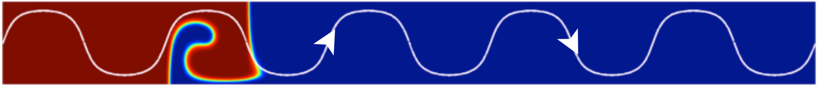
How jets and eddies influence the speed of propagation of plankton blooms

Alexandra Tzella
University of Birmingham



with Jacques Vanneste, University of Edinburgh

Transport and Mixing in Complex and Turbulent Flows
January 11 - 14, 2021

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Sharp fronts in periodic flows: FKPP vs G

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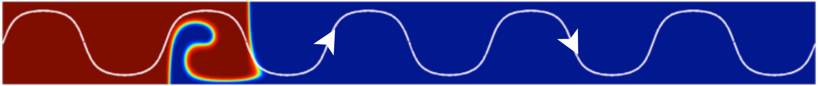

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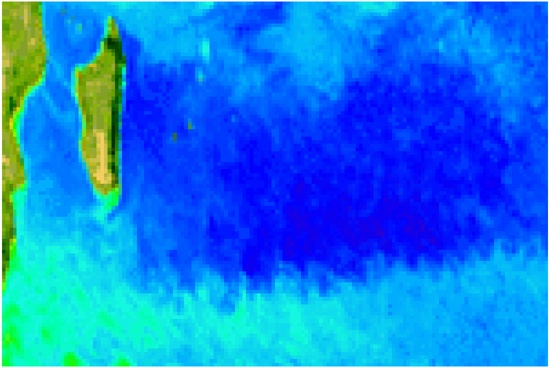
Plankton Blooms as reactive fronts

- Plankton blooms can propagate much more rapidly than a passive tracer in the same environment.
- Often in the form of localized, strongly inhomogeneous structures associated with reactive fronts.



Phytoplankton bloom off the coast of Madagascar, speed: 10-20km/day

Plankton blooms as reactive fronts

- Classic example of reactive front: the Fisher-Kolmogorov or FK front arising from reaction+diffusion:

$$\partial_t \theta = \kappa \Delta \theta + \frac{1}{\tau}\theta(1-\theta). \tag{FK}$$

- Dubois (1975) employed (FK) in the North Sea under quiescent conditions.
- Srokosz et al. (2004) extended (FK) to the Madagascar case, adopting a diffusive description with
  - a mean flow U,
  - the Okubo eddy diffusivity K
  - the phytoplankton growth rate τ^{-1}.
- However the velocity field has a non-trivial structure which affects the speed, shape and direction of propagation.

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FK fronts in shear and cellular flows

Reaction-diffusion-advection with Fisher-Kolmogorov nonlinearity

$$\partial_t \theta(\boldsymbol{x}, t) + \boldsymbol{u}(\boldsymbol{x}) \cdot \nabla \theta(\boldsymbol{x}, t) = \mathrm{Pe}^{-1} \Delta \theta(\boldsymbol{x}, t) + \mathrm{Da}\, \theta(1-\theta), \qquad \text{(FK)}$$

where $\nabla \cdot \boldsymbol{u} = 0$ and

$$\mathrm{Pe} = V\ell/\kappa \quad \text{and} \quad \mathrm{Da} = \ell/V\tau$$

in domain

$$\Omega = \mathbb{R} \times [0, \pi] \quad \text{with} \quad \boldsymbol{u} \cdot \boldsymbol{n} = \nabla\theta \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \mathcal{B}$$

and

$$\theta(x, y, 0) = \mathbb{1}_{x \le 0}, \quad \theta \to 1 \quad \text{as } x \to -\infty, \quad \theta \to 0 \quad \text{as } x \to \infty.$$

FK fronts in shear and cellular flows

For $t \gg 1$, a front is established which propagates eastward as long as $c_{\mathsf{FK}} > 0$ and is stationary if $c_{\mathsf{FK}} = 0$.

- When $\boldsymbol{u} = 0$, the front has the form $\theta(x, y, t) = \Theta(x - c_0 t)$ where

$$c_0 = 2\sqrt{\mathrm{Da}/\mathrm{Pe}} = 2\sqrt{\kappa/\tau}.$$

  For a constant flow $\boldsymbol{u} = (U, 0)$, $c_{\mathsf{FK}} = c_0 + U$; $c_{\mathsf{FK}} = 0$ when $c_0 = -U$.

