IPAM : Transport and Mixing in Complex and Turbulent Flows Jan 11-14 2021

Exhausting the background approach for bounding the heat flux in Rayleigh Benard convection

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&

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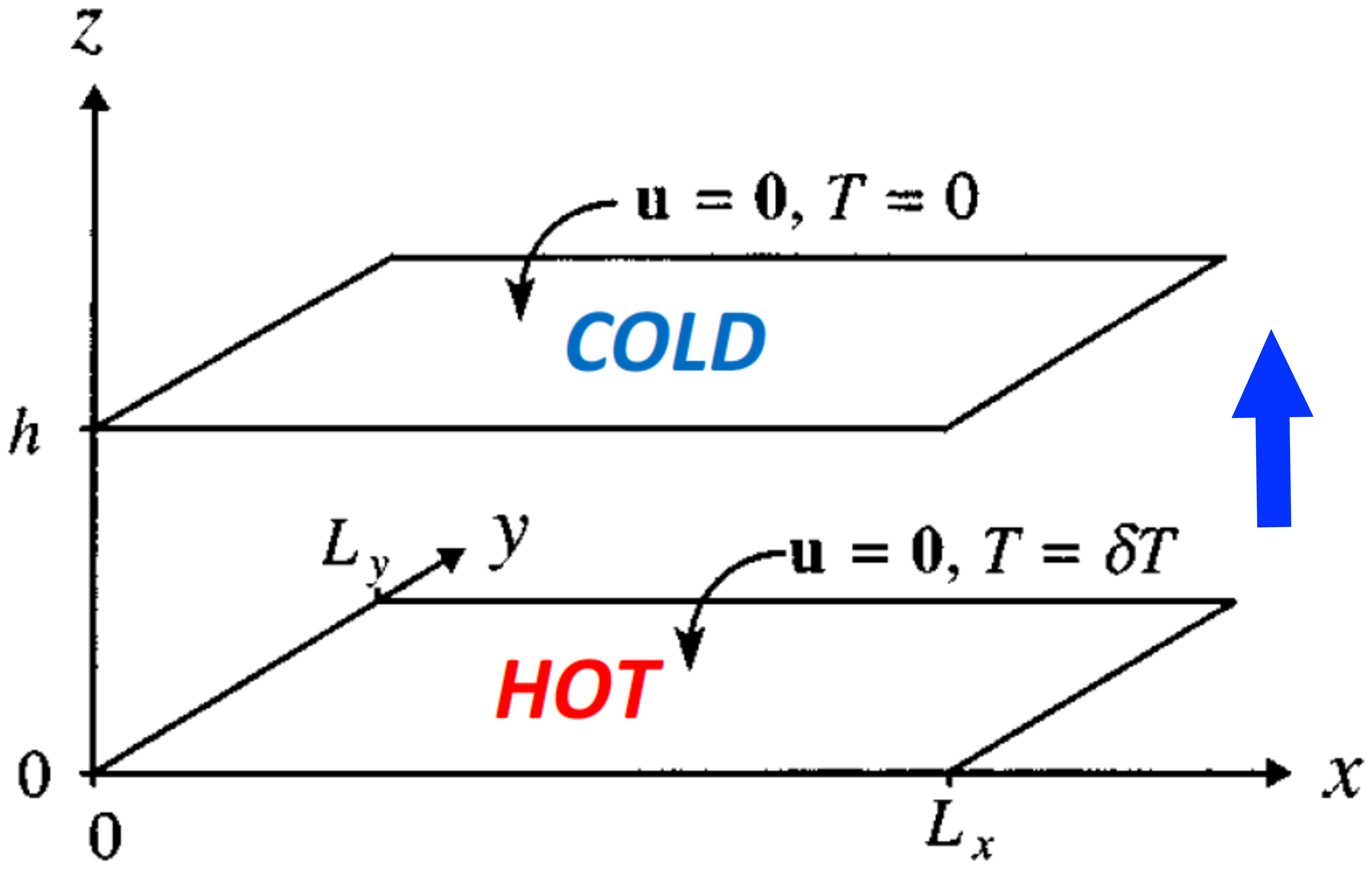
Ding & K JFM 2020

Rayleigh-Benard Convection

$$Ra = \delta T g \beta h^3 / \nu\kappa$$

$$\sigma = \nu / \kappa$$



$$Nu(Ra, \sigma) := \frac{\text{total heat flux}}{\text{conductive heat flux}}$$

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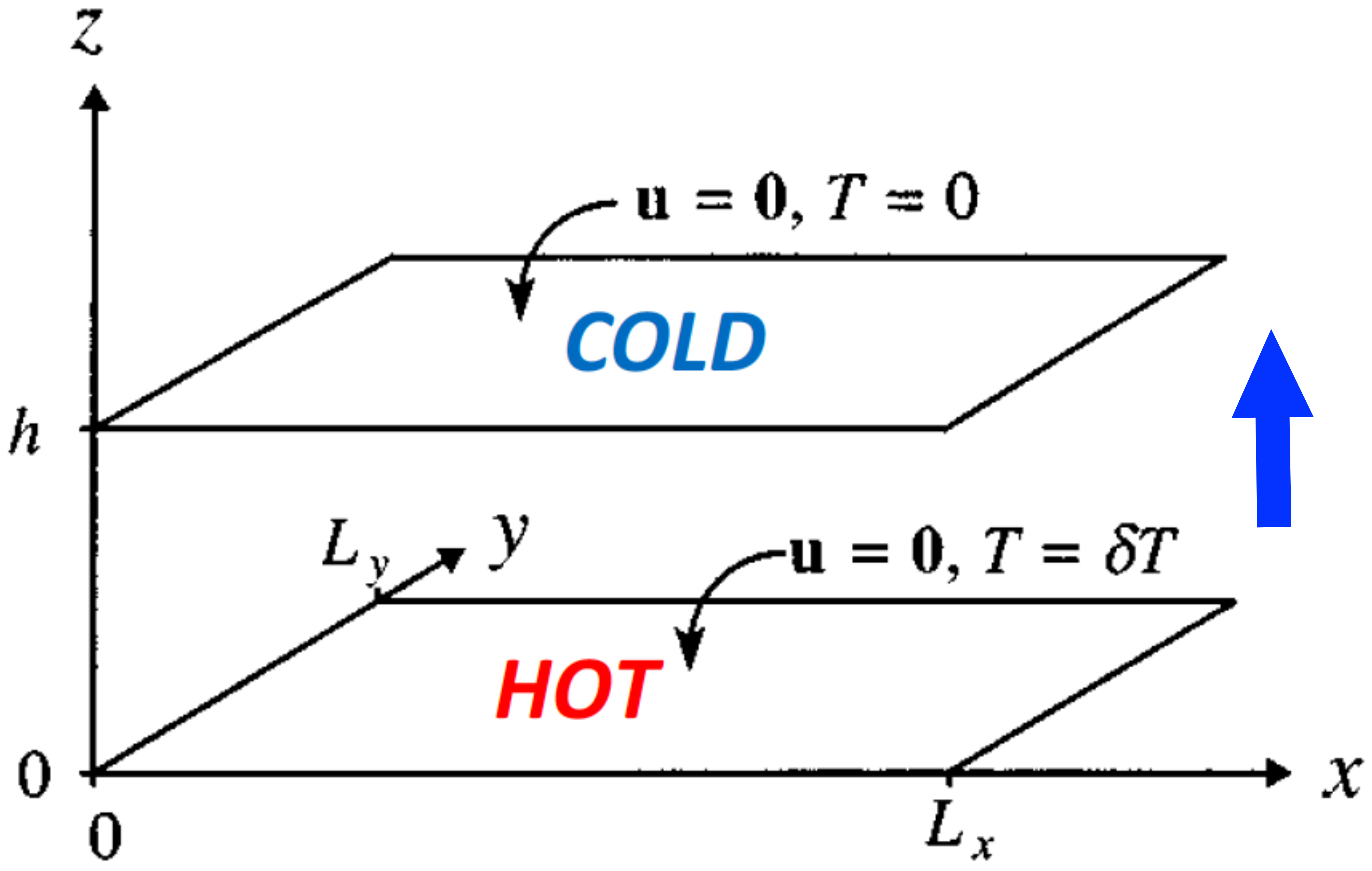

