On Universal Mixers

Andrej Zlatoš

UCSD

IPAM, January 13, 2021

Basic setup

Mixing in the absence of diffusion (e.g., two immiscible fluids), or when diffusion is negligible on relevant time scales, is modeled by the transport equation

$$\rho_t + u \cdot \nabla \rho = 0$$

where $\rho \in L^\infty$ is the concentration of a mixed substance (or takes values ± 1 for immiscible fluids) with $\rho(\cdot, 0) = \rho_0$, and u is the prescribed divergence-free (i.e.,$\nabla \cdot u = 0$) mixing flow.

- Consider $(x, t) \in Q_d \times [0, \infty)$ with $Q_d = (0, 1)^d$ being either a cube in $\mathbb{R}^d$ (then $u \cdot n = 0$ on ∂Q_d) or the torus $\mathbb{T}^d$.
- Without loss assume that ρ_0 (and hence also ρ) is mean zero, i.e., $\int_{Q_d} \rho_0 dx = 0$.

Goal: Look for incompressible flows that are as efficient as possible in mixing (in an appropriate sense, see below).

Mixing in the presence of diffusion

When diffusion is present, one can look at decay of L^p norms to quantify mixing. E.g., Constantin-Kiselev-Ryzhik-Z. (Annals of Math. 2008) characterized all time-independent flow profiles u such that for each mean-zero $\rho_0 \in L^1$ and $\delta > 0$ there is A_0 such that ρ solving

$$\rho_t + Au \cdot \nabla\rho = \Delta\rho \qquad \text{and} \qquad \rho(\cdot, 0) = \rho_0$$

satisfies $\|\rho(\cdot, \delta)\|_{L^\infty} < \delta$ for all $A > A_0$. The question is when $\|\rho(\cdot, t)\|_{H^1}$ becomes large quickly, so that $\|\rho(\cdot, t)\|_{L^2}$ decays fast.

The flows that do this arbitrarily well when A is large (i.e., relaxation-enhancing flows) are precisely those for which the operator $u \cdot \nabla$ has no H^1 eigenfunctions except constants.

Without diffusion each L^p norm of ρ is conserved, so we need measures of mixing that better capture small scale variations.

Definitions of mixing efficiency

Let $f \in L^\infty(Q_d)$ be mean-zero and $⨍_A f(x)dx := ⨍_{A \cap Q_d} f(x)dx$.

- We say that f is κ-mixed to scale ε, with $\kappa, \varepsilon \in (0,1)$, if

  $$\left| ⨍_{B_\varepsilon(y)} f(x)dx \right| \leq \kappa \|f\|_{L^\infty}$$

  for each $y \in Q_d$. The smallest such ε is the (κ-dependent) geometric mixing scale of f.
- The functional mixing scale of f is $\|f\|^2_{\dot{H}^{-1/2}} \|f\|^{-2}_{L^\infty}$.

The geometric definition is a generalization of the special case $d = 2$, $\kappa = \frac{1}{3}$, $f(Q_2) = \{-1, 1\}$ considered by Bressan. Here the "worst-mixed" region determines the mixing scale, but deviations of size roughly κ are tolerated at all scales.

The mix norm

The functional definition does not tolerate deviations but averages the degree of “mixedness” of f over all of Q_d. In fact, Matthew-Mezic-Petzold showed that $\|f\|_{\dot{H}^{-1/2}}$ is equivalent to the mix-norm

$$\Phi(f) := \left[\int_{Q_d \times (0,1)} \left(\fint_{B_r(y)} f(x)dx \right)^2 dydr \right]^{1/2}$$

for mean-zero functions f, which also gives a nice connection to the geometric definition.

Other $\dot{H}^{-s}$ norms can be used for the functional definition (e.g., $\dot{H}^{-1}$ is more standard) but they lack the equivalence with the mix-norm. Also, all our results for $\dot{H}^{-1/2}$ extend to $\dot{H}^{-1}$.