- For a shear flow $\boldsymbol{u} = (u(y), 0)$, $\theta(x, y, t) = \Theta(x - c_{\mathsf{FK}} t, y)$
- When $\boldsymbol{u}$ is 2π-periodic, the front is pulsating:

$$\theta(x, y, t) = \Theta(x - c_{\mathsf{FK}} t, x, y),$$

  where Θ is 2π-periodic in the second variable. Berestycki & Hamel (2002)

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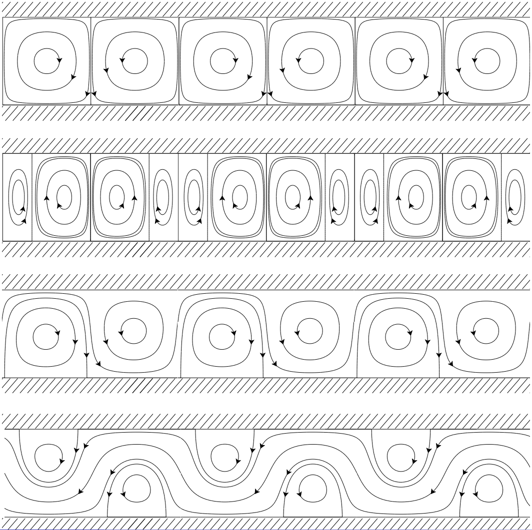
Focus on $\boldsymbol{u} = (u(y), 0)$ and $\boldsymbol{u} = \nabla^{\perp}\psi$ with

$$\psi = -Uy - (\sin(x) + A\sin(2x))\sin(y).$$



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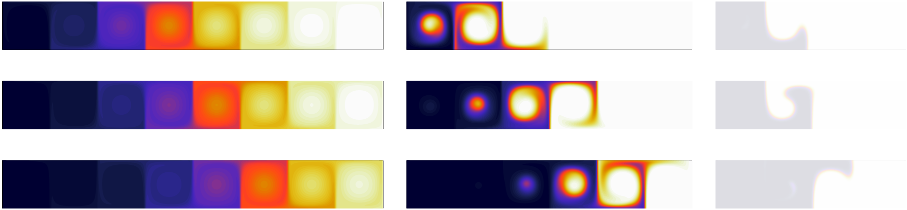

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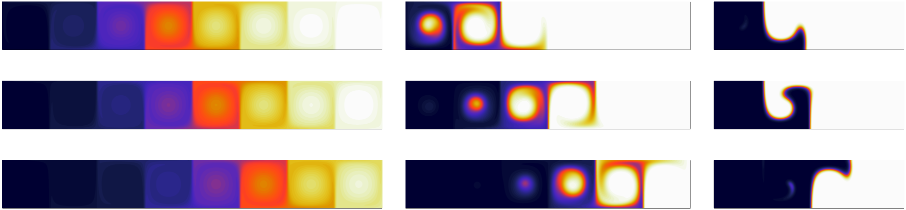

$\mathrm{Da} = 4 \times 10^{-2}$ $\mathrm{Da} = 4 \times 10^{-1}$ $\mathrm{Da} = 4$

FK front speed

- c_{FK} can be determined for arbitrary Da, Pe from eval problem derived by looking for solutions of the form $\exp(-g(x/t)t)\phi$ for linearised (FK):

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Gärtner and Freidlin (1979)

Two homogenization regimes ($\mathrm{Pe} \gg 1$):

- weak reactions $\mathrm{Da} \ll \mathrm{Pe}^{-1}$:

$$c_{\mathsf{FK}} \sim 2\sqrt{\kappa_{\mathsf{eff}}\mathrm{Da}}, \quad \kappa_{\mathsf{eff}} : \text{effective diffusivity}$$

  obtained from a linear cell problem associated with the advection-diffusion equation e.g. Fannjiang & Papanicolaou (1994), Constantin et al. (1999), Novikov & Rhyzhik (2007), Rhyzhik & Zlatos (2007), Zlatos (2010); Soward (1987), Childress & Soward (1989)

- strong reactions $\mathrm{Da} \gg \mathrm{Pe}^{-1}$: c_{FK} obtained from a nonlinear cell problem associated with a Hamilton-Jacobi equation

Freidlin (1985), Majda & Souganidis (1994)

Also some results for other Da e.g. Novikov & Rhyzhik (2007), Tzella & Vanneste (2015)

