Bounds on mixing norms for advection diffusion equations

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Camilla Nobili
University of Hamburg

In collaboration with Steffen Pottel

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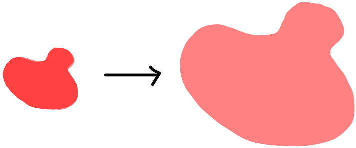
Mixing mechanisms

Molecular diffusion (homogenization) $\rightsquigarrow \partial_t f = \kappa \Delta f$



Stirring (transfer mass from low to high wavenumber modes) $\rightsquigarrow$ filamentation $\rightsquigarrow \partial_t f + \mathbf{u} \cdot \nabla f = 0$

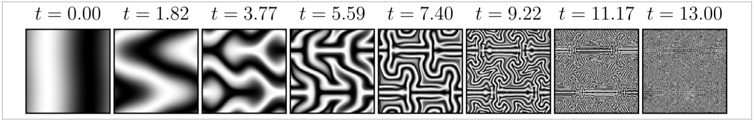


Figure: Miles & Doering 2018

In the spotlight: In a domain Ω, for a given stirring field $\mathbf{u}$, with $\nabla \cdot \mathbf{u} = 0$, consider

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = \kappa \Delta \theta \qquad \text{(AD)}$$

where $\theta = \theta(\mathbf{x}, t)$ is the concentration field.

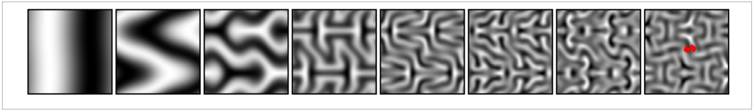


Figure: Miles & Doering 2018

What is the smallest scale that we expect to see in passive tracer mixing?

- "good" stirring creates very fine filaments that (due to incompressibility) get stretched longer, becoming thinner and thinner
- in presence of diffusion these filaments reach a minimal width

A good measure for mixing

Suppose $\Omega = \mathbb{T}^d$

- Variance is not a good measure for mixing (in regular contexts, $\mathbf{u}$ drops out)
- The norm

$$\|\nabla^{-1}\theta(t)\|_{L^2}^2 = \sum_{\mathbf{k}\neq 0} L^d \frac{|\hat{\theta}_{\mathbf{k}}(t)|^2}{|\mathbf{k}|^2}$$

  with $\hat{\theta}_{\mathbf{k}}(t) = \frac{1}{L^d}\int e^{-i\mathbf{k}\cdot\mathbf{x}}\theta(x)\,dx$ was introduced by *Mathew et.al 2005-2007* as it " downplays the role of small scales" (cit. Thiffeault). This norm captures both homogenization (pure diffusion, decrease Fourier amplitudes $|\theta_{\mathbf{k}}|$, $\mathbf{k} \neq 0$) and filamentation (pure advection, transfer mass from low to high wavenumbers modes).

Results of C.Miles and C.Doering, 2018

Quantify the ratio

$$\lambda(t) := \frac{\|\nabla^{-1}\theta(t)\|_{L^2}}{\|\theta(t)\|_{L^2}}$$

which we call *filamentation length*.

Theoretical results:

- For $\|\nabla \mathbf{u}(t)\|_{L^\infty} = 1$,

$$\lambda(t) \geq \lambda(0)\exp(-t)$$

- For $\|\mathbf{u}(t)\|_{L^\infty} = 1$,

$$\lambda(t) \gtrsim \frac{\|\theta_0\|_{L^2}}{\|\nabla\theta_0\|_{L^2}} \exp\left(-\frac{t}{2\kappa}\right)$$

Numerical results: for energy and enstrophy constrained flows, i.e. $\|\mathbf{u}(t)\|_{L^2} = C$ or $\|\nabla \mathbf{u}(t)\|_{L^2} = C$

3.1. Methodology

We solve (3) with either flow (4) or (5) by using a Fourier basis to represent the spatial domain with a 4th order Runge–Kutta time-stepping method. We slightly perturb the concentration field $\theta_0(\mathbf{x}) = \sin(2\pi x/L)$ by evolving the field according to (3) with a steady sin flow given by $\mathbf{u}(\mathbf{x}) = \sin(2\pi y/L)\hat{x}$ for a time duration of 0.01. The concentration field, resulting from this short time integration, is then used as an initial condition for the local-in-time optimisation scheme. This perturbation is necessary since the denominator is zero in both expressions (4) and (5) for pure Fourier modes such as θ_0 [2]. All simulation code is written in Python and available at http://github.com/cjm715/lit.

