Estimates on the rate of enhanced dissipation

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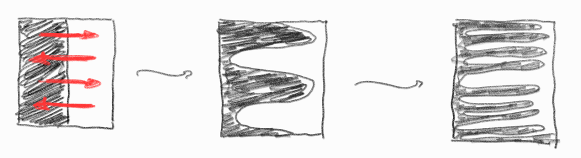
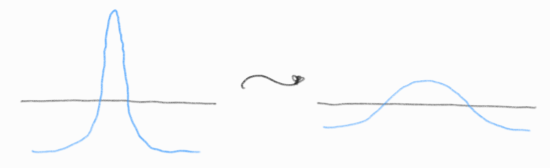
Fluid mixing of diffusive quantity

Advection: Filamentation / reduction of scales



Diffusion: Reduction of intensity (gradients)



Linear model equations

$$\partial_t \theta + u \cdot \nabla \theta = \kappa \Delta \theta$$

- For simplicity: $\mathbb{T}^d$;
- $\theta = \theta(t, x) \in \mathbb{R}$ passive scalar;
- $u = u(t, x) \in \mathbb{R}^d$ velocity, divergence-free, $\nabla \cdot u = 0$;
- $0 < \kappa \ll 1$ diffusivity constant.

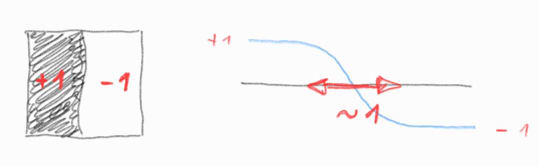
$$\int_{\mathbb{T}^d} \theta(t, x)\, dx = \int_{\mathbb{T}^d} \theta_0(x)\, dx \overset{!}{=} 0, \quad \text{say.}$$

Some oversimplified heuristics

Viewer discretion is advised...

Consider regular initial configuration, $O(1)$ variations,

$$\theta_0 \in [-1, 1], \quad \|\theta_0\| \sim 1, \quad \|\nabla\theta_0\| \sim 1.$$



Apply straining flow, not too rough, say,

$$\|\nabla u(t)\| \equiv U \sim 1.$$

Result: Initially reduction of scale by advection, later reduction of intensity by diffusion.

Early advection-dominated mixing stage

Neglect diffusion: $\kappa = 0$.

Constant intensity:

$$\|\theta(t)\| = \|\theta_0\|.$$

Reduction of scale:

$$\|\nabla^{-1}\theta(t)\| \sim e^{-cUt}.$$

Lower bound: Crippa–De Lellis '08, Brenier–Otto–S. '11, S. '11, Iyer–Kiselev–Xu '14, Leger '18.

Upper bound: Alberti–Crippa–Mazzucato '14, '19, Yao–Zlatos '17, Elgindi–Zlatos '19, Blumenthal–Bedrossian–Punshon-Smith '19.

Late diffusion-dominated mixing stage

Neglect advection: $u = 0$.

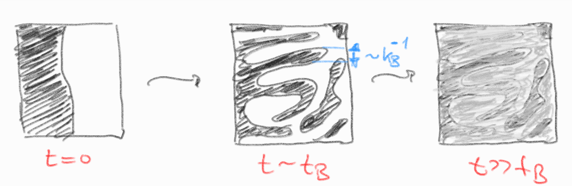
Scale-wise dispersion for heat equation. For any wavenumber k:

$$\hat{\theta}_k(t) = e^{-\kappa |k|^2 t} \hat{\theta}_k(0).$$

Decrease of variance dominated by *smallest* wavenumber k_B:

$$\|\theta(t)\| \sim e^{-\kappa |k_B|^2 t} \|\theta_0\|.$$

Crossover wavenumber k_B



Balance between advection and diffusion determined by dimensional analysis:

$$k_B \sim \sqrt{\frac{U}{\kappa}}.$$

Associated length k_B^{-1} is the Batchelor scale.