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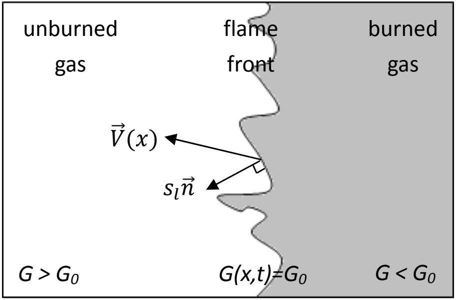
The G equation

A heuristic model often used to replace (FK) when $\mathrm{PeDa} \gg 1$, describing the front interface as the zero-level curve that satisfies

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = c_0 |\nabla \theta|, \quad \mathrm{Da}/\mathrm{Pe} = c_0^2/4 = O(1). \tag{G}$$



Williams (1985), Kerstein et al. (1988)

- When $\boldsymbol{u} = 0$, $c_G = c_{FK} = c_0$.
- When $\boldsymbol{u} = (u(y), 0)$, $c_G = c_{FK}$ Embid et al. (1995), Xin & Xu (2013)
- When $\boldsymbol{u}$ is 2π-periodic, $c_G =? c_{FK}$ (some results for $c_0 \to 0$).

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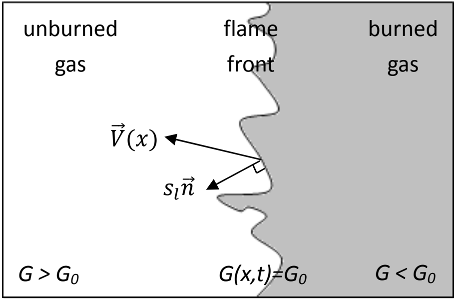

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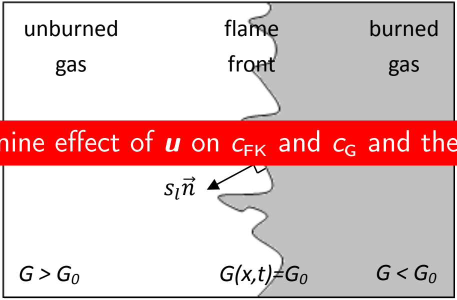

Aim: determine effect of $\boldsymbol{u}$ on c_{FK} and c_{G} and their difference

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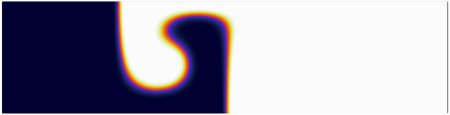
Sharp FK fronts



In the limit of $\mathrm{Pe}\,\mathrm{Da} \gg 1$, $\mathrm{Da}/\mathrm{Pe} = c_0^2/4 = O(1)$, the solution to

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = \mathrm{Pe}^{-1}\,\Delta\theta + \mathrm{Da}\,\theta$$

can be approximated using a WKB approximation:

$$\theta(\boldsymbol{x}, t) \sim \mathrm{e}^{-\mathrm{Pe}\,\mathscr{I}(\boldsymbol{x},t,c_0)}.$$

At leading order,

$$\partial_t \mathscr{I} + |\nabla \mathscr{I}|^2 + \boldsymbol{u}(\boldsymbol{x}) \cdot \nabla \mathscr{I} + c_0^2/4 = 0.$$

The front interface is given by

$$\mathscr{I}(\boldsymbol{x}, t, c_0) = 0.$$

Variational formulation for c_{FK}

The solution to the Hamilton–Jacobi equation is given by

$$\mathscr{I}(\boldsymbol{x}, T, c_0) = \frac{1}{4}\left(\inf_{\boldsymbol{X}(\cdot)} \int_0^T |\dot{\boldsymbol{X}}(t) - \boldsymbol{u}(\boldsymbol{X}(t))|^2 \,\mathrm{d}t - c_0^2 T\right),$$
$$\text{subject to } \boldsymbol{X}(0) = (0, \cdot), \quad \boldsymbol{X}(T) = \boldsymbol{x}.$$

For $T \gg 1$, the front propagates at the constant speed c_{FK} obtained from

$$\mathscr{G}(c_{\mathsf{FK}}, c_0) \coloneqq \lim_{T\to\infty} \frac{1}{T}\mathscr{I}(\boldsymbol{x}, T, c_0) = 0, \quad c_{\mathsf{FK}} = \frac{x}{T}.$$

e.g. Piatnitski (1998)

