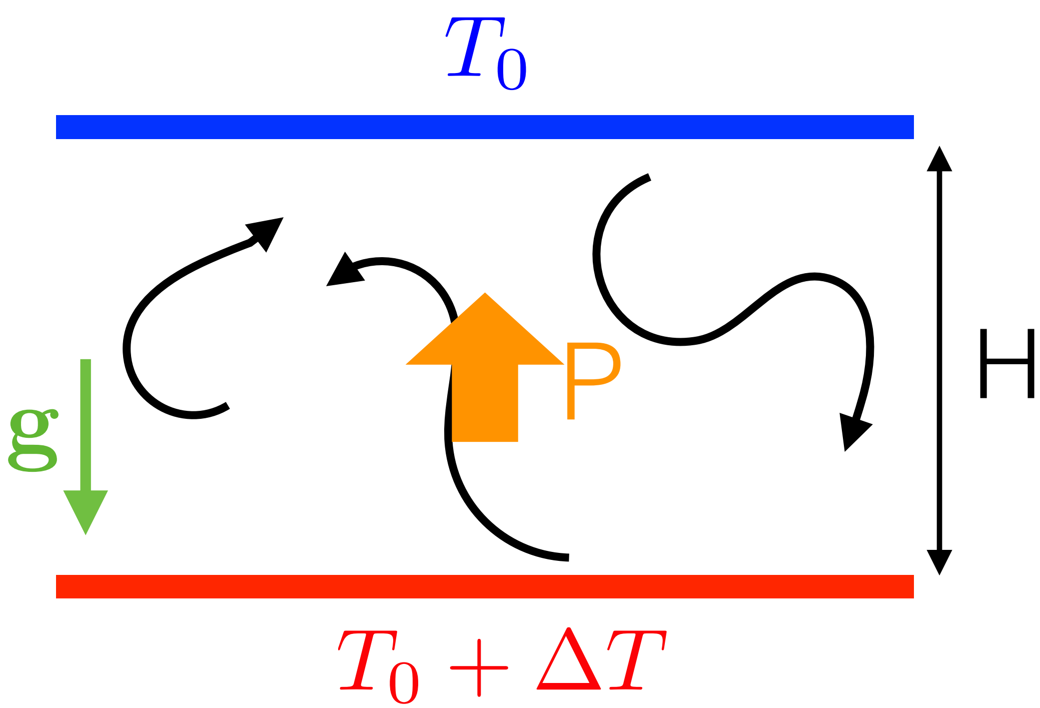
Heat transport efficiency beyond the so-called ‘ultimate’ regime?

Basile Gallet, B. Flesselles, V. Bouillaut, S. Lepot, B. Miquel, S. Aumaître, **SPEC, CEA Saclay, FRANCE.**

Rayleigh-Bénard convection



Enhanced heat transport:

$$P = f(\Delta T)\ ?$$

Dimensionless parameters:

$$Nu = \frac{PH}{\lambda \Delta T}$$

$$Ra = \frac{\alpha g \Delta T H^3}{\kappa \nu}$$

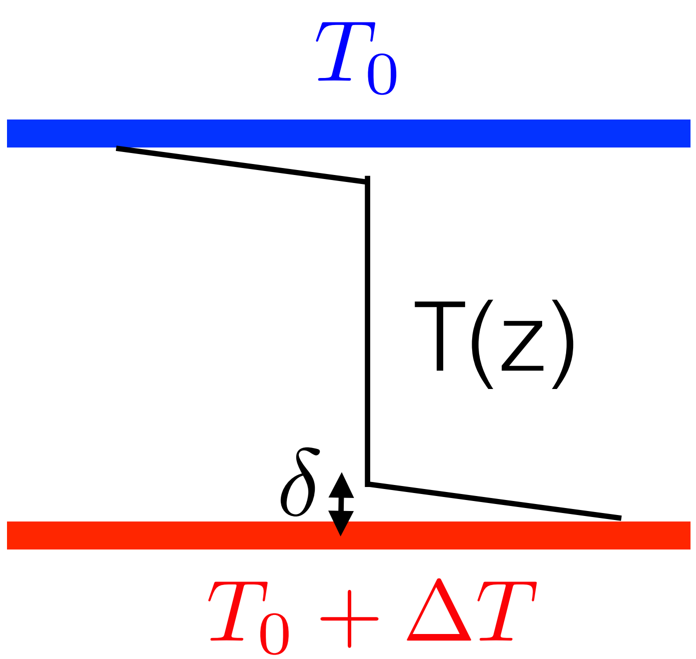
$$Pr = \frac{\nu}{\kappa}$$

$$\boxed{Nu \sim Ra^{\gamma}\, Pr^{\chi}}$$

Scaling-law that can be extrapolated to parameter values of natural flows.

Two competing predictions

- $\gamma = 1/3$ [Malkus (1954)]



Marginally stable BLs:
δ independent of H.
P independent of H.

$$Nu \sim Ra^{1/3}$$

all existing
experimental RB data!