$$Nu(Ra, \sigma) := \frac{\text{total heat flux}}{\text{conductive heat flux}}$$

$$\approx const \times Ra^{\alpha}$$

as $Ra \to \infty$

Ultimate

$$Nu \sim Ra^{1/2}$$

Kraichnan (1962), Speigel (1963)

Ahlers et al. (2017), He et al. (2012)

Upper bounding: Malkus (1954), Howard (1963), Busse (1969) - *classical*
Doering & Constantin (1996) - *background method*

Classical

$$Nu \sim Ra^{1/3}$$

Malkus (1954), Priestley (1954), Howard (1964)

Iyer et al. (2020)

Waleffe et al. (2015)

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`add more constraints’

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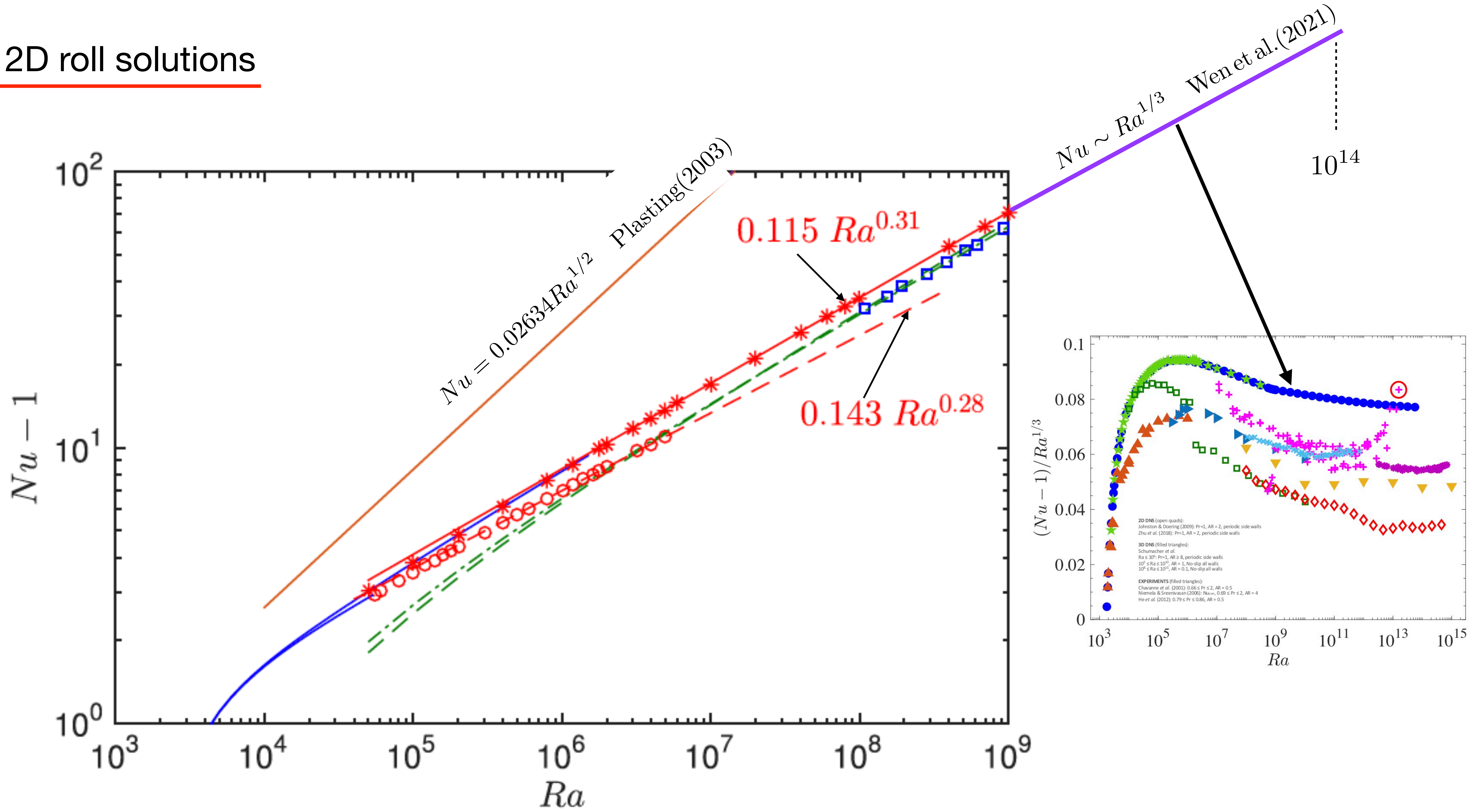
