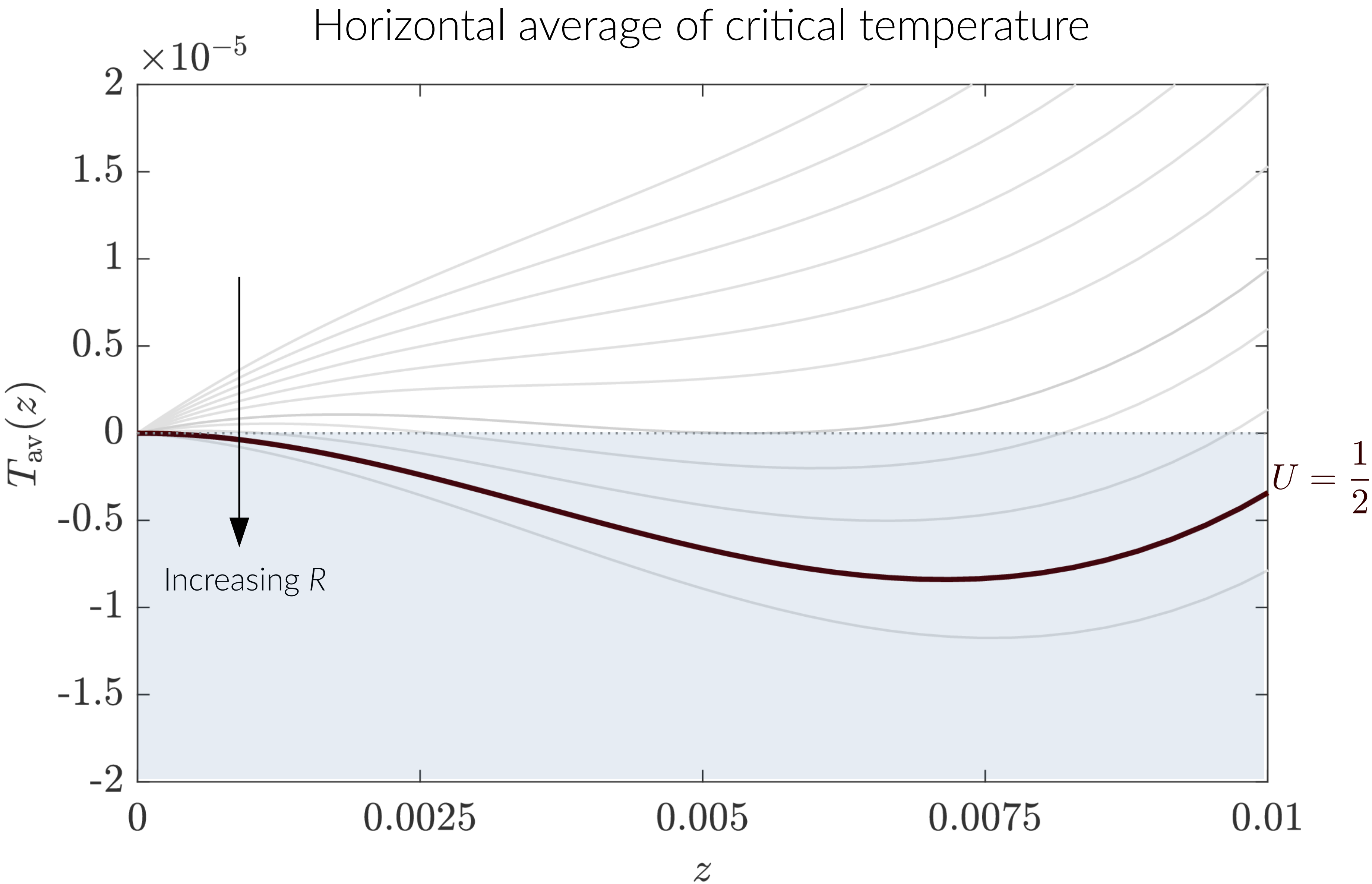


# Revised bounding framework

$$\langle wT \rangle \le \inf_{U,a,b,\psi(z)} \quad U$$

$$\text{s.t.} \quad \mathcal{S}\{\mathbf{u},T\} \ge 0 \quad \forall \mathbf{u},T : \begin{cases} \text{BCs} \\ \nabla\cdot\mathbf{u} = 0 \\ T(\mathbf{x}) \ge 0 \text{ on } \Omega \end{cases}$$

