Atmospheric Flows with Water: Multiscale Coupling

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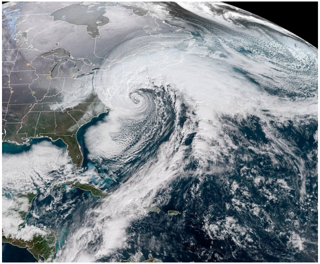


‘Cloud Waves’

‘Bomb Cyclone’

When do *high-frequency, propagating waves* influence *slowly varying, large-scale* structures, e.g., jets, vortices, layers, rivers ...?

Fast-slow (unbalanced-balanced) interactions

Much is known for the *dry* dynamics.

- Numerical Simulations: finite, small parameters

Bartello '95, S-Waleffe '02, Wingate et al. '11, Marino et al. '13

- Rigorous Proofs: limiting dynamics

Embid & Majda (EM) '96, '98, ME '98, Babin et al. '96, '97, '00

Remarkable Result:

Slow dynamics are decoupled from fast waves in the singular limit of fast oscillations

The case with phase changes is wide open!

Outline

- Moist Boussinesq dynamics with phase changes
- What is fast-wave averaging?
- Decomposition into slow and fast variables
- Results

Boussinesq with Fast Cloud Microphysics (HMSS '13)

$$\frac{D\mathbf{u}}{Dt} + f\sin(\phi)\mathbf{u}_h^{\perp} = -\nabla p + \mathbf{k}\, g\left(\frac{\theta}{\theta_o} + R_{vd} q_v - q_r\right)$$

$$\frac{Dq^{\mathrm{tot}}}{Dt} = \frac{\partial}{\partial z}(V_T q_r), \quad q_t = q_v + q_r$$

$$\nabla \cdot \mathbf{u} = 0, \qquad \frac{D\theta_e^{\mathrm{tot}}}{Dt} = 0$$

$$\theta_e = \theta + (L_v/c_p)q_v, \quad \theta_e^{\mathrm{tot}}(\mathbf{x}, t) = \tilde{\theta}_e(z) + \theta_e(\mathbf{x}, t)...$$

Abstract formulation:

$$\partial_t \mathbf{v} + \varepsilon^{-1}\mathscr{L}(\mathbf{v}) = -\mathscr{B}(\mathbf{v},\mathbf{v})$$

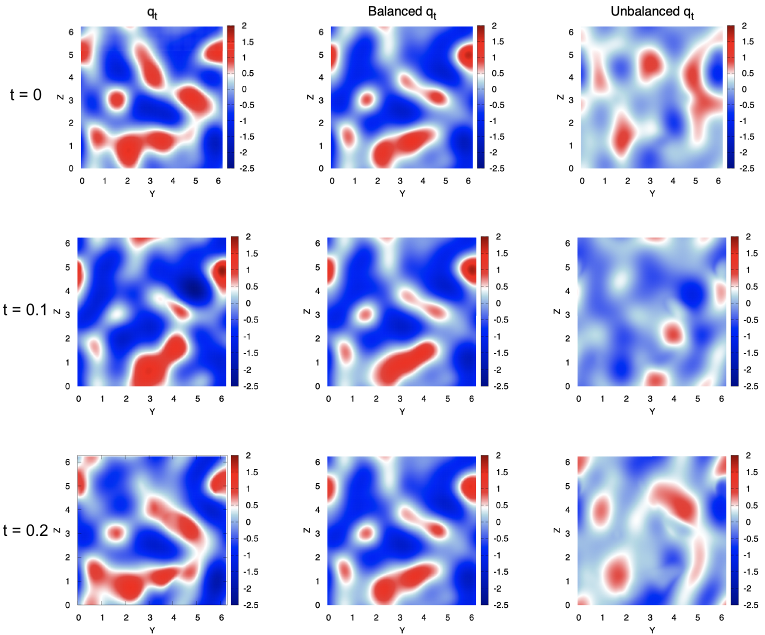
$$\mathbf{v}(\mathbf{x},0) = \bar{\mathbf{v}}(\mathbf{x},t)$$

in a periodic domain; $\mathbf{v}$ is the state vector.