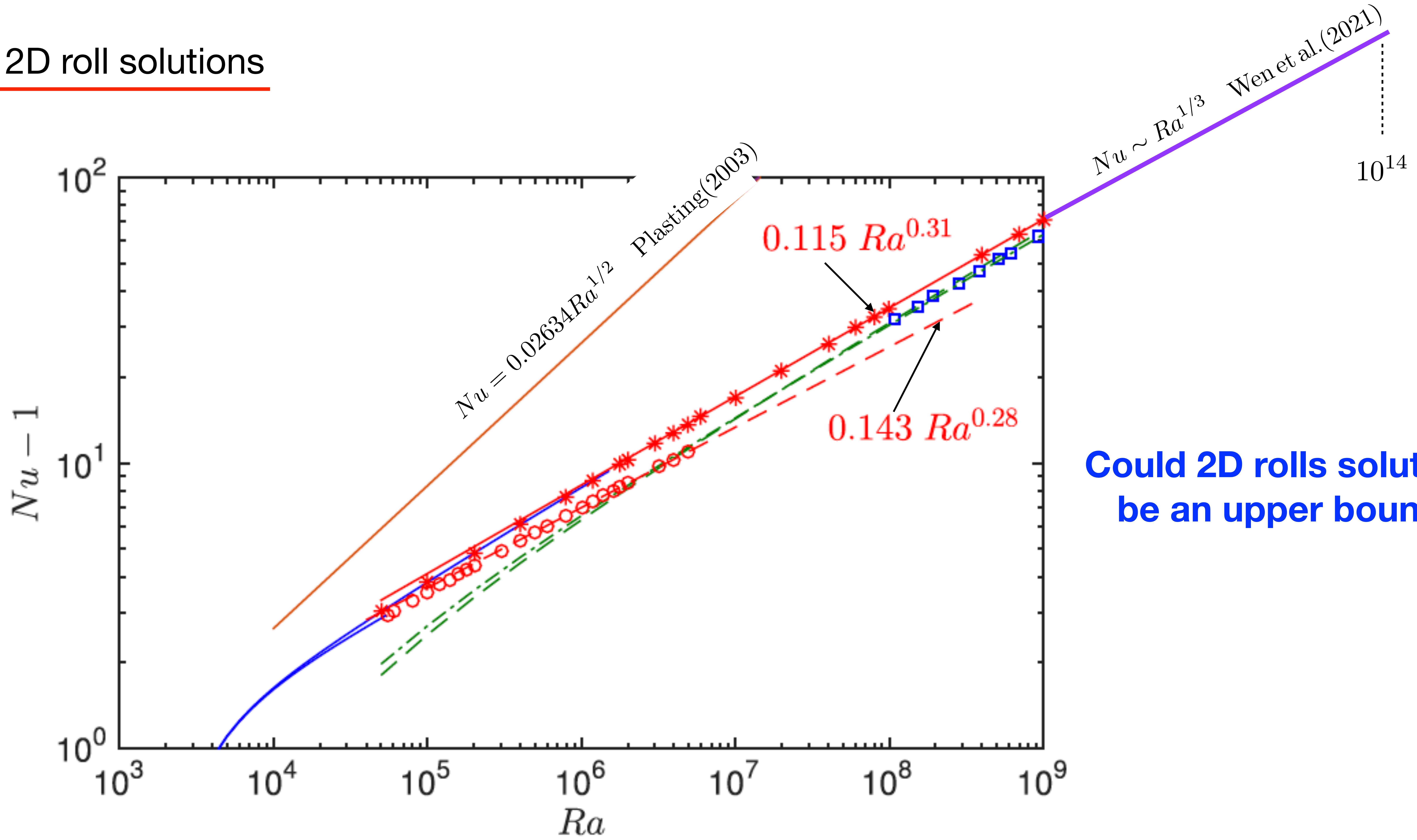
Simple 2D roll solutions



Waleffe et al (2015)

Simple 2D roll solutions



Waleffe et al (2015)

Could 2D rolls solutions be an upper bound?

Key Questions

1. Can the upper bound be lowered from `ultimate’ to `classical’ scaling by incorporating more constraints?

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1. Can the upper bound be lowered from `ultimate' to `classical' scaling by incorporating more constraints?

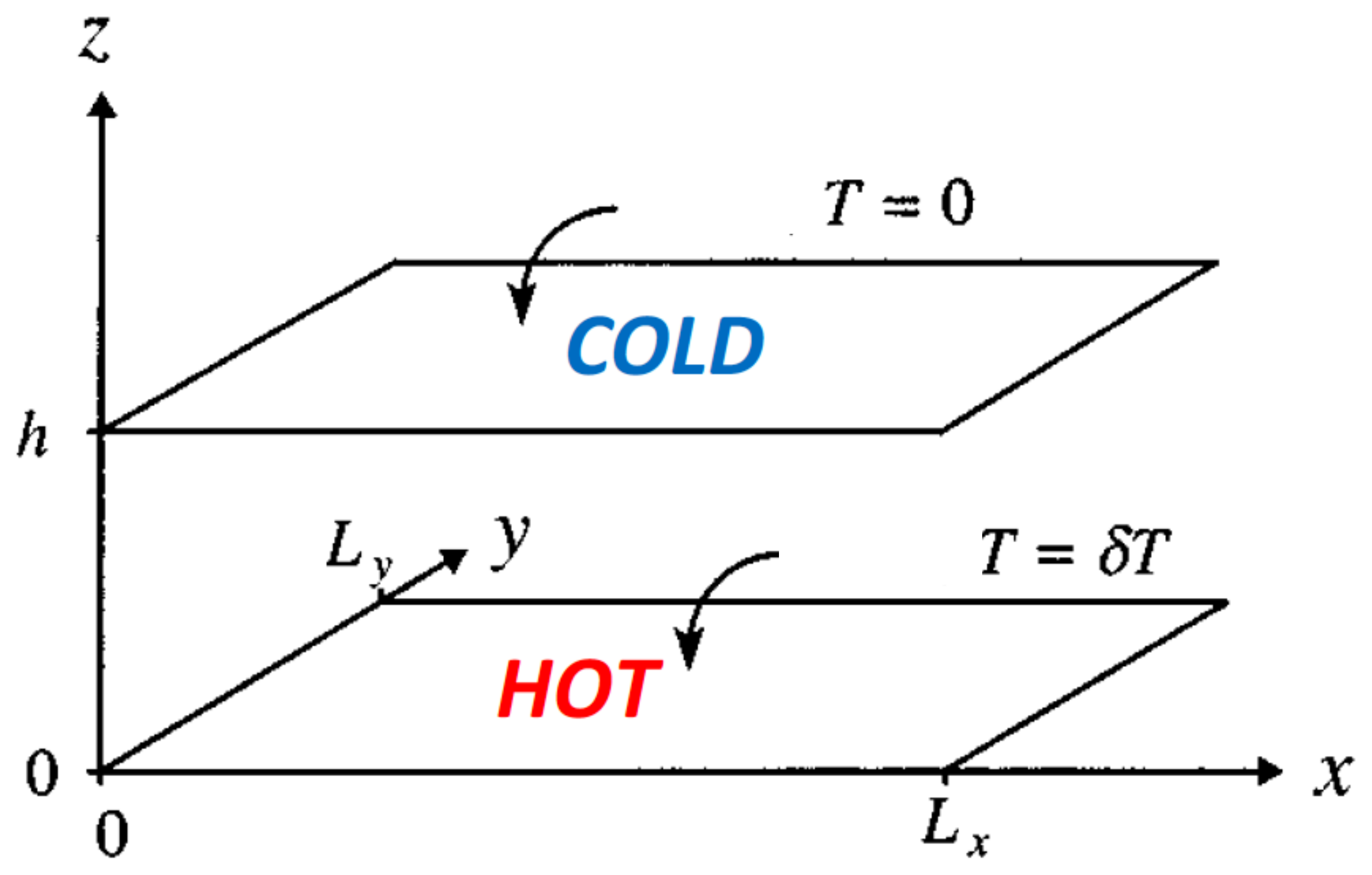
2. If ***all*** (steady) dynamical constraints imposed, could the optimal be a steady Boussinesq solution?