Note: $\mathscr{G}(c_{\mathsf{FK}}, c_0)$ is the Legendre dual of $\mathscr{H}$, the effective Hamiltonian satisfying the homogenised Hamilton-Jacobi equation

$$\partial_t \hat{\mathscr{I}} + \mathscr{H}(\nabla \hat{\mathscr{I}}) = 0$$

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Taking $T = n\tau$ with $\tau = 2\pi/c_{\text{FK}}$ (flow is periodic),

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Comparison between FK and G speeds

Taking $T = n\tau$ with $n \gg 1$, we obtain that

$$c_{\mathsf{G}} = \frac{2\pi}{\tau_{\mathrm{G}}}, \text{ where } \tau_{\mathrm{G}} = \inf_{\boldsymbol{X}(\cdot)} \tau, \quad \text{subject to } \boldsymbol{X}(\tau) = \boldsymbol{X}(0) + (2\pi, 0)$$
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- $c_{\mathsf{FK}} \geq c_{\mathsf{G}}$.
- $c_{\mathsf{G}} > 0$ implies $c_0 > -\min_x \max_y u(x, y)$.
- For shear flows $c_{\mathsf{FK}} = c_{\mathsf{G}} = u_+ + c_0$ where $u_+ = \max_y u(y)$.

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Cellular flow ($A = U = 0$): Trajectories

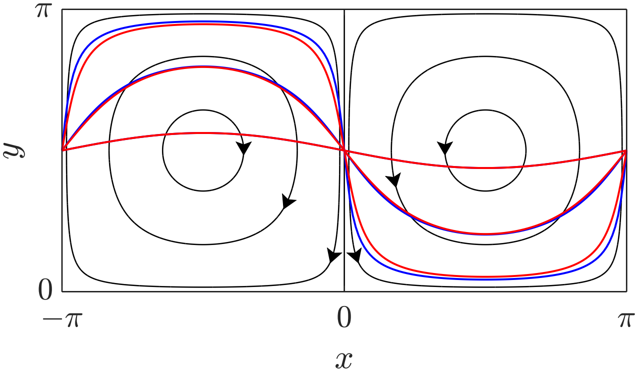
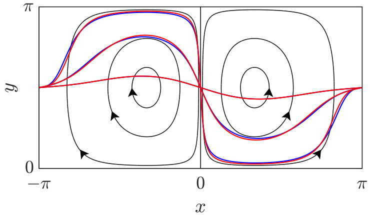


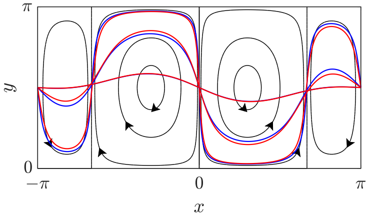
Figure: Minimising periodic trajectories calculated numerically for $c_0 = 0.1$, $c_0 = 1$ and $c_0 = 10$. They become closer to the straight line $y = \pi/2$ as c_0 increases.

c_{FK}	c_{G}	range of validity
$\pi/\mathrm{W}_p(32c_0^{-2})$	$-\pi/(2\log(\pi c_0/8))$	$c_0 \ll 1$
$c_0(1 + 3c_0^{-2}/4 - 105c_0^{-4}/64)$	$c_0(1 + 3c_0^{-2}/4 - 109c_0^{-4}/64)$	$c_0 \gg 1$

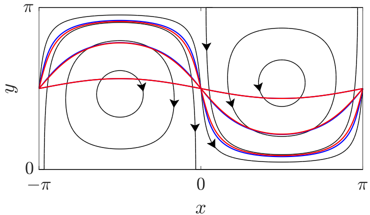
Cellular flow ($A > 0$, $U > 0$): Trajectories



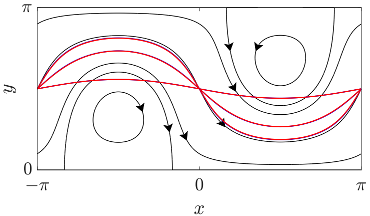
(a) $A = 0.5$, $U = 0$



(b) $A = 1$, $U = 0$



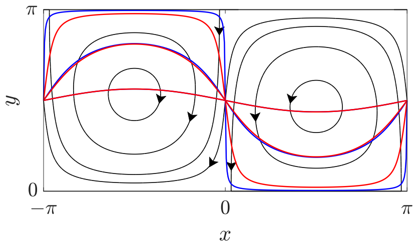
(c) $A = 0$, $U = 0.1$



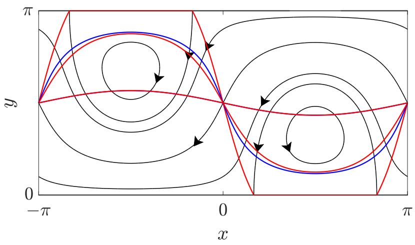
(d) $A = 0$, $U = 0.5$

Figure: Effect of small-scale perturbations and a mean flow.