We might thus expect that for large times:

$$\|\theta(t)\| \leq e^{-Dt}\|\theta_0\|$$

for some $D \sim 1$ and all $t \gg 1/D$.

Enhanced dissipation!

We expect

$$\|\theta(t)\| \leq e^{-Dt}\|\theta_0\|$$

for some $D \sim 1$ and all $t \gg 1/D$.

Recall: $D \sim \kappa$ for heat equation!

Rigorous results

What is known?

Enhanced dissipation: $\exists D \gg \kappa$, $\Lambda \geq 1$ such that

$$\|\theta(t)\| \leq \Lambda e^{-Dt} \|\theta_0\|.$$

Qualitiatve results: Constantin–Kiselev–Ryzhik–Zlatos '08. Sharp characterization of steady velocity fields.

Quantitative results for shear flows, $\Lambda \sim 1, D \sim \kappa^{\alpha}$ *for some* $\alpha \in (0,1)$*: Bedrossian–Coti Zelati '14, Wei '19, Coti Zelati–Drivas '19, Coti Zelati '19, Colombo–Coti Zelati–Widmayer '20, Coti Zelati–Dolce '20; ... for randomly forced fluid systems: Bedrossian–Blumenthal–Punshon-Smith '19:*

$$\Lambda \sim 1, D \sim \log^{-1} \frac{1}{\kappa} \quad \text{and} \quad \Lambda \sim \frac{1}{\kappa}, D \sim 1.$$

New result: Universal upper bound

Suppose $\|\theta_0\| \sim \|\nabla\theta\| \sim 1$ and let u be arbitrary, satisfying

$$\int_0^t \|\nabla u\| \, dt \lesssim 1 + t^{\alpha},$$

for some $\alpha \in (0, 1]$. (**Example:** $\|\nabla u\| \equiv U \implies \alpha = 1$.)

Theorem (S. '20). *If there exist Λ and D such that*

$$\|\theta(t)\| \leq \Lambda e^{-Dt} \|\theta_0\|,$$

then

$$D \lesssim \left(\frac{\Lambda}{\log \frac{1}{\kappa}} \right)^{\alpha}.$$

Remarks

- First absolute upper bound on rate of enhanced dissipation.
- Pivotal cases: $*$ $\Lambda \sim 1 \Longrightarrow D \lesssim \log^{-1} \frac{1}{\kappa}$. This bound is sharp by *Bedrossian et al.* modulo stochastics...
  $*$ $D \sim 1 \Longrightarrow \Lambda \gtrsim \log \frac{1}{\kappa}$. Optimal result not known.
- Similar (but suboptimal) results by *Bruè–Nguyen '20*.
- *Coti Zelati–Delgadino–Elgindi '19*: IF there exists a universal mixer with $\|\nabla u\|_{L^\infty} \leq 1$, then $D \gtrsim \log^{-2} \frac{1}{\kappa}$.
- Result is different from the lower bound

$$\|\theta(t)\| \gtrsim e^{-Dt} \quad \text{— which is certainly controversial.}$$

- $\|\theta_0\| \sim \|\nabla \theta_0\| \sim 1$ is necessary, otherwise: $\forall D \exists \theta_0 \ldots$
- The proof relies on stability estimates for advection-diffusion equations in terms of Kantorovich–Rubinstein distances with logarithmic cost (cf. *Crippa–De Lellis '08, Brenier–Otto–S. '11, S. '13, '17, '18.*)

A final remark on mixing

By a simple interpolation argument, e.g., for $\alpha = 1$.

Theorem (S. '20). *If there exist* Λ *and* D *such that*

$$\|\nabla^{-1}\theta(t)\| \lesssim \frac{\Lambda^2\sqrt{\kappa}}{D} e^{-Dt},$$

then

$$D \lesssim \frac{\Lambda}{\log\frac{1}{\kappa}}.$$

Note:

$$\frac{\Lambda^2\sqrt{\kappa}}{D} \approx \text{Batchelor scale (modulo logarithms).}$$