   - i.e could 2D rolls `satisfy' the spectral constraint?

Simplest problem:

consider 2D Rayleigh Benard Convection with stress-free b.c.s

Wall-to-Wall problem (Hassanzadeh et al. 2014
Tobasco & Doering 2017
Souza et al. 2020)



$$(\mathcal{N}) := \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} p - \sigma \nabla^2 \boldsymbol{u} - \sigma Ra T \hat{\boldsymbol{z}} = \boldsymbol{0},$$

$$(\mathcal{H}) := \frac{\partial T}{\partial t} + \boldsymbol{\nabla} \cdot (\boldsymbol{u} T - \boldsymbol{\nabla} T) = 0,$$

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Background Approach

$$\mathscr{L} := \langle |\nabla T|^2 \rangle - \langle a \boldsymbol{\nu} \cdot (\mathcal{N}) \rangle - \langle b \theta (\mathcal{H}) \rangle$$

$$\boldsymbol{\nu}(x, z, t) := \mathbf{u}(x, z, t)$$

$$\theta(x, z, t) := T(x, z, t) - \tau(x, z)$$

Background field

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$$\underline{\boldsymbol{\nu}}(x, z, t) := \mathbf{u}(x, z, t)$$

$$\underline{\theta}(x, z, t) := T(x, z, t) - \underline{\tau(x, z)}$$

$$\Rightarrow \mathcal{L} = \langle |\nabla T|^2 \rangle - \underbrace{a \langle \mathbf{u}.(\mathcal{N}) \rangle}_{\text{power}} - \overbrace{b \langle T(\mathcal{H}) \rangle}^{\text{entropy}} + \underbrace{b \langle \tau \mathcal{H} \rangle}_{\text{heat eqn}}$$

$$\Rightarrow \mathcal{L} = \langle |\nabla\tau|^2 \rangle - \mathcal{G}(\mathbf{u}, \theta; \tau)$$

where $$\mathcal{G} := \langle \frac{a}{Ra} |\nabla \mathbf{u}|^2 + |\nabla\theta|^2 + 2\theta\mathbf{u}.\nabla\tau \rangle$$

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$$\min \mathcal{G} = \begin{cases} 0 & \tau \in \Omega \\ -\infty & \tau \notin \Omega \end{cases} \quad \begin{cases} (\mathbf{u}, \theta) = (\mathbf{0}, 0) \\ (\mathbf{u}, \theta) \neq (\mathbf{0}, 0) \end{cases}$$

marginal modes

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$$\frac{\delta \mathcal{L}}{\delta \tau} = -2\nabla^2 \tau - \frac{\delta \mathcal{G}(\mathbf{u}, \theta)}{\delta \tau}$$

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problem : $$Nu \leq \min_{\tau \in \Omega} \max_{\mathbf{u}, \theta} \mathcal{L}$$

$$\lambda \mathbf{u} = \frac{\delta \mathcal{G}}{\delta \mathbf{u}} = -\theta \nabla \tau - \nabla p + \frac{a}{Ra} \nabla^2 \mathbf{u}$$

$$\lambda \theta = \frac{\delta \mathcal{G}}{\delta \theta} = -\mathbf{u}.\nabla \tau + \nabla^2 \theta$$

with $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \ldots$

$(a)\ \tau = \tau(z)$

Marginal modes are Fourier modes in x

$$k(j) \in K \qquad j = 1, 2, \dots, n$$

$$(\mathbf{u}^*, \theta^*) = \sum_{j=1}^{n} (\mathbf{u}_j, \theta_j) e^{ik(j)x} + c.c.$$

$$\mathcal{G}(\mathbf{u}^*, \theta^*) = \sum_j \left\{ \mathcal{G}_j := \langle \frac{a}{Ra} |\nabla \mathbf{u}_j|^2 + |\nabla \theta_j|^2 + 2\theta_j \mathbf{u}_j . \nabla \tau \rangle \right\}$$

$$\delta \mathcal{G}_j = 0 \Rightarrow \delta \mathcal{G}(\mathbf{u}^*, \theta^*) = 0$$