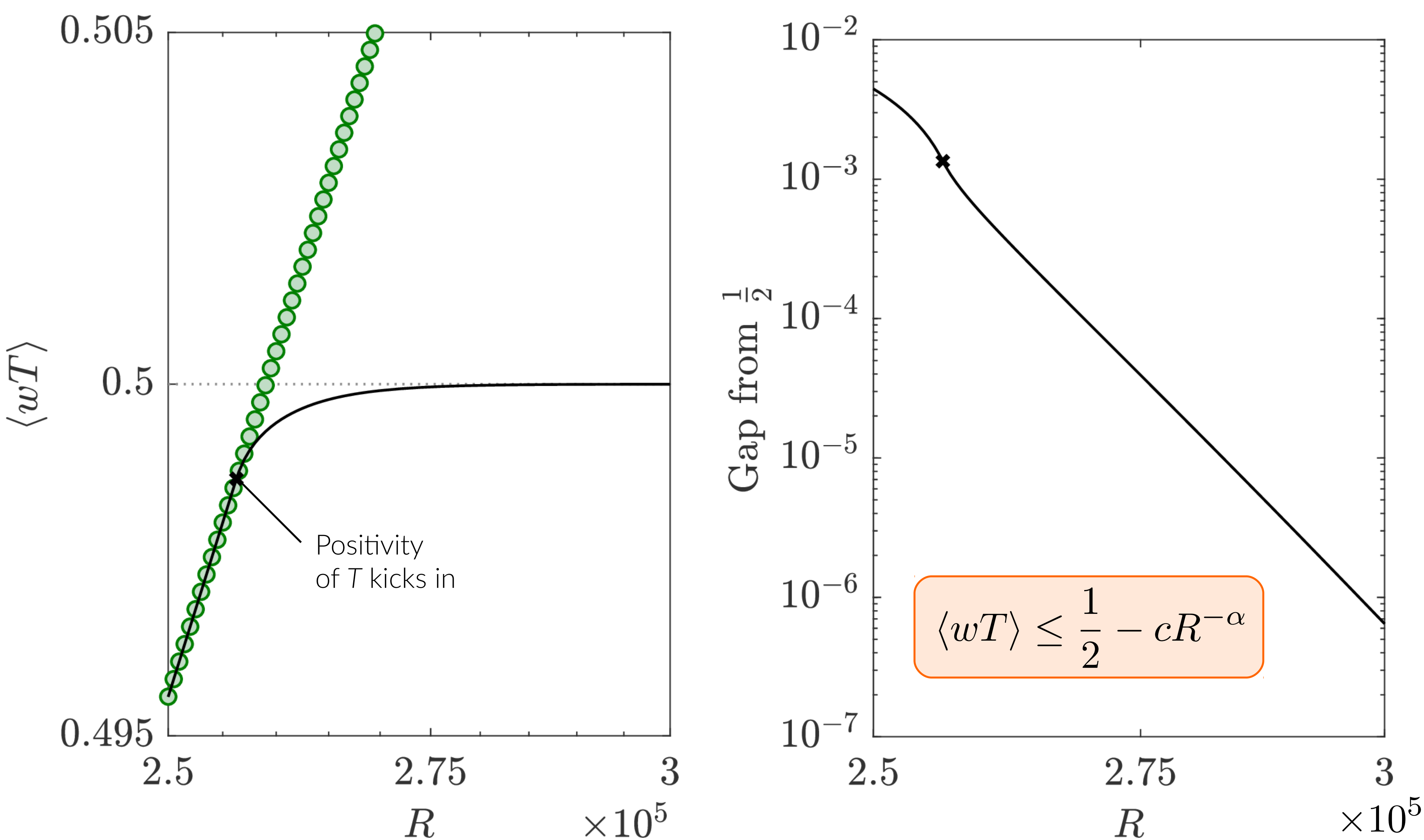
Enforce using a nonnegative Lagrange multiplier:

$$\langle wT \rangle \le \inf_{U,a,b,\psi(z),q(z)} \quad U$$

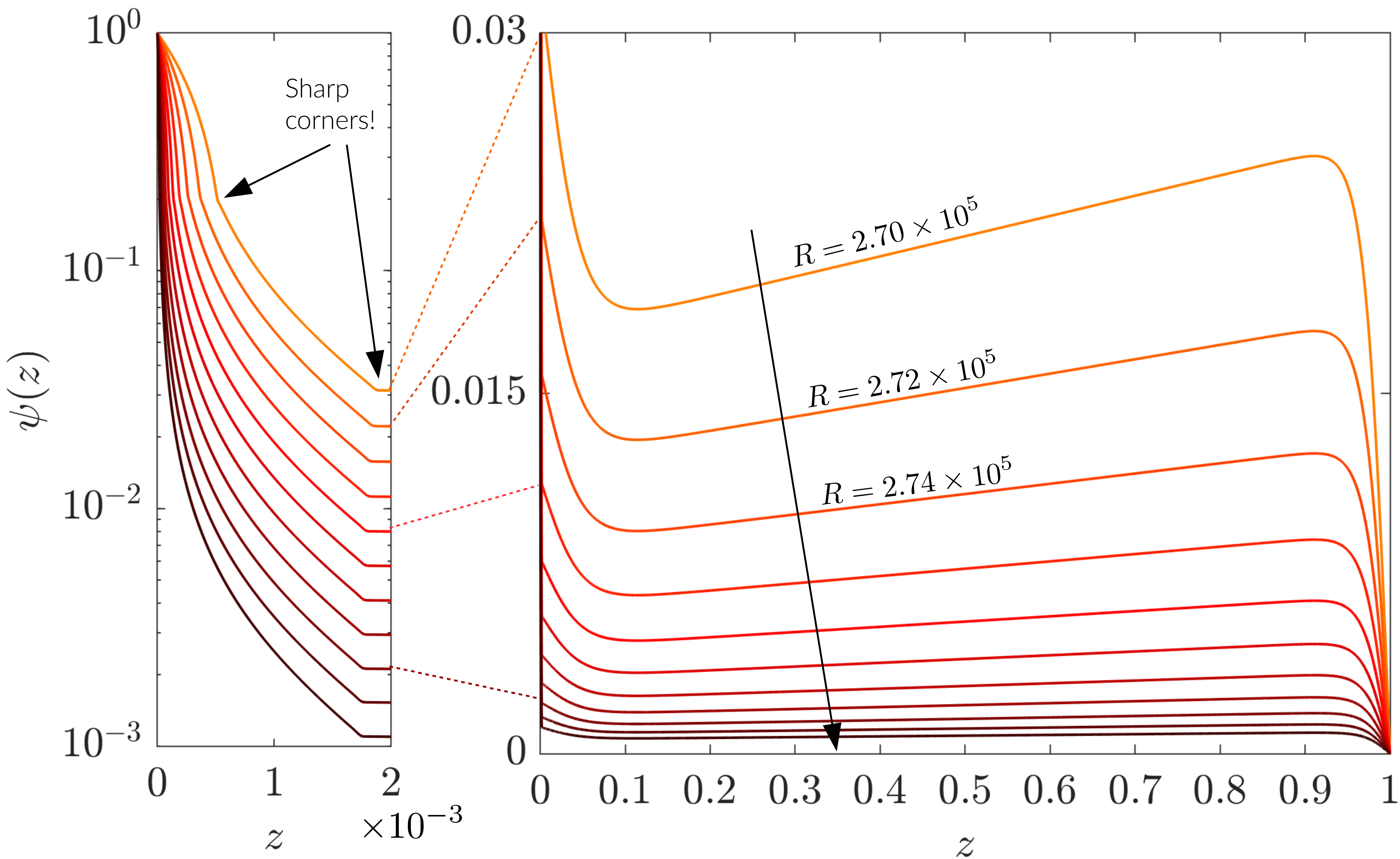
$$\text{s.t.} \quad \mathcal{S}\{\mathbf{u},T\} \ge \fint_\Omega q'(z)T\,\mathrm{d}\mathbf{x} \quad \forall \mathbf{u},T : \begin{cases} \text{BCs} \\ \nabla\cdot\mathbf{u} = 0 \end{cases}$$