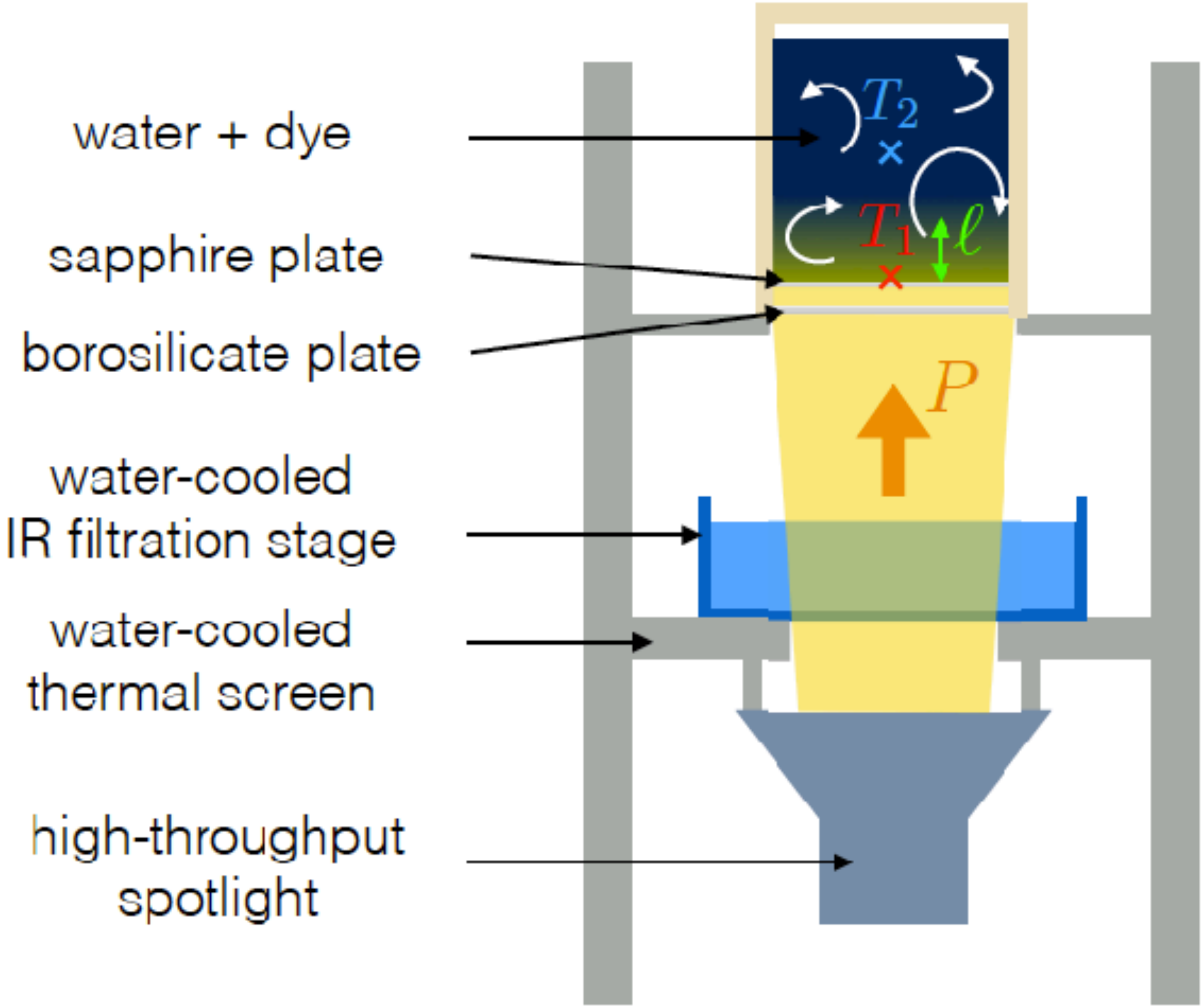
- $\gamma = 1/2$ [Spiegel, Kraichnan (1962)]

Heat flux should be independent of
molecular diffusivities κ and ν.

$$Nu \sim Ra^{1/2}\, Pr^{1/2}$$

mixing-length or
« ultimate » regime

Radiatively driven convection in the lab

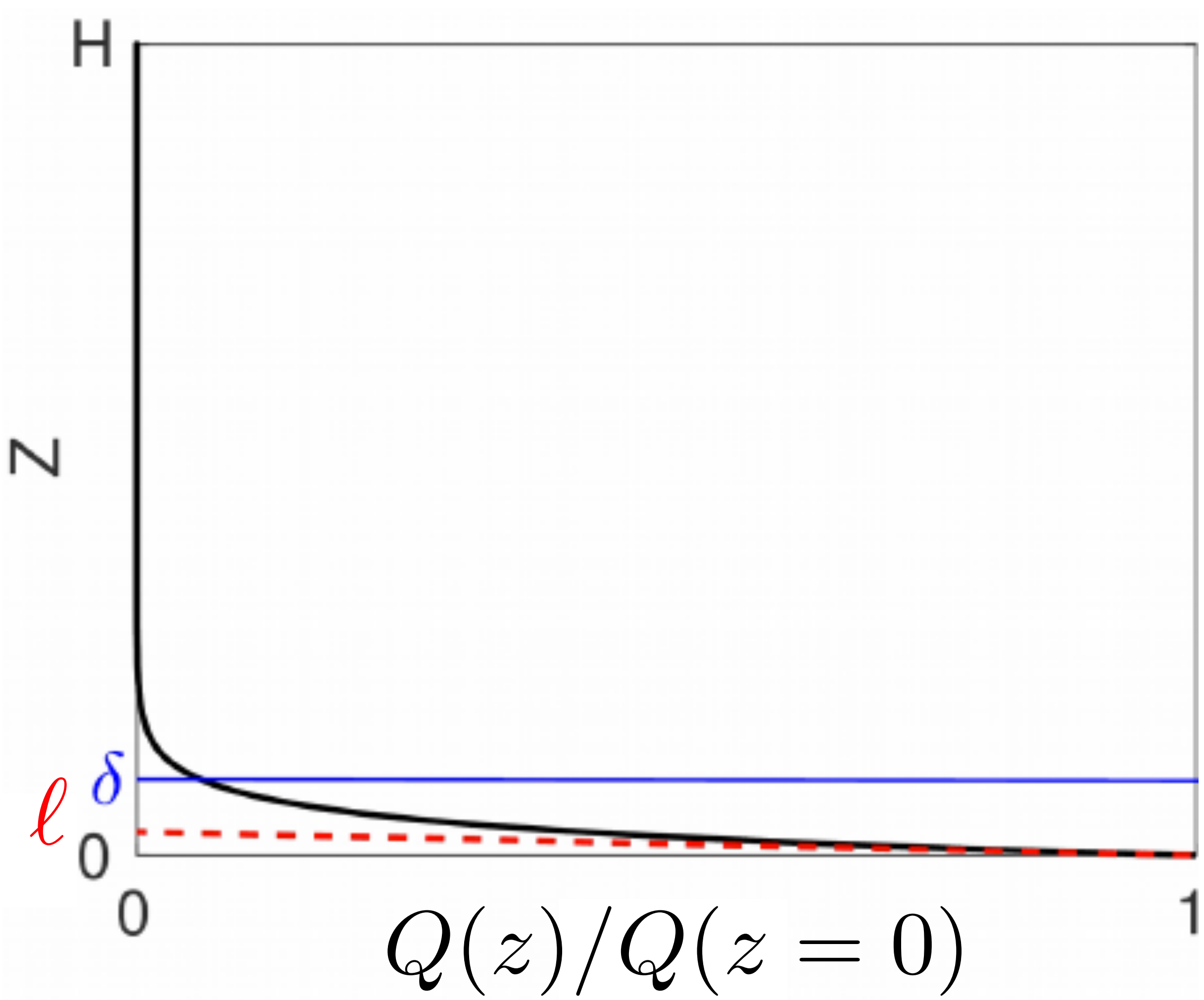


+ uniform effective cooling associated with secular heating.