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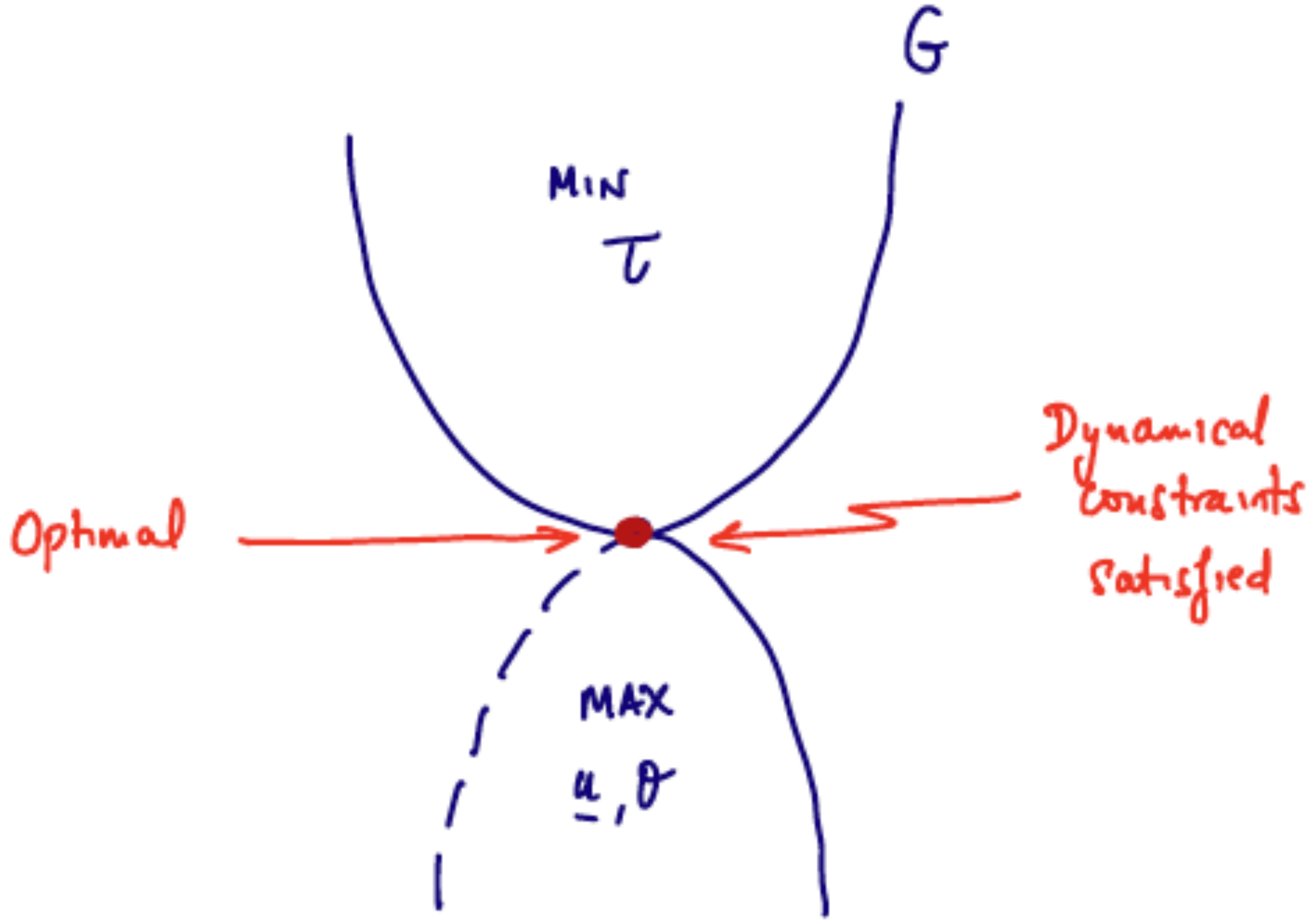

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$$\delta \mathcal{G}_j = 0 \Rightarrow \delta \mathcal{G}(\mathbf{u}^*, \theta^*) = 0$$

Optimal satisfies constraining equations

$(b) \tau = \tau(x,z)$

marginal modes $(\mathbf{u}_j, \theta_j)(x,z)$

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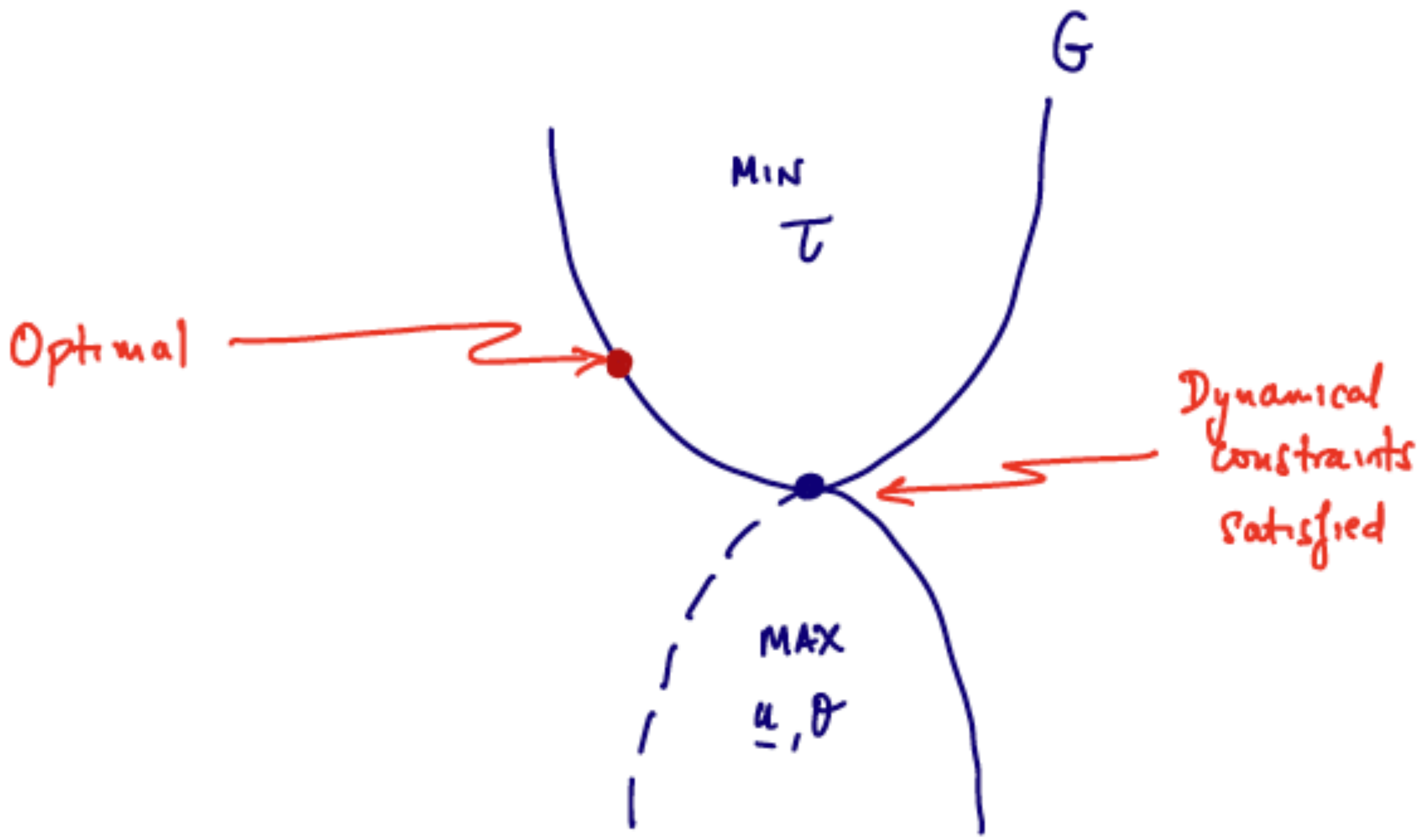