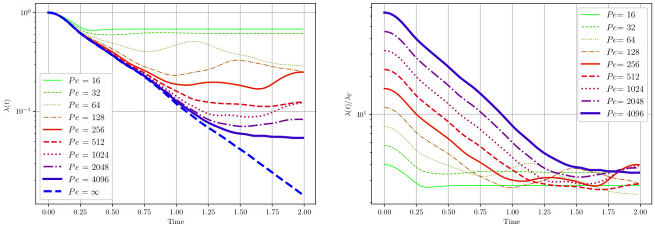


Figure: Numerical simulation with local-in-time optimal flows. Miles&Doering 2017. $\lambda(t)/\lambda_\Gamma = \lambda(t)\sqrt{\mathrm{Pe}}$ and $\mathrm{Pe} = \kappa^{-1}$ for $\|\nabla u\|_{L^\infty} = 1$

Conjecture

$$\lambda(t) := \frac{\|\nabla^{-1}\theta(t)\|_{L^2}}{\|\theta(t)\|_{L^2}} \xrightarrow{t\to\infty} \sqrt{\frac{\kappa}{\|\nabla u\|}} := \lambda_B\,.$$

Why is this conjecture hard to prove mathematically?

- While (pointwise) upper bounds on $\|\theta(t)\|_{L^2}$ are easy due to the boundary conditions and incompressibility, **lower bounds** are very difficult to prove
- The quotient in λ is very tricky. Convergence for long time to the Batchelor scale would mean that the norms $\|\theta\|_{L^2}$ and $\|\nabla^{-1}\theta\|_{L^2}$ will go *exponentially* to zero **at the same rate**!
- Will this convergence hold for *any* given flow? For which **class of flows** does this happen?
- Even on simple **geometries** like $\Omega = \mathbb{T}^d$ the situation is not clear (as, roughly speaking, energy comes back in).

What happens when $\Omega = \mathbb{R}^d$?

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta - \kappa \Delta \theta = 0 & \text{in } \mathbb{R}^d \\ \nabla \cdot u = 0 \\ \theta(x,0) = \theta_0(x) \end{cases}$$

Interpolation estimate:

$$\frac{\|\nabla^{-1}\theta(t)\|_2}{\|\theta(t)\|_2} \geq \frac{\|\theta(t)\|_2}{\|\nabla\theta(t)\|_2} \tag{1}$$

Need:

- A pointwise (in time) lower bound for $\|\theta(t)\|_2$
- A pointwise (in time) upper bound for $\|\nabla\theta(t)\|_2$

Looking for a lower bound for $\|\theta(t)\|_2$...

Decompose:

$$\theta(x,t) = h(x,t) + \eta(x,t)$$

where

$$\begin{cases} \partial_t h - \kappa \Delta h = 0 \\ h(x,0) = \theta_0(x) \end{cases} \qquad \begin{cases} \partial_t \eta - \kappa \Delta \eta = -u \cdot \nabla \theta \\ \eta(x,0) = 0 \end{cases}$$

then

$$\boxed{\|\theta(t)\|_{L^2} \geq \|h(t)\|_{L^2} - \|\eta(t)\|_{L^2}}$$

Lower bound for $\|h(t)\|_{L^2}$: Assume $|\hat{\theta}_0(\xi)| \geq M$ for $|\xi| \leq \delta$, then

$$\begin{aligned} \int_{\mathbb{R}^d} |h(x,t)|^2 \, dx &= \int_{\mathbb{R}^d} |\hat{h}(\xi,t)|^2 \, d\xi \\ &= \int_{\mathbb{R}^d} |\hat{\theta}_0(\xi)|^2 e^{-2\kappa|\xi|^2 t} \, d\xi \\ &\geq M^2 \int_{|\xi|\leq\delta} e^{-2\kappa|\xi|^2 t} \, d\xi \\ &\sim M^2 \delta^d e^{-2\kappa\delta^2} t^{-\frac{d}{2}} \end{aligned}$$