Tuning the absorption length

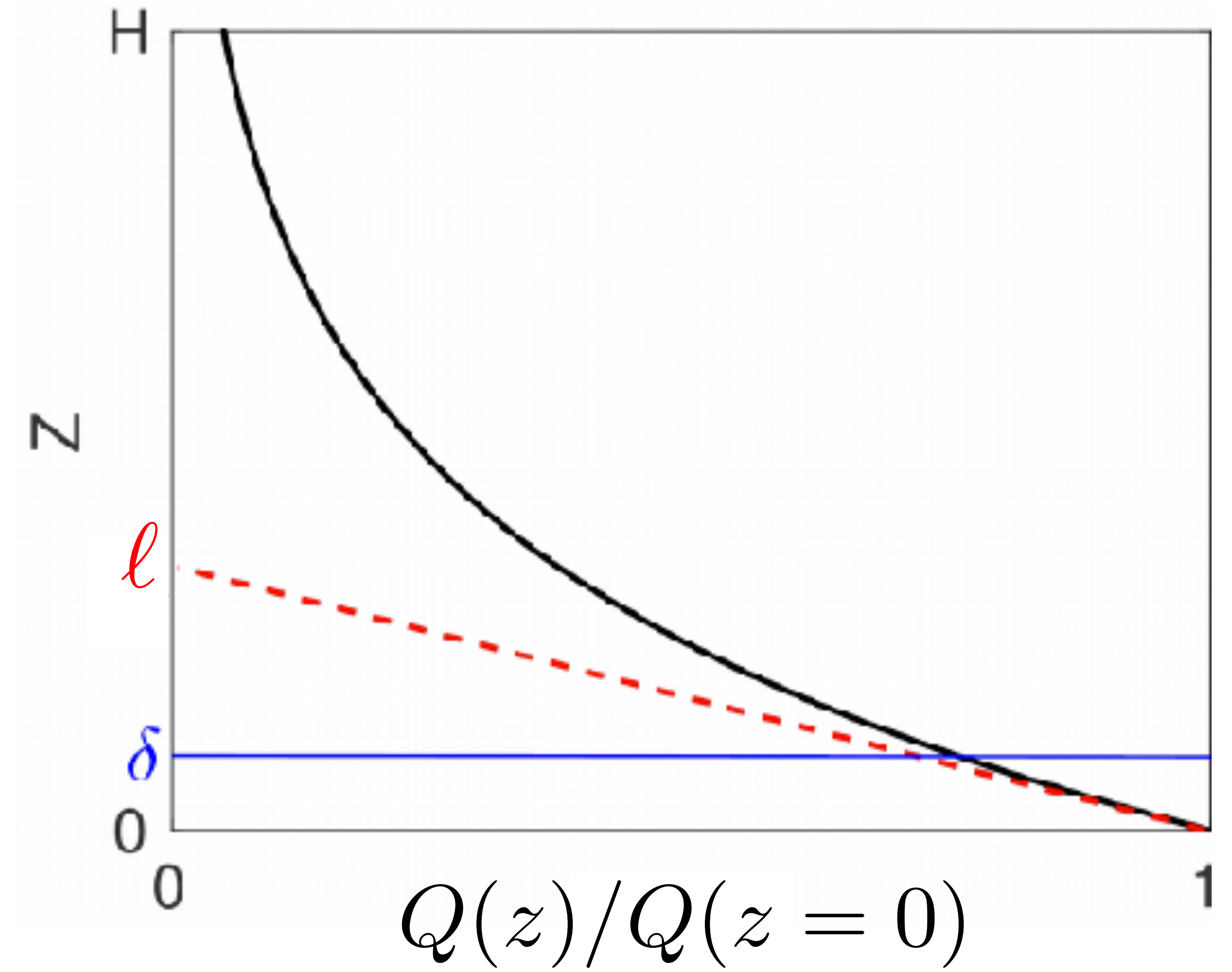
- Internal heat source: $Q(z) = \frac{P}{\ell} e^{-z/\ell}$ tuned through dye concentration
- Two limiting cases