Souza et al. 2020

$\delta \mathcal{G}_j = 0 \not\Rightarrow \delta \mathcal{G}(\mathbf{u}^*, \theta^*) = 0$ Optimal does **not** satisfy constraining equations

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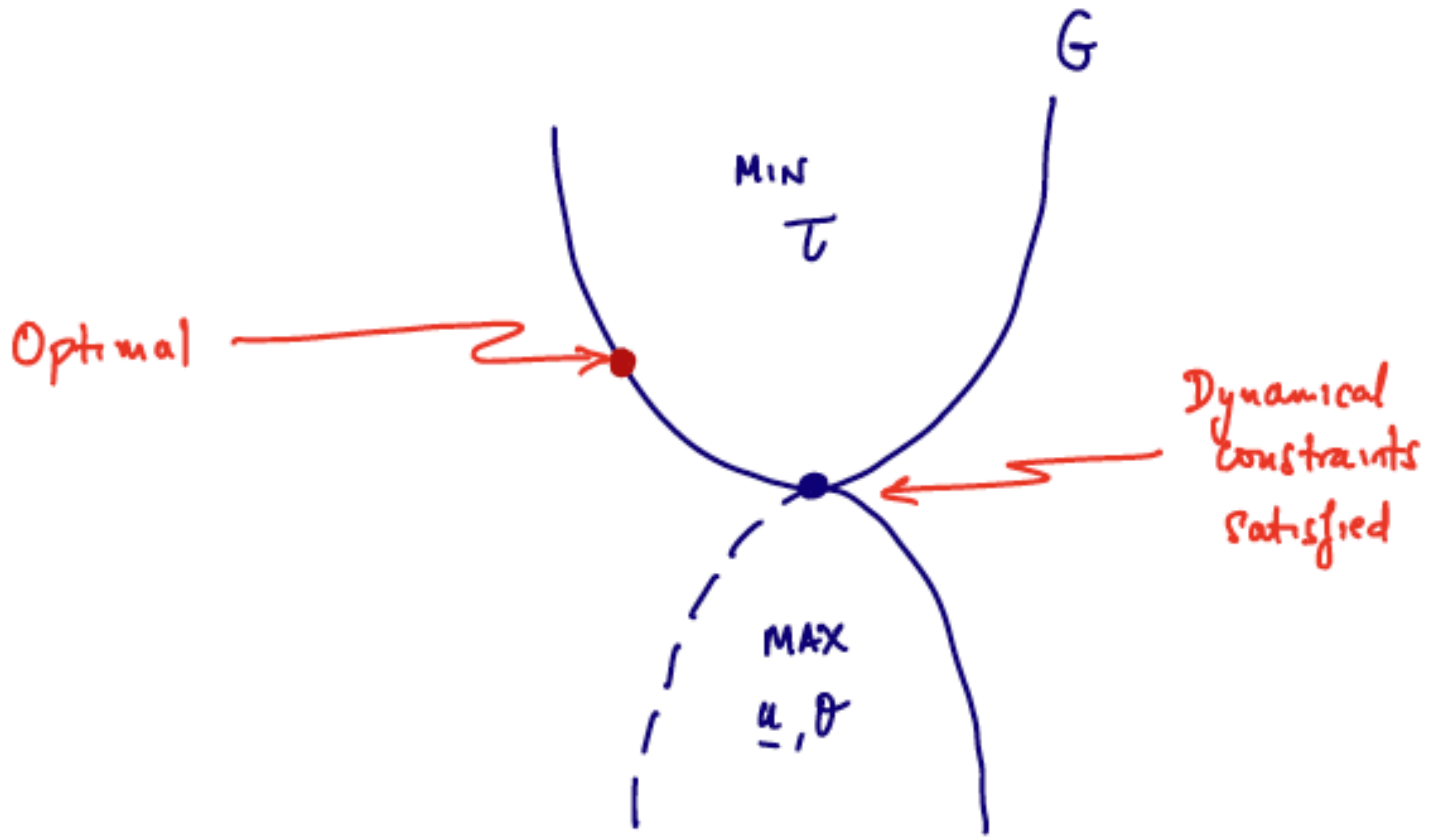

