Heat transport efficiency beyond the so-called ‘ultimate’ regime?

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Rayleigh-Bénard convection

Enhanced heat transport:

\[ P = f(\Delta T) \]

Dimensionless parameters:

\[ Nu = \frac{PH}{\lambda \Delta T} \]

\[ Ra = \frac{\alpha g \Delta TH^3}{\kappa \nu} \]

\[ Pr = \frac{\nu}{\kappa} \]

Scaling-law that can be extrapolated to parameter values of natural flows.
Two competing predictions

- $\gamma = 1/3$  
  [Malkus (1954)]

  Marginally stable BLs:
  $\delta$ independent of $H$.  
  $P$ independent of $H$.  

  $Nu \sim Ra^{1/3}$
  all existing experimental RB data!

- $\gamma = 1/2$  
  [Spiegel, Kraichnan (1962)]

  Heat flux should be independent of molecular diffusivities $\kappa$ and $\nu$.  

  $Nu \sim Ra^{1/2} Pr^{1/2}$

  mixing-length or « ultimate » regime
Radiatively driven convection in the lab

- water + dye
- sapphire plate
- borosilicate plate
- water-cooled IR filtration stage
- water-cooled thermal screen
- high-throughput spotlight

+ uniform effective cooling associated with secular heating.
Tuning the absorption length

- Internal heat source: \( Q(z) = \frac{P}{\ell} e^{-z/\ell} \) tuned through dye concentration

- Two limiting cases

  \( \ell \ll \delta \): similar to RB

  \( \ell \gg \delta \): bypassing the BL.

we expect \( \gamma = 1/3 \).

Heat flux governed by bulk turbulence?
Nusselt vs Rayleigh

\[ \gamma = 0.31 \]
\[ \ell < 10^{-4} H \]

Experimental - RB case

[Lepot, Aumaître, Gallet, PNAS, 2018]
Nusselt vs Rayleigh

Experimental mixing-length or ultimate scaling!

\[ \gamma = 0.54 \]
\[ \ell = 0.05H \]

\[ \gamma = 0.31 \]
\[ \ell < 10^{-4}H \]

[Leport, Aumaître, Gallet, PNAS, 2018]
Nusselt vs Rayleigh

3D DNS (respective cases)

[LePot, Aumaître, Gallet, PNAS, 2018]
‘Ultimate’ in what sense?

- Our setup leads to $\text{Nu} \sim \sqrt{\text{Ra}}$: scaling relation between $P$ and $\Delta T$ does not involve the tiny molecular diffusivities $\kappa$ and $\nu$.

  ‘fully turbulent’ flow (good news for a model of geo- and astrophysical flows!).

- $\text{Nu} \lesssim \sqrt{\text{Ra}}$ is a rigorous upper bound for RB convection [Howard, Busse, Doering & Constantin, etc]. Does the bound hold for our alternate setup?

  What is the maximum heat transport efficiency $\text{Nu}$ for a given $\text{Ra}$?
Boussinesq equations

\[ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \text{Pr}(\nabla^2 \mathbf{u} + T \mathbf{e}_z) \quad \nabla \cdot \mathbf{u} = 0 \]

\[ \partial_t T + \mathbf{u} \cdot \nabla T = \nabla^2 T + \text{Ra}_P S(z) \]

\[ \text{Ra}_P = \frac{\alpha g PH^4}{\rho C \kappa^2 \nu} \]

\[ S(z) = \frac{e^{-z/\ell}}{\ell} - \left(1 - e^{-1/\ell}\right) \]

\[ \text{Pr} = \nu / \kappa \]

+ insulating, either stress-free or no-slip boundaries at \( z=0;1 \).

Emergent Rayleigh number based on rms temperature:

\[ \text{Ra} = \langle T^2 \rangle^{1/2} \quad \text{Nu} = \frac{\text{Ra}_P}{\text{Ra}} \]

Rk: The results to come also hold for the following definitions:

\[ \text{Ra}_{\text{max}} = \sqrt{\max_z \{T^2(z)\}} \quad \text{Nu}_{\text{max}} = \frac{\text{Ra}_P}{\text{Ra}_{\text{max}}} \]

(T being mean-zero)
A simple upper bound

\[ \partial_t u + (u \cdot \nabla) u = -\nabla p + \Pr (\nabla^2 u + T e_z) \]

\[ \partial_t T + u \cdot \nabla T = \nabla^2 T + \Ra P S(z) \]

Multiply by \( z \), average over space and time. Keeping only dominant terms at large \( \Ra \), for brevity:

\[ \Ra P \langle z S(z) \rangle \simeq \langle w T \rangle \leq \langle w^2 \rangle^{1/2} \Ra \quad \Longrightarrow \quad \Nu \lesssim \frac{\langle w^2 \rangle^{1/2}}{\langle z S \rangle} \]

Bound rms vertical velocity according to:

\[ \langle w^2 \rangle \leq \frac{\langle (\partial_z w)^2 \rangle}{\pi^2} \leq \frac{\langle (\nabla u)^2 \rangle}{\pi^2} = \frac{\langle w T \rangle}{\pi^2} \leq \frac{\Ra}{\pi^2} \langle w^2 \rangle^{1/2} \quad \Longrightarrow \quad \langle w^2 \rangle^{1/2} \leq \frac{\Ra}{\pi^2} \]

Finally:

\[ \Nu \lesssim \frac{\Ra}{\pi^2 \langle z S \rangle} \simeq \frac{2}{\pi^2} \Ra \quad \text{for} \quad \ell \ll 1 \]

much greater than \( \Ra^{1/2} \)!
Beyond the ‘ultimate’ heat transport efficiency?