$\ell \ll \delta$: similar to RB



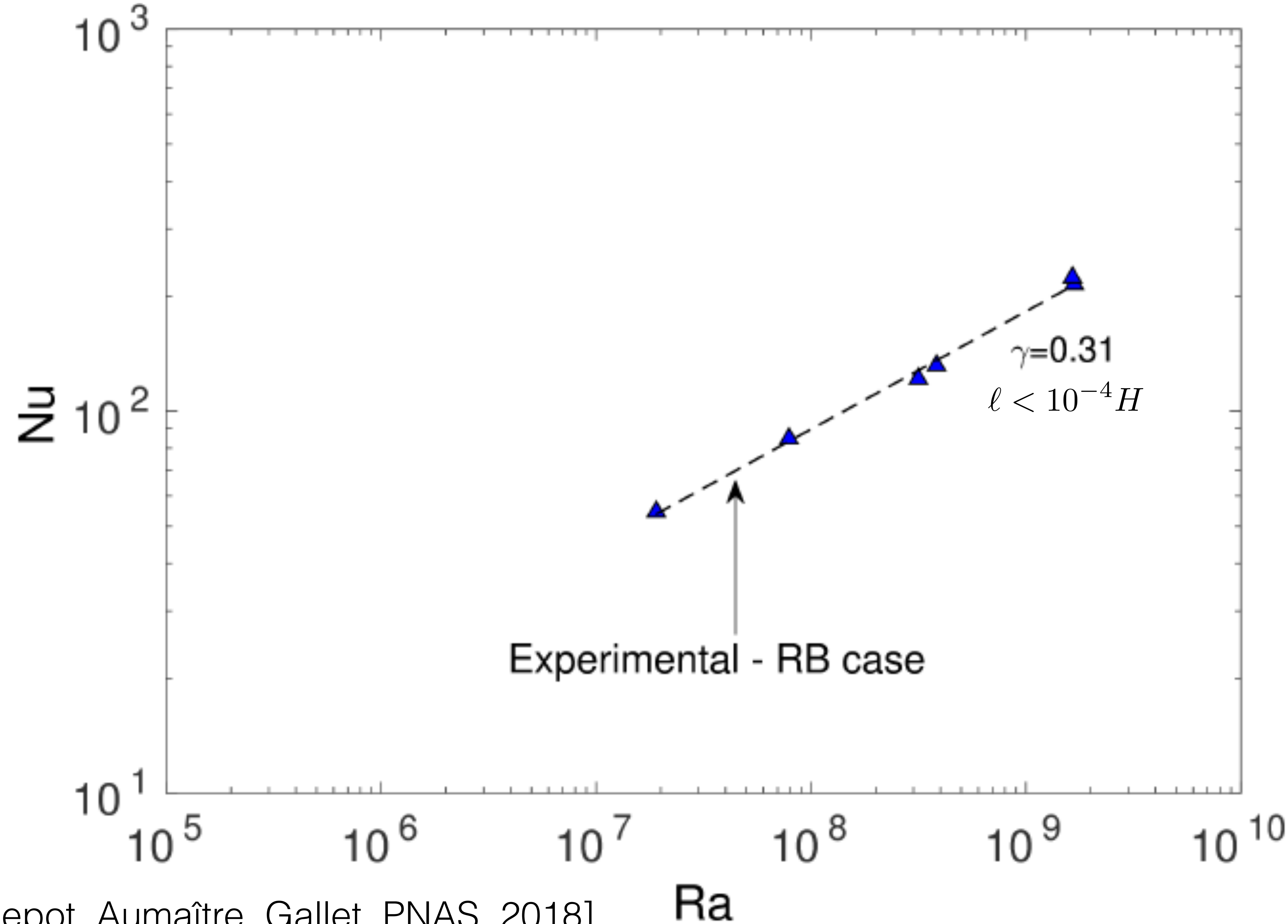
we expect $\gamma = 1/3$.

$\ell \gg \delta$: bypassing the BL.



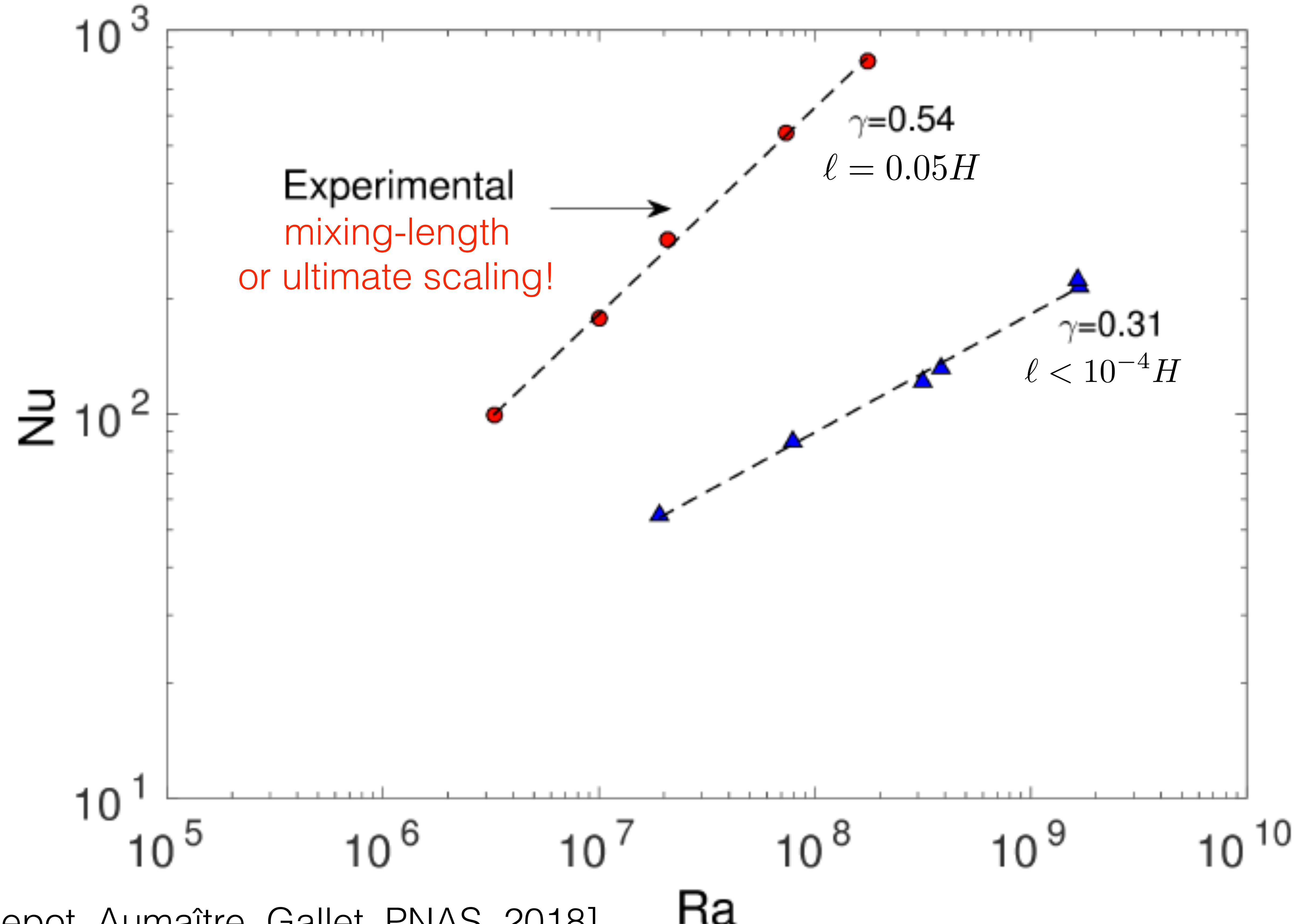
Heat flux governed by bulk turbulence?