$$q'(z) \ge 0 \quad \forall z \in [0,1]$$

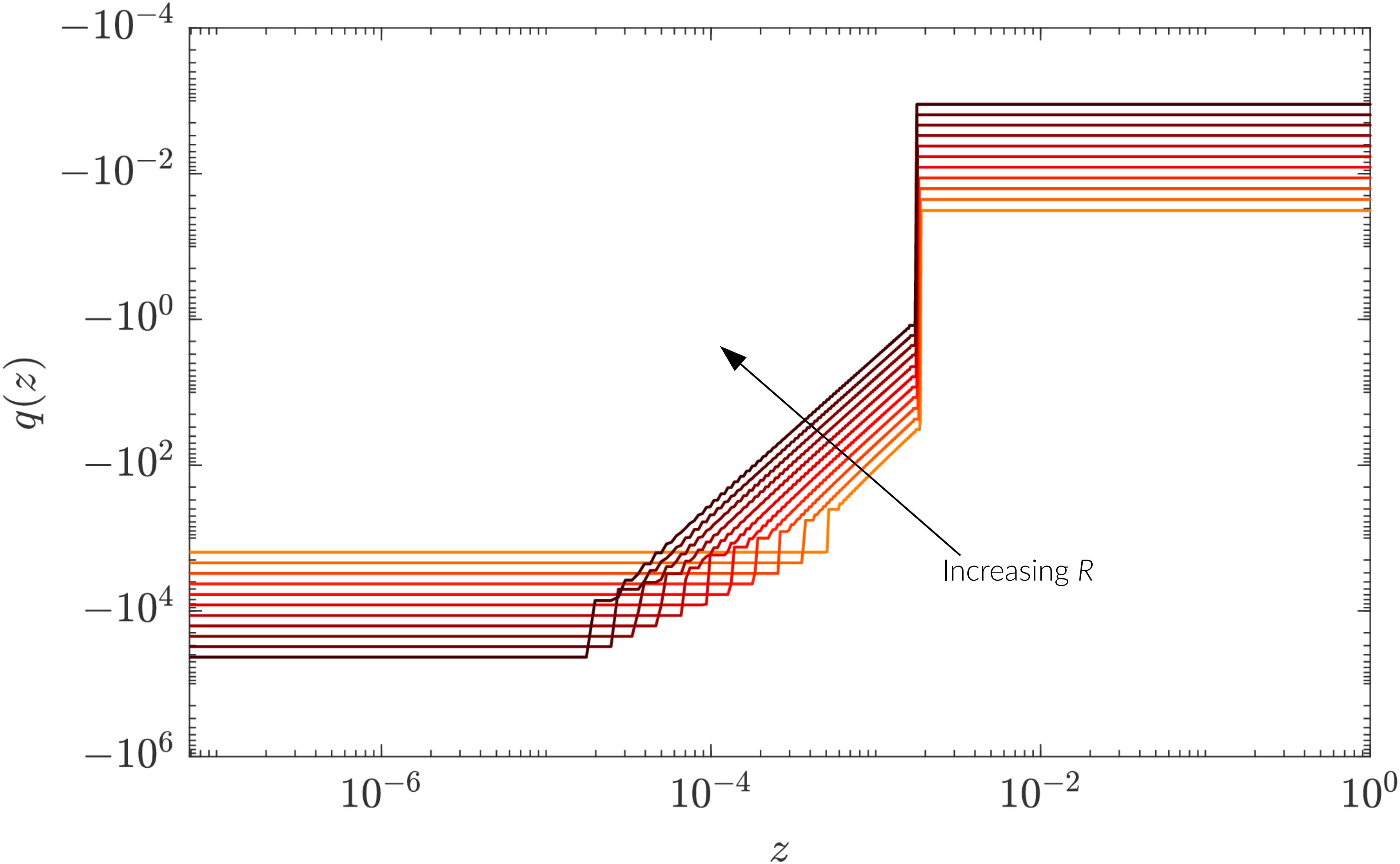