Cellular flow ($U < 0$): Trajectories



(e) $A = 0$, $U = -0.1$
(minimum $c_0 = 0.11$)

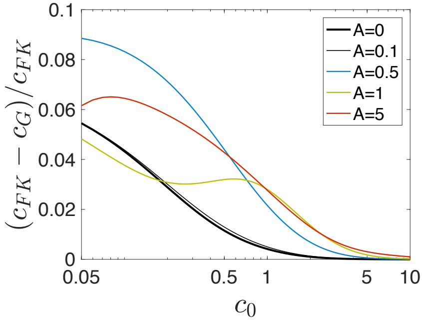


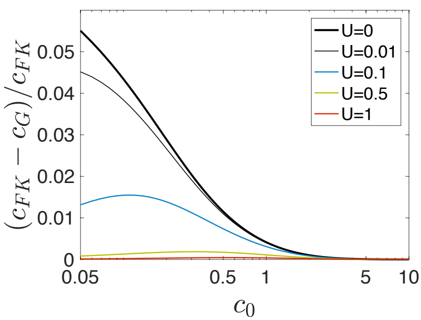
(f) $A = 0$, $U = -0.5$
(minimum $c_0 = 0.19$)

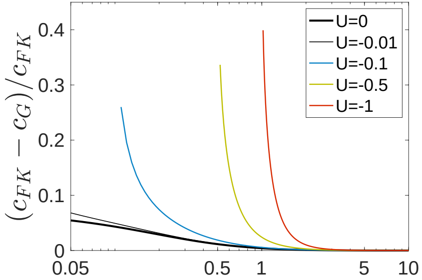
Figure: Effect of an opposing mean flow.

There exist $c_{FK} > 0$ for $c_0 + U \leq 0$ (i.e. when there is no $c_G > 0$)!

Cellular flows







Summary

- We studied the effect of shear & cellular flows on the sharp fronts arising in the (FK) model for strong reactions and small diffusivity and on their heuristic approximation by the (G) equation.
- The front speed is determined by a single periodic trajectory minimising the time of travel across a period under a constraint.
- The difference between the two models is due to the difference between the pointwise and time-integrated constraint.
- For the class of cellular flows, the difference increases with decreasing c_0, remaining small for $U \geq 0$. For $U < 0$ the difference can be dramatic: the two fronts may propagate in very different directions!

Tzella & Vanneste (2014) Phys. Rev. E 90, 011001(R);

Tzella & Vanneste (2019) SIAM J. Appl. Math, 79(1), 131-152;

Outlook

- **Challenge:** Prove that in all periodic flows, the minimising trajectories inherit the spatial periodicity of the background flow.
- We have focused on the front speed. We can extend minimal-time trajectory calculations to explain front shape.
- More complicated unbounded/time-dependent flows and/or reactions?
- Can 2D mesoscale flows+mixing diffusion+reaction really explain the Madagascar bloom?

Thank you for your attention!

The eigenvalue problem for the speed

We solve for $g(c)$ via an eigenvalue equation

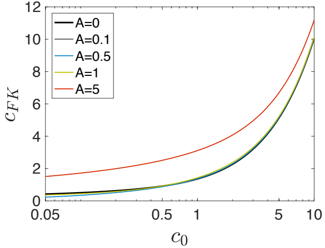
$$f(q)\phi = \mathrm{Pe}^{-1}\Delta\phi - (u_1, u_2)\cdot\nabla\phi - 2\mathrm{Pe}^{-1}q\partial_x\phi + (u_1 q + \mathrm{Pe}^{-1}q^2)\phi,$$

where f is the Legendre transform of g

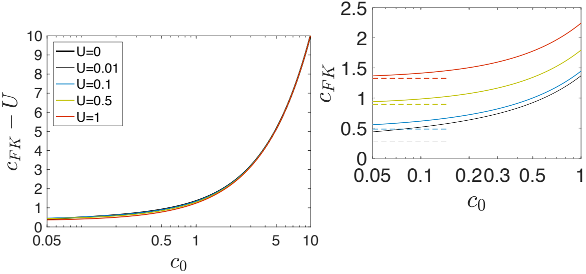
$$g(c) = \sup_{q>0}(qc - f(q)),$$

and $\phi(x, y)$ is 2π-periodic in x with $\partial_y\phi = 0$ at $y = 0, 1$.