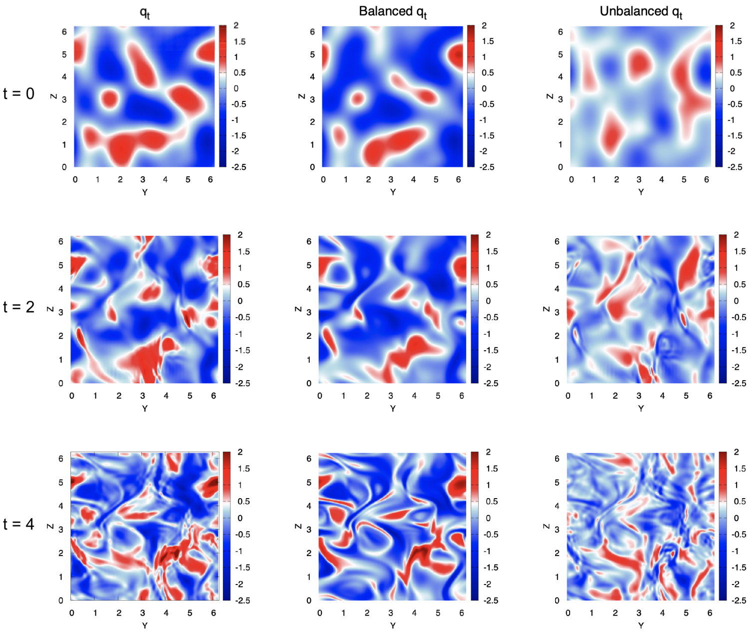
$\mathscr{L}$ includes rotation and buoyancy; $\mathscr{B}$ blinear

$$Ro = Fr_1 = Fr_2 = \varepsilon \to 0, \quad {N_1}^2 = \frac{g}{\theta_0}\frac{d\tilde{\theta}_e}{dz}, \quad {N_2}^2 = -\frac{gL_v}{c_p\theta_0}\frac{d\tilde{q}_t}{dz}$$

Random i.c.s, $\varepsilon = 0.13$, sat. deficit $= 0.5$, $V_T = 0$



For $t = O(1)$, is there coupling?



The idea of fast-wave averaging for $\varepsilon \to 0$

$$\partial_t \mathbf{v} + \varepsilon^{-1}\mathscr{L}(\mathbf{v}) = -\mathscr{B}(\mathbf{v},\mathbf{v}), \quad \mathbf{v}(\mathbf{x},0) = \bar{\mathbf{v}}(\mathbf{x},t)$$

$$\mathbf{v}^{\varepsilon}(\mathbf{x},t,\tau) = \mathbf{v}^{0}(\mathbf{x},t,\tau)\big|_{\tau=t/\varepsilon} + \varepsilon\mathbf{v}^{1}(\mathbf{x},t,\tau)\big|_{\tau=t/\varepsilon} + \cdots$$

$$\frac{\partial \mathbf{v}^{0}}{\partial \tau} + \mathscr{L}(t,\tau)(\mathbf{v}^{0}) = 0$$

$$\mathbf{v}^{0}(\mathbf{x},t,\tau) = e^{-\int_0^{\tau}\mathscr{L}(t,\tau')d\tau'}\bar{\mathbf{v}}(\mathbf{x},t)$$

$$\frac{\partial \mathbf{v}^{1}}{\partial \tau} + \mathscr{L}(t,\tau)(\mathbf{v}^{1}) = -\left(\frac{\partial \mathbf{v}^{0}}{\partial t} + \mathscr{B}(\mathbf{v}^{0},\mathbf{v}^{0})\right)$$

The condition for sub-linear growth of $\mathbf{v}^1 ==>$

the fast-wave averaging equation:

$$\frac{\partial \bar{\mathbf{v}}(\mathbf{x},t)}{\partial t} = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} \left(\int_0^{s} \frac{\partial \mathscr{L}(t,s')}{\partial t} ds' \right) \bar{\mathbf{v}} ds$$

$$- \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^{\tau} \mathscr{A}\mathscr{B}(\mathscr{A}\bar{\mathbf{v}}, \mathscr{A}\bar{\mathbf{v}}) ds,$$

$$\mathscr{A} = \exp\left(- \int_0^{s} \mathscr{L}(t,s') ds' \right)$$

Project onto slow modes; assess coupling

Dry Dynamics with $\mathscr{L}$ constant

$$\xi + \frac{\partial \theta}{\partial z} = PV \quad (\mathit{slow\ mode})$$

$$\frac{\partial}{\partial t} PV = -\mathbf{u}_{PV} \cdot \nabla PV, \quad (\mathit{FWA\ equation})$$

$$(= \text{slow} - \text{slow})$$

$PV, \mathbf{u}_{PV}$ decoupled from fast waves

Moist Dynamics with Phase Changes

The buoyancy function changes at phase boundaries

$$\frac{\partial \mathbf{v}}{\partial t} + \varepsilon^{-1}\Big(H_u(\mathbf{v})\mathscr{L}_u(\mathbf{v}) + H_s(\mathbf{v})\mathscr{L}_s(\mathbf{v}) \Big) = -\mathscr{B}(\mathbf{v},\mathbf{v})$$

$$H_u = \begin{cases} 1 \text{ for } q_t^{tot} < q_{vs} \\ \\ 0 \text{ for } q_t^{tot} \geq q_{vs} \end{cases} \qquad \text{and } H_s = 1 - H_u$$

Phase interfaces H_u, H_s are determined by the full state vector $\longrightarrow$ a nonlinear operator, depends on τ!