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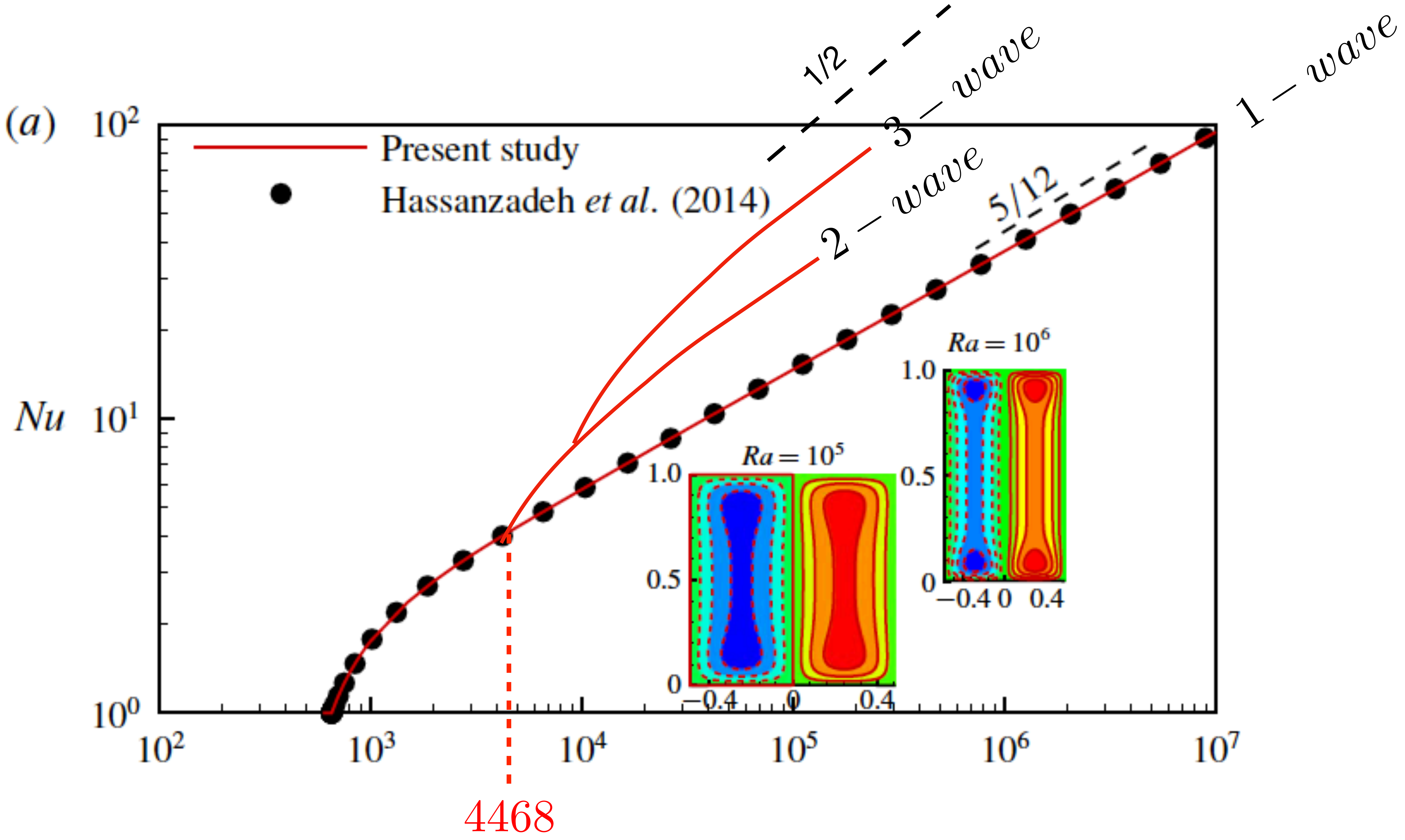
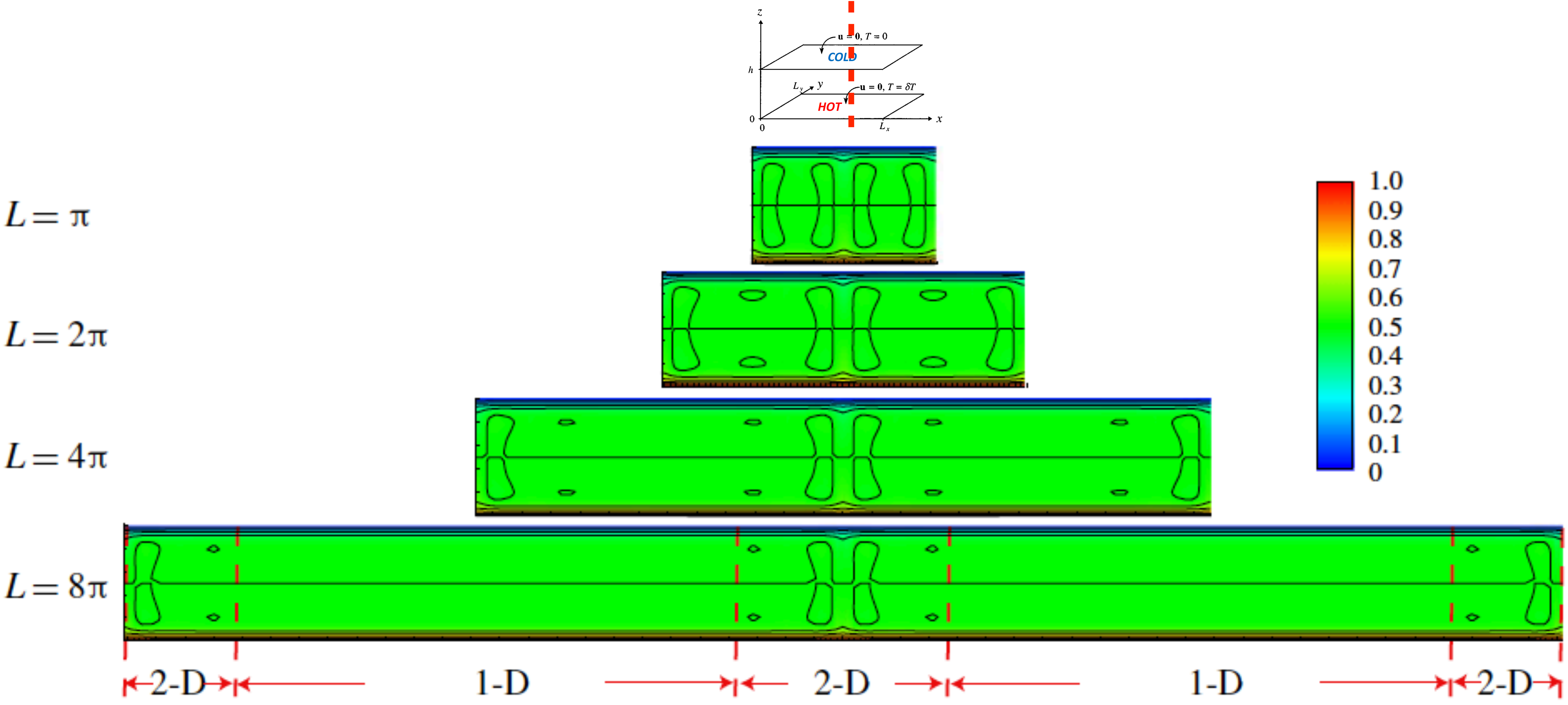
A steady Boussinesq solution cannot be optimal - 2D rolls not upper (background) bound



(a)
Present study
Hassanzadeh et al. (2014)
1/2
3 − wave
2 − wave
1 − wave
5/12
Nu
Ra = 10^5
Ra = 10^6
4468



$$\tau(x,z) \to \tau(z) \;\text{ as }\; L \to \infty$$

Convexity: Doering & Constantin (1996)

.....no Improvement on Bound...