# Computational upper bounds



# Optimal $\psi(z)$



# Optimal Lagrange multipliers



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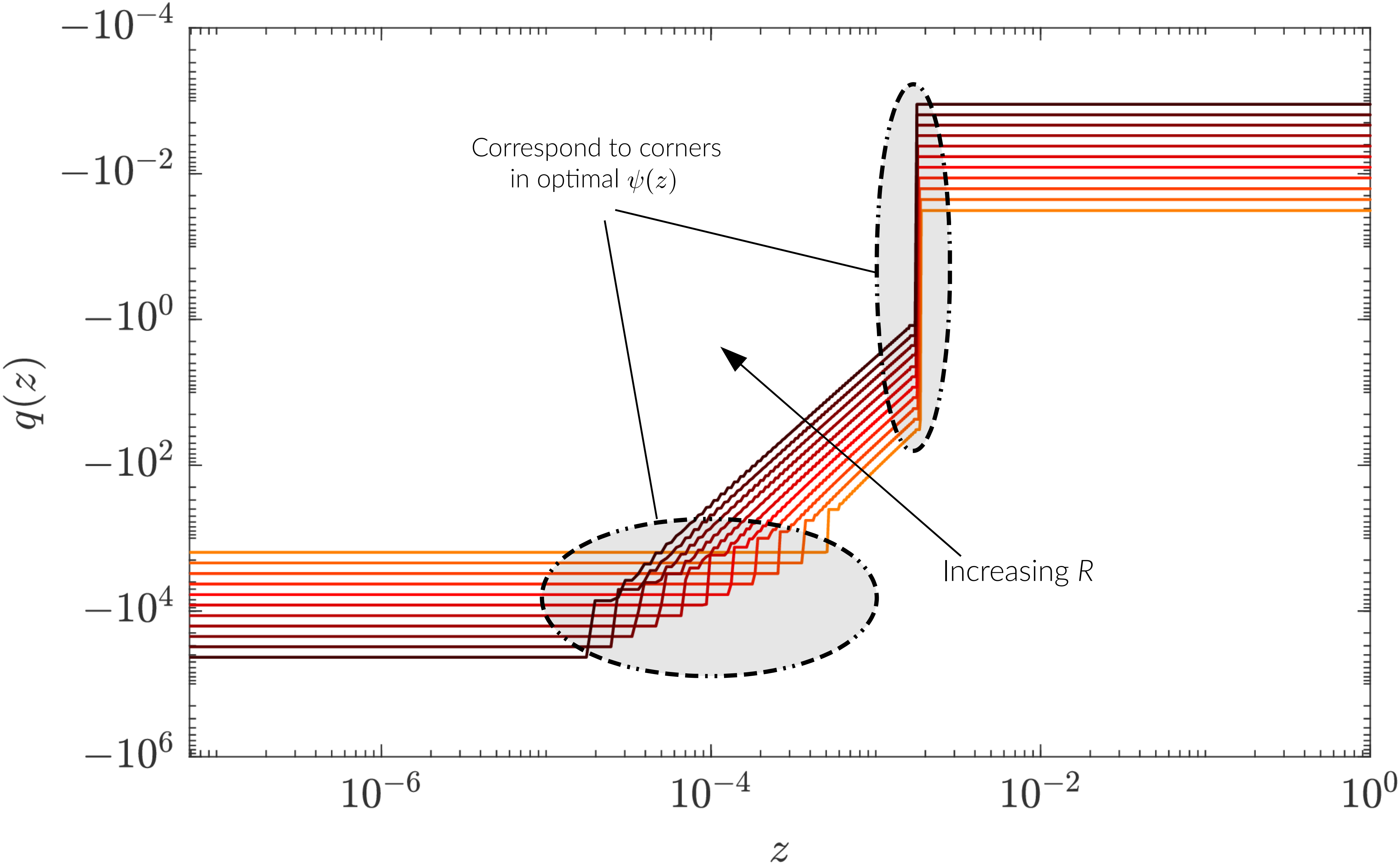
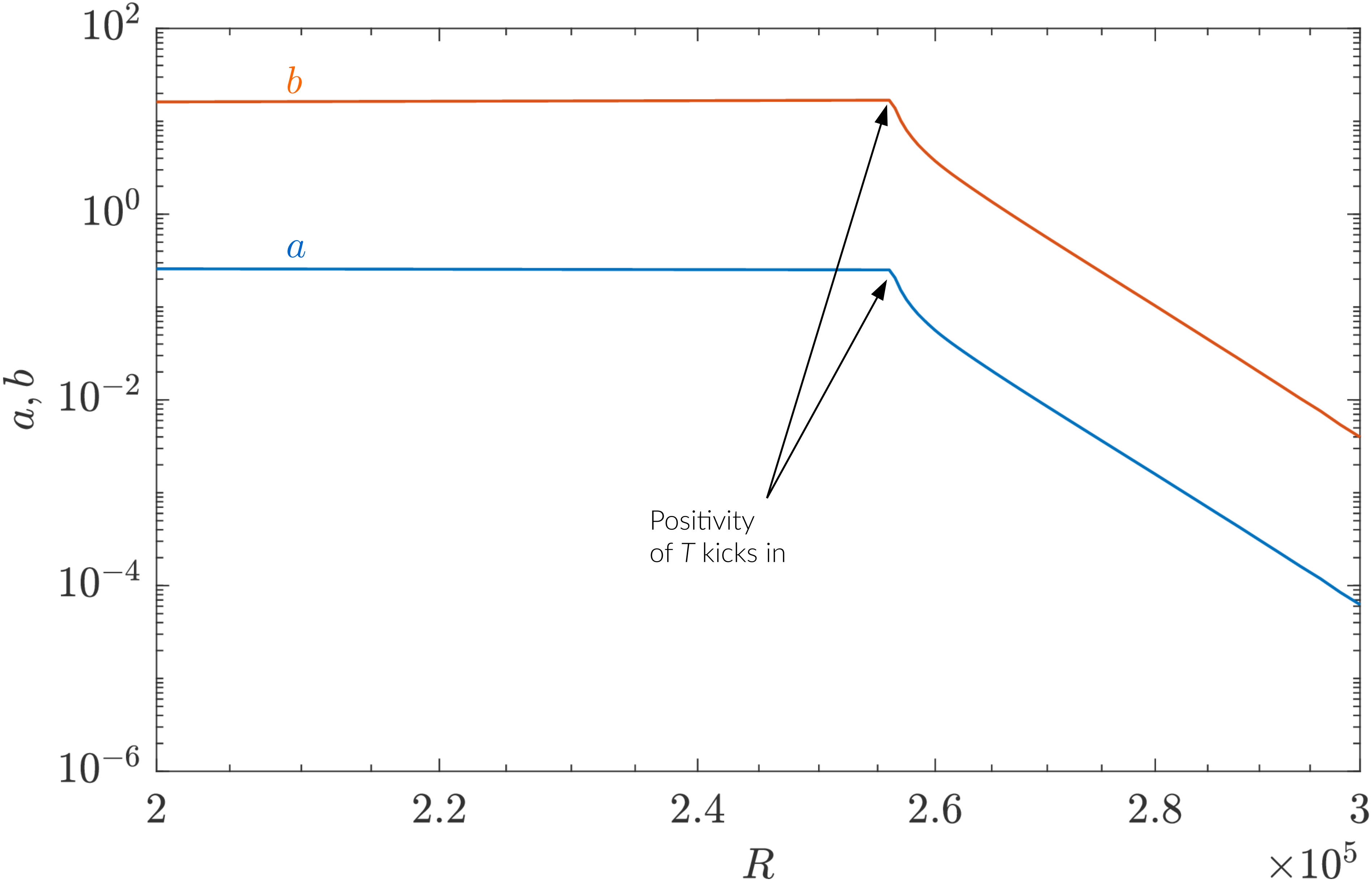