Steps for fast-wave averaging with phase changes:

- Treatment of the Heaviside functions

- Find the slow (and fast) variables

- Evaluate terms in the fast-wave averaging equation (cannot use Fourier analysis)

Slow Variables in Moist Boussinesq

quantities that do not change in time as $\varepsilon \to 0$

$$M = q_t + \theta_e, \quad PV_e = \xi + \frac{\partial \theta_e}{\partial z}$$

$$\theta_e = \theta + q_v, \quad \xi = \nabla \times \mathbf{u}_h$$

The moist Fast-Wave-Averaging equations

$$\frac{\partial M}{\partial t} = -\langle \mathbf{u}_{(M,PV_e)} \rangle \cdot \nabla M - \langle \mathbf{u}_{(W\ldots)} \rangle \cdot \nabla M$$

$$(= \text{slow-slow} + \text{fast-slow}\ )$$

$$\frac{\partial PV_e}{\partial t} = -\langle \mathbf{u}_{(M,PV_e)} \rangle \cdot \nabla PV_e + \langle \frac{\partial \mathbf{u}_{(M,PV_e)}}{\partial z} \cdot \nabla \theta_{e(M,PV_e)} \rangle$$

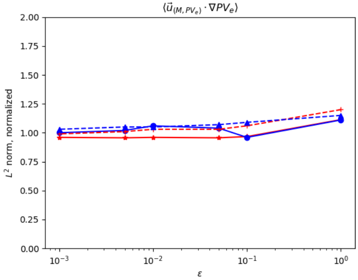
$$-\langle \mathbf{u}_{(W\ldots)} \rangle \cdot \nabla PV_e + \text{other fast terms}$$

$$(= \text{slow-slow} + \text{slow-slow} + \text{fast-slow} + \text{other fast})$$

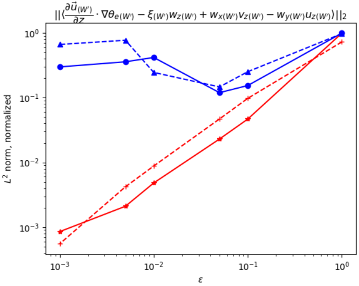
$$\langle f \rangle(\mathbf{x}, t) = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(\mathbf{x}, t, s)\, ds.$$

Time averages over $t = O(1)$ with $10^{-3} \le \varepsilon \le 1$

Slow-Slow



Fast-Fast



Dry in red, Phase Change in blue,

128^3 and 256^3 different symbols

A simple ODE model

$$\frac{dw}{dt} = b = b_u H_u + b_s H_s$$

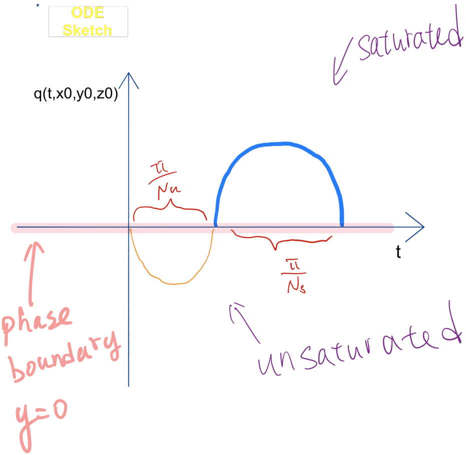
$$\frac{db_u}{dt} + N_u^2 w = 0, \qquad \frac{db_s}{dt} + N_s^2 w = 0$$

$$H_u = H(b_u - b_s), \quad H_s = 1 - H_u$$

$dM/dt = 0$; $M = N_u^{-2} b_u - N_s^{-2} b_s$ is a parameter

While unsaturated, use (w, b_u); evolve forward in time until saturation; switch to (w, b_s) while saturated

The basic mechanism for non-zero averages involving fast modes:



$==>$ coupling of fast-slow components as $\varepsilon \to 0$

Conclusions and Future Work

- Fast-wave averaging suggests coupling as $\varepsilon \to 0$ when phase changes are present

- Numerical simulations with decreasing ε also suggest coupling

- Future Work: rigorous results for the limiting dynamics; statistical steady states for finite parameter values