For convection driven by internal heat sources and sinks:

\[ \text{Nu} \lesssim \frac{2}{\pi^2} \text{Ra} \]

• Is this the true scaling behavior of the system? Or is it just a limitation of the bounding procedure?

• Is it sharp? Can we exhibit solutions that display this scaling behavior?

• If so, what does the flow look like?
A simple example

Consider the simpler source/sink distribution \( S(z) = \sqrt{2} \sin(2\pi z) \) with stress-free boundary conditions.

Look for 2D steady solutions using a streamfunction:

\[
J(\psi, \Delta \psi) = \Pr \left[ \Delta^2 \psi + \partial_x T \right] \\
J(\psi, T) = \Delta T + Ra_P \cdot S(z)
\]

[Introduce asymptotic expansion for strong heating \( Ra_P \gg 1 \)]

\[
\psi = Ra_P^{1/2} \psi_0 + \psi_1 + Ra_P^{-1/2} \psi_2 + \ldots \\
T = Ra_P^{1/2} T_0 + T_1 + Ra_P^{-1/2} T_2 + \ldots
\]

[Chini & Cox 2009] [Sondak, Smith, Waleffe 2015] [Wen et al., 2020]
A simple example

To highest order in $Ra_P$

$$J(\psi_0, \Delta\psi_0) = 0, \quad J(\psi_0, T_0) = S(z) = \sqrt{2} \sin(2\pi z)$$

Solution of the form:

$$\begin{align*}
\psi_0 &= \psi_m \sin(\pi x) \sin(\pi z) \\
T_0 &= T_m \cos(\pi x) \sin(\pi z)
\end{align*}$$

with $T_m = 2\sqrt{2}/\pi^2 \psi_m$

Solvability condition at next order (or simply power integral) yields:

$$\psi_m = \pm 2^{-1/4} \pi^{-5/2} \quad T_m = \pm 2^{7/4} \pi^{1/2}$$

On this asymptotic branch of solutions,

$$Ra = \langle T^2 \rangle^{1/2} \sim Ra_P^{1/2} = \sqrt{Nu Ra} \quad \Rightarrow \quad Nu \sim Ra$$
A challenge is to prove the stability of this asymptotic solution up to arbitrary Rayleigh number in 2D...

[Miquel, Bouillaut, Lepot, Gallet, PRF, 2020]
Heat transport beyond the ‘ultimate’ regime

\[ N_u = \frac{R_a}{\left(\frac{2^3}{2\pi}\right)} \]

- \( T|_{z=0:1} = 0 \)
- \( \partial_z T|_{z=0:1} = 0 \)
- Upper bound

[Miquel, Bouillaut, Lepot, Gallet, PRF, 2020]
Focusing on turbulent flows

This is all fine, but \( \text{Nu} \sim \text{Ra} \) is associated with laminar flows, whereas the 3D solutions and laboratory flows are strongly turbulent.

Can we compute an upper bound that focuses on turbulent flows instead?

Define turbulent: ‘zeroth law of turbulence’

Dissipation coefficient

\[
C = \frac{\nu \left\langle |\nabla \mathbf{u}|^2 \right\rangle H}{\left\langle u^2 \right\rangle^{3/2}} \quad \text{const.} \neq 0
\]

\[
\text{Re} \quad \rightarrow \quad \infty
\]
Back to the bound

Combining the definition of $C$ with the energy power integral.

$$\langle w^2 \rangle^{1/2} \leq \sqrt{\frac{Pr Ra}{C}}$$

Substitution into $\text{Nu} \lesssim \frac{\langle w^2 \rangle^{1/2}}{\langle zS \rangle}$ yields:

$$\text{Nu} \lesssim \sqrt{\frac{Pr Ra}{C}}$$

On a turbulent branch of solutions, characterized by an asymptotically constant dissipation coefficient $C$, the heat transport efficiency cannot exceed the ‘ultimate’ scaling-law.
Summary

Numerically and experimentally, we measure $C \simeq 2$, independent of $\ell$. 

$\text{Nu} \sim \text{Ra}$

No solutions

non-turbulent flows only

$\text{Nu} \sim \sqrt{\text{Ra}}$

turbulent flows

data:

$\ell/H = 0.1$

$\ell/H = 0.05$
Conclusions

• Our setup leads to $\text{Nu} \sim \sqrt{\text{Ra}}$ in the laboratory and in 3D DNS.

• Rigorous upper bound $\text{Nu} \lesssim \text{Ra}$ over all solutions.

  heat transport beyond the ultimate efficiency is achieved by analytical laminar solutions, realized in 2D stress-free DNS.

• The better upper bound $\text{Nu} \lesssim \sqrt{\text{Ra}}$ holds for any turbulent branch of solutions.

  (any branch of solutions with an asymptotically constant dissipation coefficient).

• Any scaling exponent greater than $1/2$ is necessarily associated with non-turbulent flows.