Nusselt vs Rayleigh



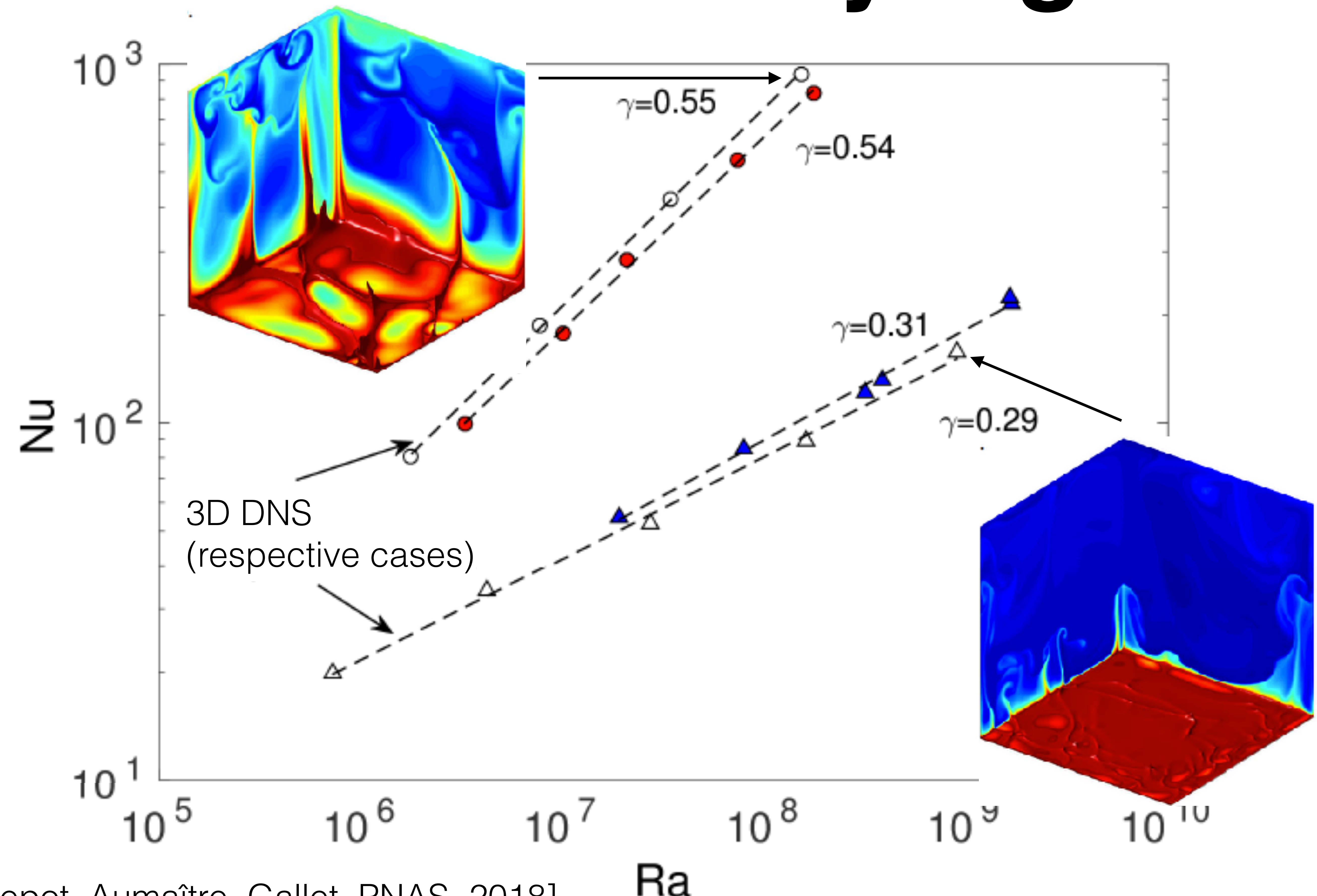
[Lepot, Aumaître, Gallet, PNAS, 2018]

Nusselt vs Rayleigh



[Lepot, Aumaître, Gallet, PNAS, 2018]

Nusselt vs Rayleigh



[Lepot, Aumaître, Gallet, PNAS, 2018]

‘Ultimate’ in what sense?

- Our setup leads to $\mathrm{Nu} \sim \sqrt{\mathrm{Ra}}$: scaling relation between P and ΔT does not involve the tiny molecular diffusivities κ and ν.

➡ **‘fully turbulent’** flow (good news for a model of geo- and astrophysical flows!).

- $\mathrm{Nu} \lesssim \sqrt{Ra}$ is a rigorous upper bound for RB convection [Howard, Busse, Doering & Constantin, etc]. Does the bound hold for our alternate setup?

➡ **What is the maximum heat transport efficiency Nu for a given Ra?**

Boussinesq equations

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \mathrm{Pr}(\boldsymbol{\nabla}^2\mathbf{u} + T\mathbf{e}_z) \qquad \boldsymbol{\nabla}\cdot\mathbf{u} = 0$$

$$\partial_t T + \mathbf{u}\cdot\boldsymbol{\nabla} T = \boldsymbol{\nabla}^2 T + \mathrm{Ra}_P S(z)$$

$$\mathrm{Ra}_P = \frac{\alpha g P H^4}{\rho C \kappa^2 \nu} \qquad S(z) = \frac{e^{-z/\ell}}{\ell} - \left(1 - e^{-1/\ell}\right)$$

$$\mathrm{Pr} = \nu/\kappa$$

+ insulating, either stress-free or no-slip boundaries at z=0;1.