Bressan's conjecture

Let ρ solve $\rho_t + u \cdot \nabla \rho = 0$, with $\rho(\cdot, 0) = \rho_0 \in L^\infty$ mean zero. If $\rho(\cdot, t)$ is κ-mixed to scale ε, we say that the flow u κ-mixes ρ_0 to scale ε in time t. Bressan conjectured in 2003 that if a divergence-free u on Q_2 satisfying the constraint

$$\sup_{t>0} \|\nabla u(\cdot, t)\|_{L^1} \leq 1$$

$\frac{1}{3}$-mixes $\rho_0 = \chi_{(0,1/2)\times(0,1)} - \chi_{(1/2,1)\times(0,1)}$ to scale ε in time t, then $t \geq C|\log \varepsilon|$ (with some ε-independent $C < \infty$). That is, under the above constraint, the geometric mixing scale of $\rho(\cdot, t)$ cannot decrease faster than exponentially (for $\kappa = \frac{1}{3}$).
This conjecture is still open but its version with constraint

$$\sup_{t>0} \|\nabla u(\cdot, t)\|_{L^p} \leq 1$$

was proved for any $p > 1$ by Crippa-De Lellis (J. Reine Angew. Math. 2008). Same result holds for functional mixing scales, by Seis (Nonlinearity 2013) and Iyer-Kiselev-Xu (Nonlinearity 2014).
Note: if u solves stationary N-S, $\nu\|\nabla u\|_{L^2}^2$ is energy intake rate.

Exponential mixing

Is exponential mixing possible under the above constraints?
If so, for which mean zero $\rho_0 \in L^\infty$ and $p \in [1, \infty]$?

- Yao-Z. (JEMS 2017) showed exponential mixing for any such ρ_0 and all $p \in [1, \frac{3+\sqrt{5}}{2})$ (incl. $p = 2$).
  For $p \geq \frac{3+\sqrt{5}}{2}$ we proved that geometric (not functional) mixing scales of size $O(e^{-t^{\nu_p}/C_p})$ with $\nu_p > \frac{3}{4}$ are possible.
- Alberti-Crippa-Mazzucato (JAMS 2019) showed exponential mixing for all $p \geq 1$, but only for special ρ_0 (characteristic function of half-torus).

In both cases the flow intricately depends on ρ_0, both results only hold in 2 dimensions, and in both decay becomes only algebraic for constraint $\sup_{t>0} \|u(\cdot, t)\|_{W^{s,p}} \leq 1$ with $s > 1$ (flows have self-similar structure, with exponentially decaying scales).

Are there universal (exponential) mixers? In which dimensions? How about for $W^{s,p}$ constraints on u with $s > 1$?

Main results

Theorem (Elgindi-Z., Adv. Math 2019)

For any $d \geq 2$, there is a divergence-free time-periodic u on Q_d satisfying no-flow/periodic boundary conditions such that $u \in L^\infty([0,\infty); W^{s,p}(Q_d))$ for any $p \in [1, \frac{3+\sqrt{5}}{2})$ and $s < \frac{1}{p} + \frac{\sqrt{5}-1}{2}$, and u is a universal mixer (i.e., mixing scale converges to 0 as $t \to \infty$ for each mean zero $\rho_0 \in L^\infty$) as well as an almost-universal exponential mixer (mixing scale decays exponentially for any $\rho_0 \in H^\sigma$, $\sigma > 0$, with the exponential rate depending only on σ) in both geometric and functional sense.

This allows constraints with any $s < \frac{1+\sqrt{5}}{2}$.
The decay starts after an initial delay that depends on ρ_0.
The geometric mixing result holds for all $\kappa > 0$, with the exponential rate independent of κ.

Theorem (Elgindi-Z., Adv. Math 2019)

No universal mixer has a mixing rate (exponential or otherwise) that works for all mean zero $\rho_0 \in L^\infty$ (even after an initial delay).

Relation to dynamical systems results

Definitions of exponential mixing in dynamical systems literature typically relate to our definition of almost-universal exponential mixers in functional sense: for all $\rho_0 \in X$ and $f \in Y$ we have $\left| \int_{Q_d} \rho(x,t) f(x)\, dx \right| \lesssim e^{-ct} \|\rho_0\|_X \|f\|_Y$.