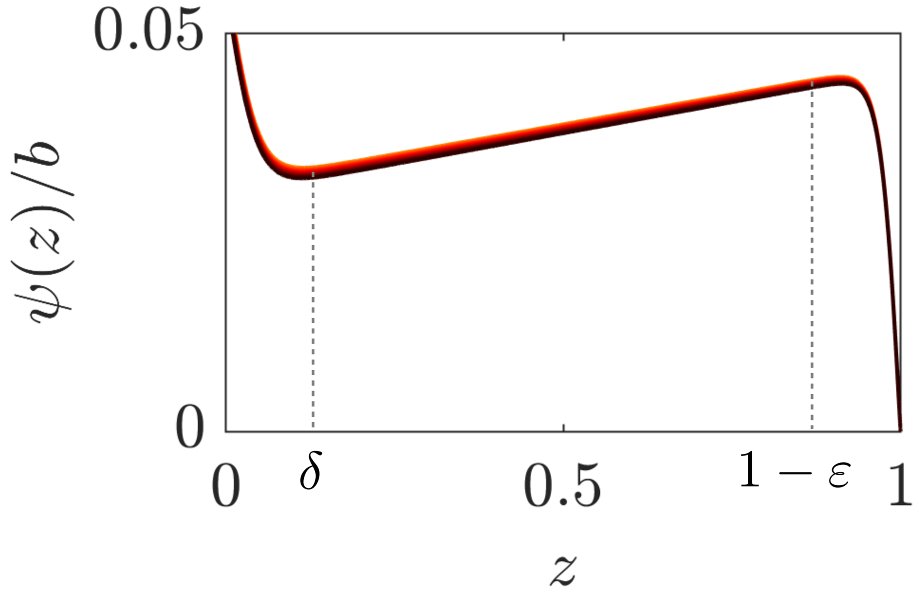
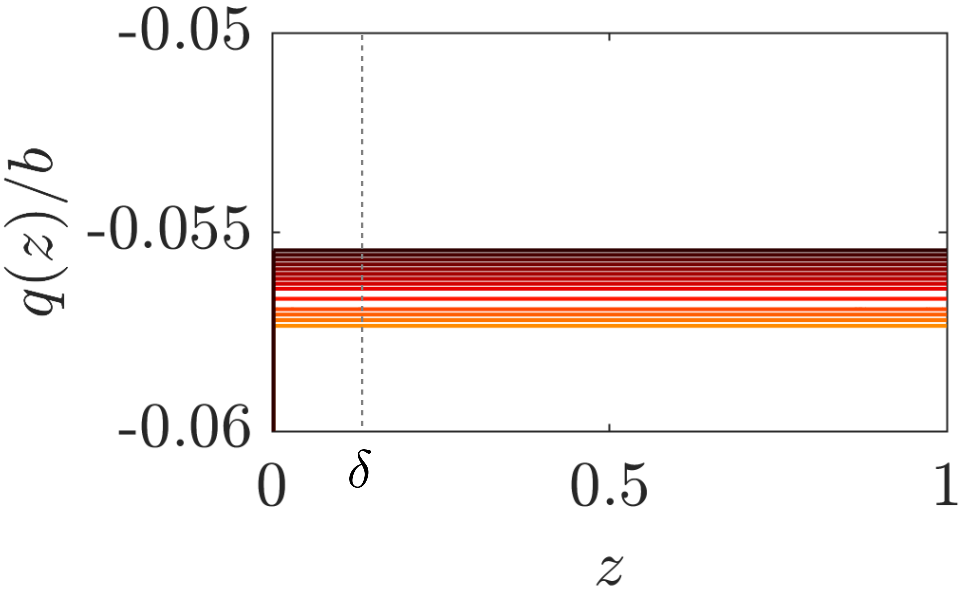