What about a background velocity field?

$$\boldsymbol{\nu}(x,z,t) := \mathbf{u}(x,z,t)$$
$$\theta(x,z,t) := T(x,z,t) - \tau(x,z)$$

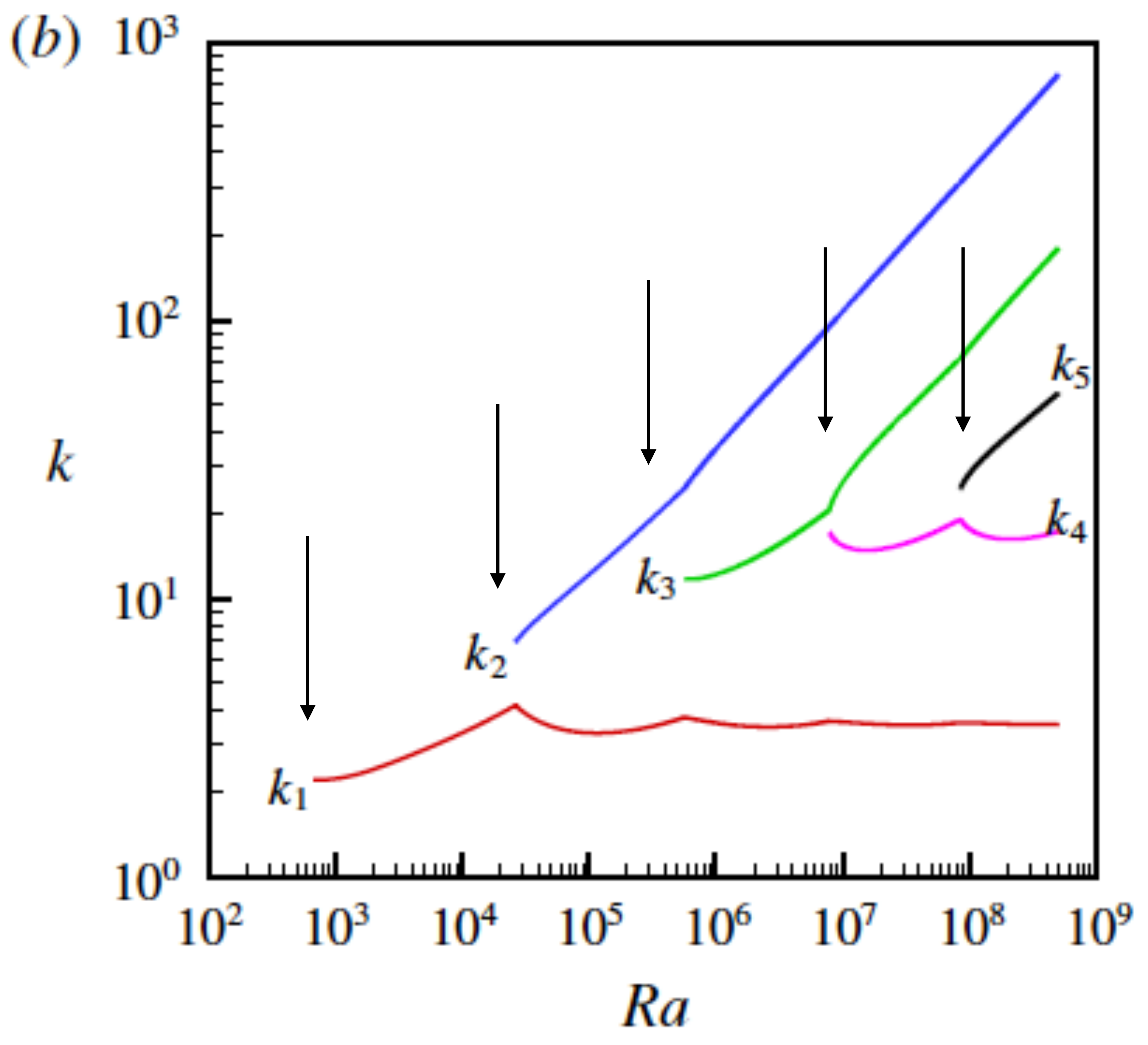
Now

$$\nu(x,z,t) := \mathbf{u}(x,z,t) - \boldsymbol{\phi}(x,z)$$
$$\theta(x,z,t) := T(x,z,t) - \tau(x,z)$$

Attempt an inductive bifurcation analysis:

If $\boldsymbol{\phi}(x,z) = \mathbf{0}$ before the bifurcation then

$\boldsymbol{\phi}(x,z) = \mathbf{0}$ afterwards



Conclusions

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** 3D optimal solution found (Motoki et al. 2018) which approaches $Nu \leq 0.02634 Ra^{1/2}$

↓

3D Boussinesq multiscale solution (Motoki et al. 2021) $Nu \sim Ra^{1/3}$

* Extra enstrophy constraint possible for 2D stress-free convection

$$Nu \leq 0.106 Ra^{5/12} \quad \text{Wen et al. (2015)}$$

Inverse engineering…

$$\boldsymbol{\nu}(x,z,t) := \mathbf{u}(x,z,t) + c\nabla \times \nabla \times \mathbf{u}(x,z,t)$$

$$(\mathcal{N}) := \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p - \sigma \nabla^2 \boldsymbol{u} - \sigma Ra T \hat{\boldsymbol{z}} = \boldsymbol{0},$$

$$(\mathcal{H}) := \frac{\partial T}{\partial t} + \nabla \cdot (\boldsymbol{u} T - \nabla T) = 0,$$

Hassanzadeh et al. 2014

$$\Theta(x,z) := T(x,z) - (1-z)$$

$$\mathcal{L}_{\mathcal{H}} := \langle w\Theta \rangle - \langle \Phi(x,z)(\mathcal{H})_s \rangle - \mu \langle |\nabla \mathbf{v}|^2 - Pe^2 \rangle$$

$$\text{problem}: \quad \max_{\mathbf{v},\Theta,\Phi,\mu} \mathcal{L}_{\mathcal{H}}$$

$$\langle (\ldots) \rangle := \lim_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} \frac{1}{V} \int (\ldots) \, \mathrm{d}V \, \mathrm{d}t,$$