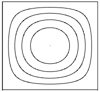
Exponentially mixing maps well known in dynamical systems literature, but not quite exponentially mixing flows.

- Anosov flows on special, at least 3D manifolds, e.g., geodesic flows on unit tangent bundles of some compact Riemannian manifolds with negative curvature.
- Low regularity (only BV, not continuous) exponentially mixing time-periodic flows exist on $\mathbb{T}^d$ ($d \geq 2$), e.g., realizing Arnold's cat map as 1-period flow map.
- Ours seem to be the only known examples on Q_d with $W^{1,1}$ or better regularity (in fact, even continuous) and satisfying uniform-in-time constraints. Also the only examples on physical domains (subsets of $\mathbb{R}^2$ or $\mathbb{R}^3$).
- Stochastic flows on $\mathbb{T}^d$ (constraint holds only in average) by Bedrossian, Blumenthal, Punshon-Smith (preprint 2019)

Regularity properties of our universal mixers

The basic building block of our flows will be a time-independent 2D flow that rotates the square Q_2 by 90° in time 1.
This was already constructed by Yao-Z. but we need to adjust the construction slightly for a better control of the regularity of u (Yao-Z. only needed $u \in W^{1,p}$). Let

$$\psi_\alpha(x, y) := \frac{\sin(\pi x)\sin(\pi y)}{(\sin(\pi x) + \sin(\pi y))^\alpha}$$



for $\alpha \in [0, 1]$. Then the divergence-free flow $\nabla^\perp \psi_\alpha$ rotates each level set $\{\psi_\alpha = r\}$ by 90° in time

$$T_\alpha(r) := \frac{1}{4} \int_{\psi_\alpha = r} \frac{1}{|\nabla \psi_\alpha|} d\sigma$$

So take $u_\alpha := T_\alpha(\psi_\alpha)\nabla^\perp \psi_\alpha$ for α that makes it most regular.

Regularity properties of $u_\alpha = T_\alpha(\psi_\alpha)\nabla^\perp\psi_\alpha$

Regularity of u_α is limited by behavior near ∂Q_2 (when $\psi_\alpha \approx 0$). Regularity of $\nabla^\perp\psi_\alpha$ there is worse for $\alpha \approx 1$, but regularity of $T_\alpha(\psi_\alpha)$ there is worse for $\alpha \approx 0$ (in fact, $T_0(0) = \infty$).

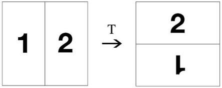
The optimal value is $\alpha_* := 3 - \sqrt{5}$ and one can prove that u_{α_*} belongs to all the spaces from the theorem on Q_2 (via a control on the rate of blowup of Du_{α_*} and $D^2u_{\alpha_*}$ near ∂Q_2 and an interpolation argument). This is only true when Q_2 is the square (one needs to adjust this construction on $\mathbb{T}^2$).

With extra work (technical and annoying!) one can also show that the flow maps of u_α are uniformly (in time) Hölder continuous, which is not obvious because u_α is only Hölder continuous. We will control ρ well at integer times, and this can then be used to get sufficient control in between.

Connection to the folded Baker's map

Now use scaled copies of u_{α_*} to rotate left/right halves on Q_2 counterclockwise/clockwise by 90° in time $\frac{1}{2}$ (these still belong to the same $W^{s,p}$ spaces) and then rotate all of Q_2 counterclockwise by 90° in time $\frac{1}{2}$.

This map T — folded Baker's map — is the time-1 map of our time-1-periodic universal mixer on Q_2 (not on $\mathbb{T}^2$).



Each dyadic square is stretched twice horizontally and squeezed twice vertically by T (and possibly rotated by 180°).

Repeated applications of T eventually turn it into a horizontal strip of length 1 and then into exponentially growing number of horizontal strips relatively regularly distributed over Q_2.
So if Q, Q' are any two dyadic squares of side length 2^{-k}, then $T^{2k}(Q') \cap Q$ is a single rectangle of size $2^{-k} \times 2^{-3k}$.

Higher dimensions and tori similar, with some adjustments.