Upper bound for $\|\eta(t)\|_{L^2}$:

$$\frac{d}{dt}\|\eta(t)\|_2^2 \leq -2\kappa\|\nabla\eta(t)\|^2 + 2\|\nabla h(t)\|_\infty \|u(t)\|_2 \|\theta(t)\|_2$$

Need an upper bound for $\|\theta(t)\|_2^2$

...through an upper bound for $\|\theta(t)\|_2^2$

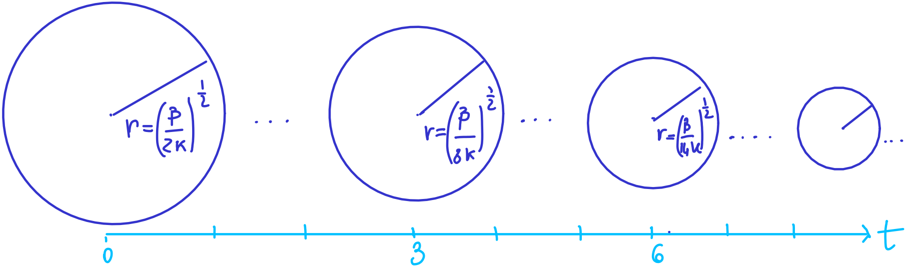
Claim:

$$\|\theta(t)\|_2^2 \lesssim \kappa^{-\ell}(1+t)^{-\frac{d}{2}}$$

<u>Sketch</u>: Fourier-splitting method (Schonbeck'85)

$$\frac{d}{dt}((1+t)^{\beta}\|\theta(t)\|_2^2) \leq \beta(1+t)^{\beta}\int_{\mathcal{S}(t)}|\hat{\theta}(\xi,t)|^2\,d\xi$$

where $\mathcal{S}(t) = \left\{\xi \in \mathbb{R}^d \middle| \quad |\xi| \leq \left(\frac{\beta}{2\kappa(1+t)}\right)^{\frac{1}{2}}\right\}$



Bound on $\int_{\mathcal{S}(t)} |\hat{\theta}(\xi,t)|^2 \, d\xi$: Representation formula, in Fourier variables

$$\hat{\theta}(\xi,t) = e^{-\kappa|\xi|^2}\hat{\theta}_0(\xi) + \int_0^t e^{-\kappa|\xi|^2(t-s)}(-\widehat{u \cdot \nabla\theta})(\xi,s)ds$$

Square and integrate over $S(t)$ and use

$$|\widehat{u \cdot \nabla\theta}| \leq C|\xi| \|\theta(t)\|_2 \|u(t)\|_2 \, .$$

Need $\|u(t)\|_{L^2} \sim (1+t)^{-\alpha}$ for some $\alpha > \frac{1}{2}$ for the claim!

Upper bound for $\|\theta(t)\|_{L^2} \rightsquigarrow$ upper bound on $\|\eta(t)\|_{L^2}$

$$\|\eta(t)\|_{L^2} \lesssim_\kappa \begin{cases} (1+t)^{-\frac{d}{4}+\frac{1}{4}-\frac{\alpha}{2}} & \frac{1}{2} < \alpha < \frac{3}{2} \\ (1+t)^{-\frac{d}{4}-\frac{1}{2}} & \alpha \geq \frac{3}{2} \end{cases}$$

Put together:

$$\|\theta(t)\|_{L^2} \geq \|h(t)\|_{L^2} - \|\eta(t)\|_{L^2} \gtrsim_{M,\delta,\kappa} (1+t)^{-\frac{d}{4}}(1-(1+t)^{-\frac{\alpha}{2}+\frac{1}{4}})$$

Our result

Theorem (Nobili & Pottel, '20)