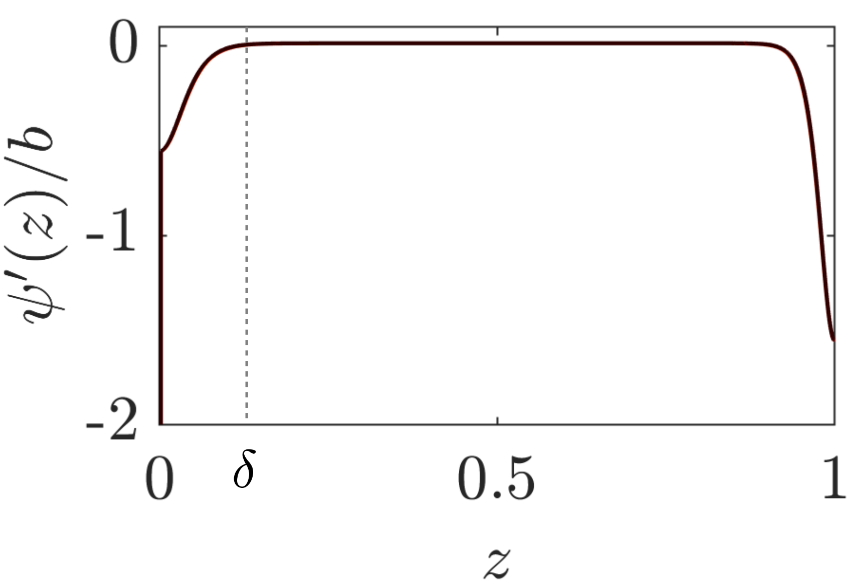
# Optimal balance parameters



# Approx. scaling with $b$ away from $z$=0




$$\|\psi\|_{L^\infty(\delta,1)} \leq \mathcal{O}(b)$$
$$\|\psi'\|_{L^\infty(\delta,1)} \leq \mathcal{O}(b)$$
$$q(z) = \text{const} \quad \text{if } z > \delta$$
$$\psi'(z) \approx \text{const} \quad \text{in the bulk}$$

$\delta =$ width of bottom boundary layer
$\varepsilon =$ width of top boundary layer

# Analysis?

$$\langle wT \rangle \leq \frac{1}{2} + \frac{1}{4b} \left\| bz - \frac{b}{2} - \psi' + q \right\|_2^2 - \int_0^1 \psi \mathrm{d}z$$

Provided that

$$\psi(0) \leq 1,$$

$$\psi(1) \leq 0,$$

$$q'(z) \geq 0,$$

$$\int_0^1 q(z)\mathrm{d}z = \psi(1) - \psi(0),$$

$$\int_0^1 \frac{2a}{R}|\partial_z w|^2 + b|\partial_z T|^2 - (a - \psi')wT\,\mathrm{d}z \geq 0 \quad \forall w, T: \text{ BCs}$$

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# Simplest idea fails!

Suppose that:

Agree well with numerics

1) $\psi(0) = 1,\ \psi(1) = 0$
2) $\|\psi\|_{L^\infty(\delta,1)},\ \|\psi'\|_{L^\infty(\delta,1)} \leq \mathcal{O}(b)$
3) $\psi'(z) \geq 0$ on an interval $(\delta, 1-\varepsilon)$
4) $\psi'(z)$ and $q(z)$ are <u>constant</u> on some interval $(\frac{1}{2} - c, \frac{1}{2} + c)$ with $c = \mathcal{O}(1)$
5) $a \leq b$
6) We use the "classical" estimates, e. g.

$$\left|\int_0^\delta (a - \psi')wT\,\mathrm{d}z\right| \leq \delta^2 \|a - \psi'\|_{L^\infty(0,\delta)}\ \|\partial_z w\|_2\ \|\partial_z T\|_2$$

$$\left|\int_{1-\varepsilon}^1 (a - \psi')wT\,\mathrm{d}z\right| \leq \varepsilon^2 \|a - \psi'\|_{L^\infty(1-\varepsilon,1)}\ \|\partial_z w\|_2\ \|\partial_z T\|_2$$

Then, the best possible bound $U^*$ that one can prove satisfies

$$U^* \geq \frac{1}{2} + b\left[\frac{c^3}{6} - \mathcal{O}(R^{-\frac{1}{4}})\right] > \frac{1}{2} \text{ if } R \gg 1$$

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# Conclusions

- Bounds on vertical heat transfer for convection with internal heating are elusive!
- "Modern background method" leads to a sensible and computationally tractable problem
- New analytical bound

$$\langle wT \rangle \leq \frac{1}{16}[8(3\sqrt{2}-4)^2]^{\frac{1}{5}} R^{\frac{1}{5}} \quad \left( \leq \frac{1}{2} \text{ if } R \leq 69\,572 \right)$$

- Positivity of temperature is necessary to obtain

$$\langle wT \rangle \leq \frac{1}{2} - cR^{-\alpha} < \frac{1}{2} \qquad \forall R$$

- Simplest type of analytical constructions fail
  a) More subtle properties of optimal solution?
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**Thank you!**

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