Emergent Rayleigh number based on rms temperature:

$$\mathrm{Ra} = \left\langle T^2 \right\rangle^{1/2} \qquad \mathrm{Nu} = \mathrm{Ra}_P/\mathrm{Ra}$$

Rk: The results to come also hold for the following definitions:

$$\mathrm{Ra}_{\max} = \sqrt{\max_z\{\overline{T^2}(z)\}} \qquad \mathrm{Nu}_{\max} = \mathrm{Ra}_P/\mathrm{Ra}_{\max}$$

(T being mean-zero)

A simple upper bound

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \mathrm{Pr}(\boldsymbol{\nabla}^2\mathbf{u} + T\mathbf{e}_z)$$

$$\partial_t T + \mathbf{u}\cdot\boldsymbol{\nabla} T = \boldsymbol{\nabla}^2 T + \mathrm{Ra}_P S(z)$$

Multiply by z, average over space and time.
Keeping only dominant terms at large Ra, for brevity:

$$\mathrm{Ra}_P \langle zS(z)\rangle \simeq \langle wT\rangle \le \left\langle w^2\right\rangle^{1/2}\mathrm{Ra} \quad \Rightarrow \quad \mathrm{Nu} \lesssim \frac{\left\langle w^2\right\rangle^{1/2}}{\langle zS\rangle}$$

Bound rms vertical velocity according to:

$$\left\langle w^2\right\rangle \le \frac{\left\langle(\partial_z w)^2\right\rangle}{\pi^2} \le \frac{\left\langle(\boldsymbol{\nabla}\mathbf{u})^2\right\rangle}{\pi^2} = \frac{\langle wT\rangle}{\pi^2} \le \frac{\mathrm{Ra}}{\pi^2}\left\langle w^2\right\rangle^{1/2} \quad \Rightarrow \quad \left\langle w^2\right\rangle^{1/2} \le \frac{\mathrm{Ra}}{\pi^2}$$

Finally:

$$\boxed{\mathrm{Nu} \lesssim \frac{\mathrm{Ra}}{\pi^2\langle zS\rangle} \simeq \frac{2}{\pi^2}\,\mathrm{Ra}} \quad \text{for} \quad \ell \ll 1$$

much greater than $\mathrm{Ra}^{1/2}$!

Beyond the 'ultimate' heat transport efficiency?

For convection driven by internal heat sources and sinks:

$$\mathrm{Nu} \lesssim \frac{2}{\pi^2}\ \mathrm{Ra}$$

- Is this the true scaling behavior of the system? Or is it just a limitation of the bounding procedure?
- Is it sharp? Can we exhibit solutions that display this scaling behavior?
- If so, what does the flow look like?

A simple example

Consider the simpler source/sink distribution $S(z) = \sqrt{2}\sin(2\pi z)$ with stress-free boundary conditions.

Look for 2D steady solutions using a streamfunction:

$$J(\psi, \Delta\psi) = \mathrm{Pr}\ [\Delta^2\psi + \partial_x T]$$
$$J(\psi, T) = \Delta T + \mathrm{Ra}_P\ S(z)$$

[Chini & Cox 2009]
[Sondak, Smith, Waleffe 2015]
[Wen et al., 2020]

Introduce asymptotic expansion for strong heating $Ra_P \gg 1$

$$\psi = \mathrm{Ra}_P{}^{1/2}\psi_0 + \psi_1 + \mathrm{Ra}_P{}^{-1/2}\psi_2 + \dots$$
$$T = \mathrm{Ra}_P{}^{1/2}T_0 + T_1 + \mathrm{Ra}_P{}^{-1/2}T_2 + \dots$$

A simple example

To highest order in Ra_P

$$J(\psi_0, \Delta\psi_0) = 0\,, \qquad J(\psi_0, T_0) = S(z) = \sqrt{2}\sin(2\pi z)$$

Solution of the form:

$$\left.\begin{aligned} \psi_0 &= \psi_m \sin(\pi x)\sin(\pi z) \\ T_0 &= T_m \cos(\pi x)\sin(\pi z) \end{aligned}\right\} \text{ with } T_m = 2\sqrt{2}/\pi^2\psi_m$$

Solvability condition at next order (or simply power integral) yields:

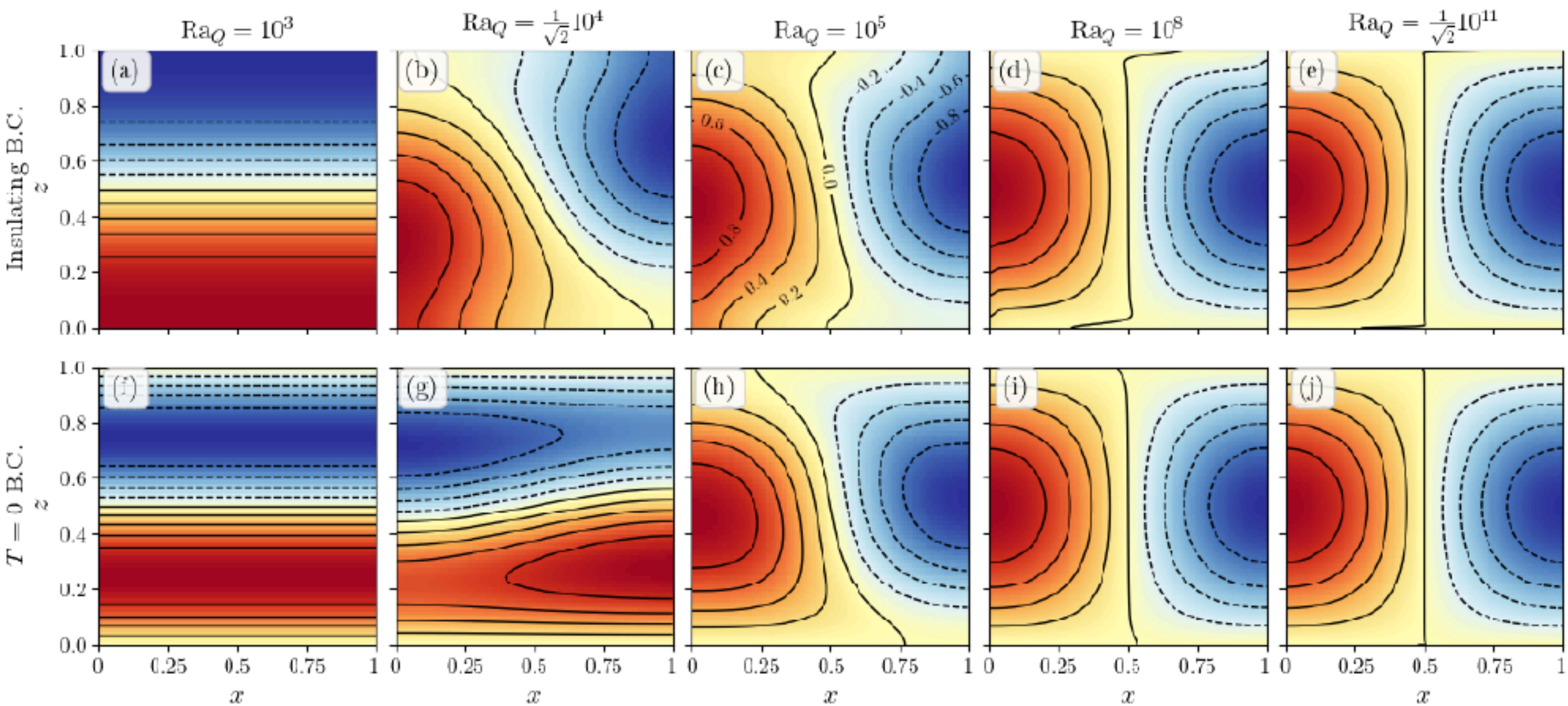
$$\psi_m = \pm 2^{-1/4}\pi^{-5/2} \qquad T_m = \pm 2^{7/4}\pi^{1/2}$$