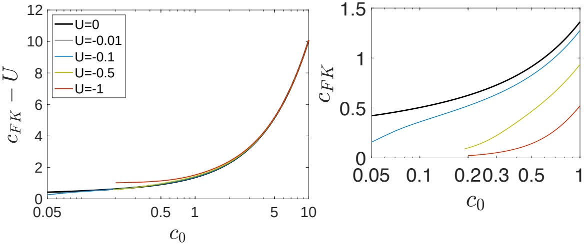
Cellular flow: Speed



(a) $A \neq 0,\ U = 0$



(b) $A = 0,\ U > 0$



(c) $A = 0,\ U < 0$

Variational formulation for c_{G}

The front reaches location $\boldsymbol{x}$ after a travel time

$$\mathscr{T}(\boldsymbol{x}, c_0) = \inf_{\boldsymbol{X}(\cdot)} T \quad \text{with} \quad \boldsymbol{X}(0) = (0, \cdot),\ \boldsymbol{X}(T) = \boldsymbol{x},$$
$$\text{subject to } |\dot{\boldsymbol{X}}(t) - \boldsymbol{u}(\boldsymbol{X}(t))|^2 = c_0^2 \text{ for } t \in [0, T],$$

For $T \gg 1$, x is large and the front moves at a constant speed given by

$$c_{\mathsf{G}} = \lim_{x \to \infty} \frac{x}{\mathscr{T}((x, y), c_0)},$$

where once more the dependence on y drops out.

Numerical procedure

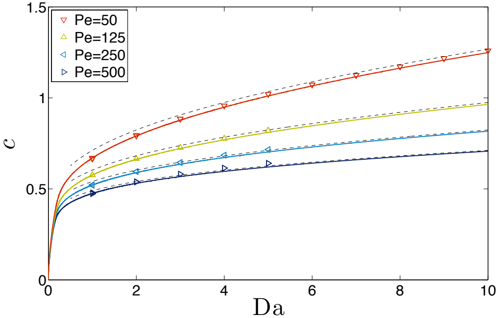
- The periodic trajectory $\boldsymbol{X}(t)$ is approximated by $\boldsymbol{X}_d$ on time grid $\{t_l = l\Delta t\}_{l=0}^{N}$ where $t_N = \tau = 2\pi/c$.
- The action functional is approximated by the sum over the discrete Lagrangians

$$G_d(\{\boldsymbol{X}_l\}_{l=0}^{N}) = \frac{1}{\tau}\sum_{l=0}^{N-1} L_d(\boldsymbol{X}_l, \boldsymbol{X}_{l+1}) - c_0^2$$

$$\text{where} \quad L_d(\boldsymbol{X}_l, \boldsymbol{X}_{l+1}) \approx \int_{l\Delta t}^{(l+1)\Delta t} |\dot{\boldsymbol{X}}(t) - \boldsymbol{u}(\boldsymbol{X}(t))|^2 \mathrm{d}t$$

  and $\boldsymbol{X}_l = \boldsymbol{X}_d(l\Delta t)$ is an approximation to $\boldsymbol{X}(t_l)$.
- Midpoint rule for $L_d(\boldsymbol{X}_l, \boldsymbol{X}_{l+1})$.
- Use MATLAB to find the optimal trajectories that minimise the value of $G_d(\{\boldsymbol{X}_l\}_{l=0}^{N})$ and deduce c_0
  (a first guess obtained from asymptotics).

Comparison with simulations: Cellular flow ($U = 0$)



All simulations obtained for $c_0 \ll 1$ since $c_0 = 2\sqrt{\frac{\mathrm{Da}}{\mathrm{Pe}}}$.

- Difficult to go for larger c_0, due to sharp gradients in the concentration.
- Range of validity bounded below. For $U = 0$ we need $\mathrm{Da} \gg (\log \mathrm{Pe})^{-1}$.