In $\mathbb{R}^3$, let the following conditions be satisfied

- $|\hat{\theta}_0(\xi)| \geq M$ *for* $|\xi| \leq \delta$
- $\|u(t)\|_{L^2} \sim (1+t)^{-\alpha}$ *for some* $\alpha > \frac{1}{2}$
- $\|\nabla u(t)\|_{L^\infty} \sim (1+t)^{-\nu}$

then

$$\lambda(t) \gtrsim_{M,\delta,\kappa} (1+t)^{\frac{1}{2}} f(t)\,,$$

where

$$(1+t)^{\frac{1}{2}} f(t) \overset{t\to\infty}{\to} \begin{cases} +\infty & \text{for } \nu \geq 1 \quad \textit{dispersion} \\ 0 & \text{for } \nu \in [0,1) \quad \textit{mixing} \end{cases}$$

Remarks

- Result agrees with *diffusive dispersion*:

$$\boxed{\mathbb{E}(\mathbf{X}_t^2) = 2\kappa t}.$$

- Decay of velocity field at infinity is crucial.
- It would be interesting to know whether the results are sharp for some velocity field.
- Passing from $\mathbb{R}^d$ to $\mathbb{T}^d$ (within the method) would require some new idea...

An (connected and) very intersting open problem...

The limit for very small κ is relevant, as the molecular diffusivity is generally very small. But how can the (Batchelor-scale) conjecture be justified in the limit $\kappa \to 0$?
L^2 energy decay of solutions to (AD):

$$\frac{1}{2}\|\theta^k(t)\|_{L^2}^2 = \frac{1}{2}\|\theta^k(0)\|_{L^2}^2 - \kappa \int_0^t \|\nabla \theta^\kappa(s)\|_{L^2}^2 \, ds$$

In certain turbulent regimes it is expected

$$\kappa \int_0^t \|\nabla \theta^\kappa(t)\|_{L^2}^2 \geq c > 0$$

with c **independent of** κ.

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THE EUROPEAN PHYSICAL JOURNAL SPECIAL TOPICS

Regular Article

Singular eigenvalue limit of advection-diffusion operators and properties of the strange eigenfunctions in globally chaotic flows*

Stefano Cerbelli[1,a], Massimiliano Giona[1,b], Olexander Gorodetskyi[2] and Patrick D. Anderson[2,c]

[1] Dipartimento di Ingegneria Chimica DICMA Facoltà di Ingegneria, La Sapienza Università di Roma via Eudossiana 18, 00184, Roma, Italy
[2] Materials Technology, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands

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Abstract. Enforcing the results developed by Gorodetskyi et al. [O. Gorodetskyi, M. Giona, P. Anderson, Phys. Fluids **24**, 073603 (2012)] on the application of the mapping matrix formalism to simulate advective-diffusive transport, we investigate the structure and the properties of strange eigenfunctions and of the associated eigenvalues up to values of the Péclet number $Pe \sim \mathcal{O}(10^8)$. Attention is focused on the possible occurrence of a singular limit for the second eigenvalue, ν_2, of the advection-diffusion propagator as the Péclet number, Pe, tends to infinity, and on the structure of the corresponding eigenfunction. Prototypical time-periodic flows on the two-torus are considered, which give rise to toral twist maps with different hyperbolic character, encompassing Anosov, pseudo-Anosov, and smooth nonuniformly hyperbolic systems possessing a hyperbolic set of full measure. We show that for uniformly hyperbolic systems, a singular limit of the dominant decay exponent occurs, $\log|\nu_2| \to \text{constant} \neq 0$ for $Pe \to \infty$, whereas $\log|\nu_2| \to 0$ according to a power-law in smooth non-uniformly hyperbolic systems that are not uniformly hyperbolic. The mere presence of a nonempty set of nonhyperbolic points (even if of zero Lebesgue measure) is thus found to mark the watershed between regular vs. singular behavior of ν_2 with Pe as $Pe \to \infty$.

Simulations shows that $\lim_{\mathrm{Pe}\to\infty} \Lambda(\mathrm{Pe}) = \mathrm{c} > 0$ where Λ is the dominant decay exponent of (AD).

Interpretation: in terms of the "Batchelor-conjecture" it means that the Batchelor scale saturates at a non-zero value in the limit of small κ.


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Thank you for your attention!