On this asymptotic branch of solutions,

$$\mathrm{Ra} = \langle T^2 \rangle^{1/2} \sim \mathrm{Ra}_P^{1/2} = \sqrt{\mathrm{Nu}\,\mathrm{Ra}} \;\Rightarrow\; \boxed{\mathrm{Nu} \sim \mathrm{Ra}}$$

Stable and realized in 2D!

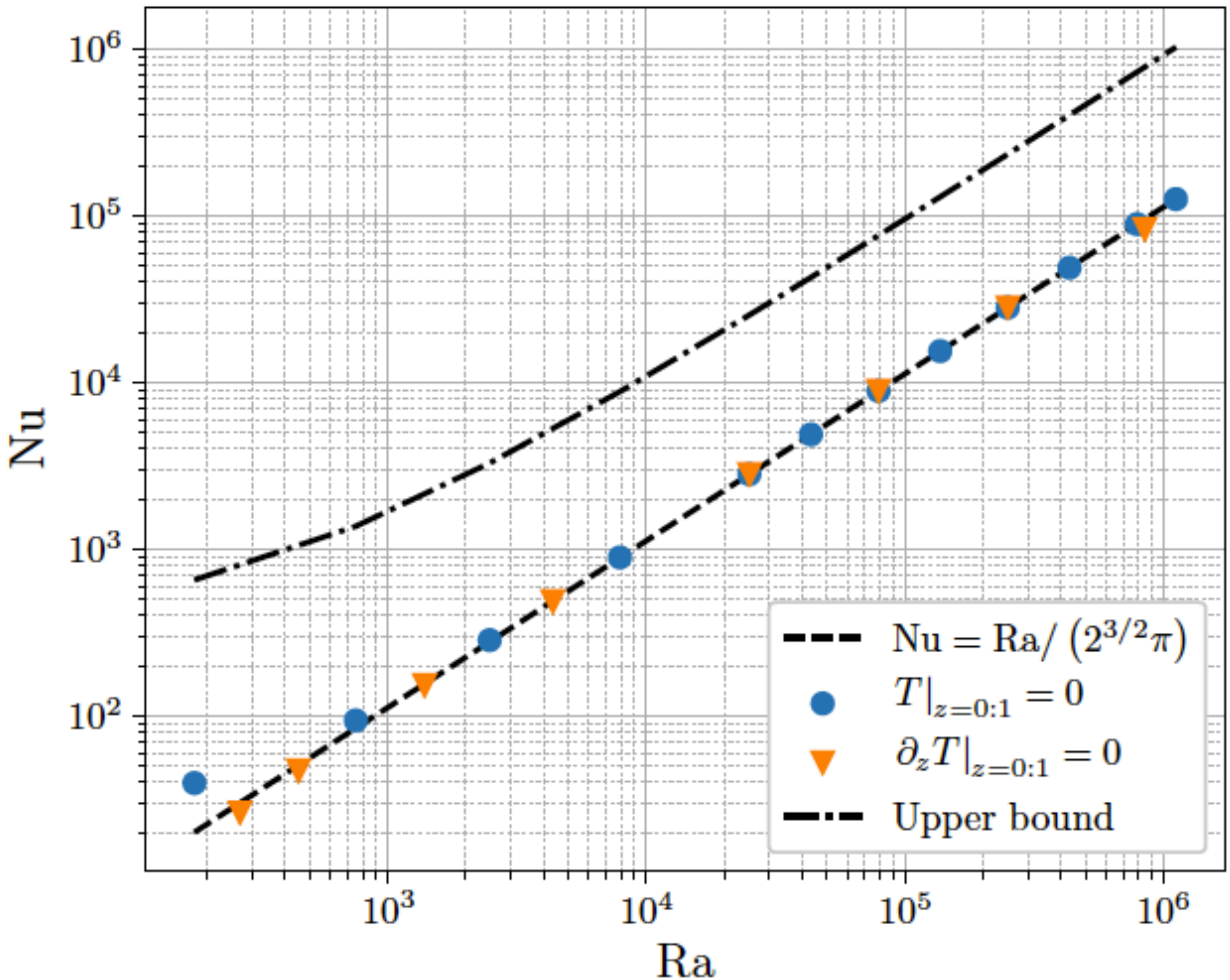
Temperature field from 2D DNS using spectral solver Coral [B. Miquel]



A challenge is to prove the stability of this asymptotic solution up to arbitrary Rayleigh number in 2D…

[Miquel, Bouillaut, Lepot, Gallet, PRF, 2020]

Heat transport beyond the ‘ultimate’ regime



[Miquel, Bouillaut, Lepot, Gallet, PRF, 2020]

Focusing on turbulent flows

This is all fine, but $\mathrm{Nu} \sim \mathrm{Ra}$ is associated with laminar flows, whereas the 3D solutions and laboratory flows are strongly turbulent.

Can we compute an upper bound that focuses on turbulent flows instead?

➡ Define turbulent: ‘zeroth law of turbulence’

Dissipation coefficient $\mathcal{C} = \dfrac{\nu \left\langle |\boldsymbol{\nabla} \mathbf{u}|^2 \right\rangle H}{\left\langle \mathbf{u}^2 \right\rangle^{3/2}} \longrightarrow \mathrm{const.} \neq 0$

$$\mathrm{Re} \longrightarrow \infty$$

Back to the bound

Combining the definition of $\mathcal{C}$ with the energy power integral.

$$\left\langle w^2 \right\rangle^{1/2} \leq \sqrt{\frac{\mathrm{Pr}\,\mathrm{Ra}}{\mathcal{C}}}$$

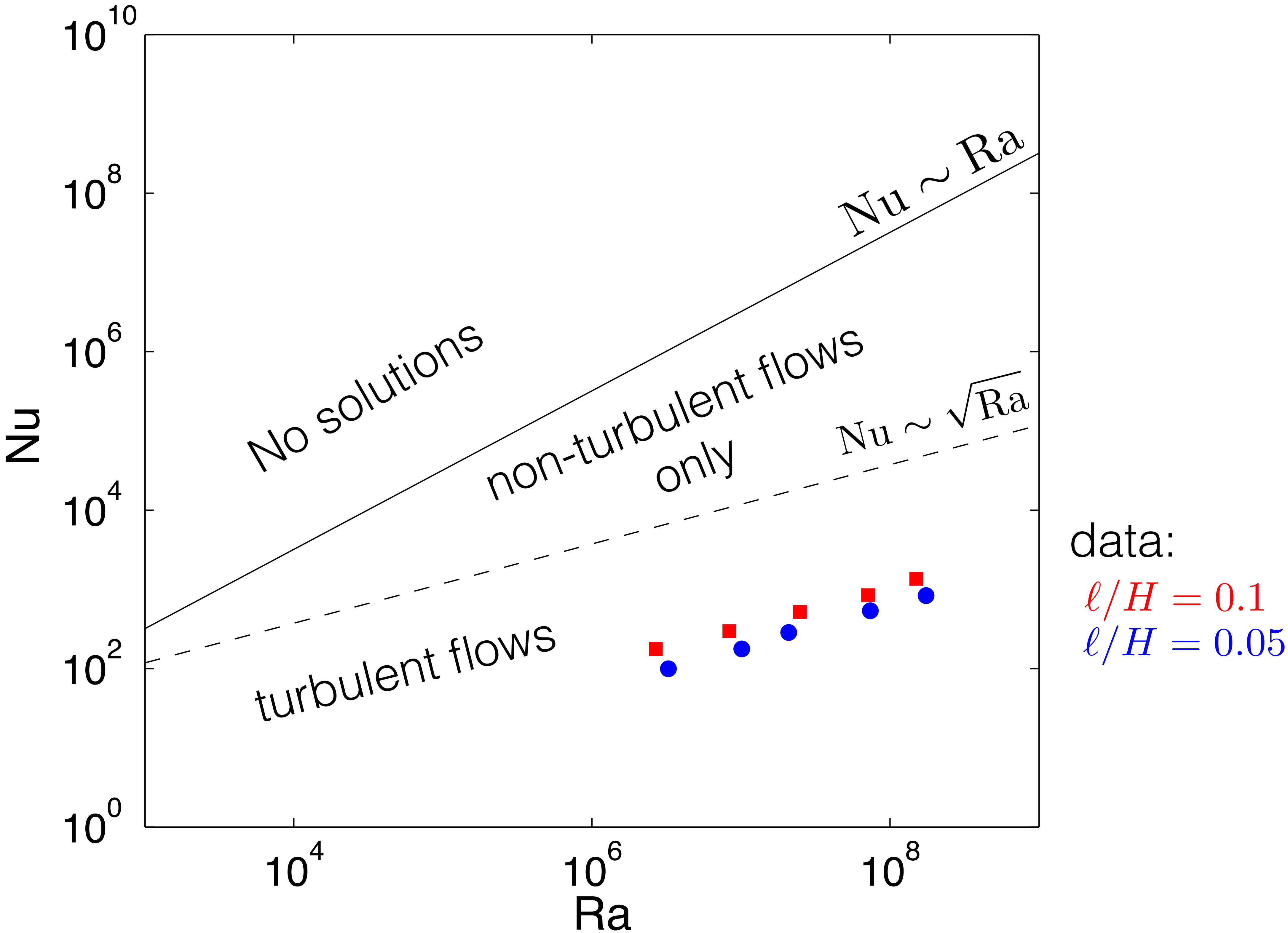
Substitution into $\mathrm{Nu} \lesssim \frac{\left\langle w^2 \right\rangle^{1/2}}{\langle zS \rangle}$ yields:

$$\boxed{\mathrm{Nu} \lesssim \sqrt{\frac{\mathrm{Pr}\,\mathrm{Ra}}{\mathcal{C}}}}$$

On a turbulent branch of solutions, characterized by an asymptotically constant dissipation coefficient $\mathcal{C}$, **the heat transport efficiency cannot exceed the ‘ultimate’ scaling-law**.

Summary

Numerically and experimentally, we measure $\mathcal{C} \simeq 2$, independent of ℓ.



Conclusions

- Our setup leads to $\mathrm{Nu} \sim \sqrt{\mathrm{Ra}}$ in the laboratory and in 3D DNS.

- Rigorous upper bound $\mathrm{Nu} \lesssim \mathrm{Ra}$ over all solutions.

  ➡ heat transport beyond the ultimate efficiency is achieved by analytical laminar solutions, realized in 2D stress-free DNS.

- The better upper bound $\mathrm{Nu} \lesssim \sqrt{\mathrm{Ra}}$ holds for any turbulent branch of solutions.

  (any branch of solutions with an asymptotically constant dissipation coefficient).

- Any scaling exponent greater than 1/2 is necessarily associated with non-turbulent flows.