Maximum Amplification of Enstrophy in Navier-Stokes Flows and the Hydrodynamic Blow-Up Problem

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Reference

J. Fluid Mech. (2020), *vol.* 893, A22. **893** A22-1
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Maximum amplification of enstrophy in three-dimensional Navier–Stokes flows

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- Navier-Stokes system ($\Omega = [0, L]^d$, $d = 2, 3$)

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} p - \nu \Delta \mathbf{u} = \mathbf{0}, & \text{in } \Omega \times (0, T] \\ \boldsymbol{\nabla} \cdot \mathbf{u} = 0, & \text{in } \Omega \times (0, T] \\ \mathbf{u} = \mathbf{u}_0 & \text{in } \Omega \text{ at } t = 0 \\ \text{Periodic Boundary Condition} & \text{on } \Gamma \times (0, T] \end{cases}$$

- **The Big Question:**

  Given a smooth initial condition $\mathbf{u}_0$*, does the Navier-Stokes system always admit smooth solutions* $\mathbf{u}(t)$ *for arbitrarily long times* t*?*

  (solutions which are not “smooth” are not physically meaningful ...)

- One of the Clay Institute “Millennium Problems” ($ 1M prize!)

  `http://www.claymath.org/millennium/Navier-Stokes_Equations`

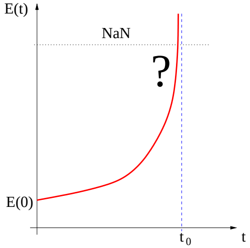
What is known? — Available Estimates

- A Key Quantity — Enstrophy

$$\mathcal{E}(t) \triangleq \int_\Omega |\boldsymbol{\nabla} \times \mathbf{u}|^2 \, d\Omega \qquad (= \|\boldsymbol{\nabla}\mathbf{u}\|_2^2)$$

- Smoothness of Solutions $\Longleftrightarrow$ Bounded Enstrophy (Foias & Temam, 1989)

$$\max_{t\in[0,T]} \mathcal{E}(t) < \infty \quad ???$$



- Can estimate $\frac{d\mathcal{E}(t)}{dt}$ using the momentum equation, Sobolev's embeddings, Young and Cauchy-Schwartz inequalities, ...
  - Remark: incompressibility not used in these estimates ....

- Bounds on the rate of growth of enstrophy — general form

$$\frac{d\mathcal{E}}{dt} < C\,\mathcal{E}^{\alpha}, \quad C > 0, \quad \alpha = \alpha(d) > 0$$

- Energy equation ($\mathcal{K}(t) \triangleq \int_{\Omega} \mathbf{u}^2 \, d\Omega$)

$$\frac{d\mathcal{K}}{dt} = -2\nu\mathcal{E}$$

$$\mathcal{K}(t) - \mathcal{K}(0) = -2\nu \int_0^t \mathcal{E}(\tau)\, d\tau \quad \Longrightarrow \quad \int_0^t \mathcal{E}(\tau)\, d\tau \le \frac{1}{2\nu}\mathcal{K}_0$$

- When $\alpha <= 2$, by Grönwall's inequality: $\mathcal{E}(t) \le \mathcal{E}_0 \exp\left[\frac{C\mathcal{K}_0}{2\nu}\right]$

  $\Longrightarrow$ Enstrophy bounded for *all* times

- When $\alpha > 2$, no finite a priori bound on enstrophy ...

- 2D Case:

$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{C^2}{\nu}\mathcal{E}(t)^2$$

  - Grönwall's lemma and energy equation yield $\forall_t \ \mathcal{E}(t) < \infty$
  - smooth solutions exist for all times

- 3D Case:

$$\frac{d\mathcal{E}(t)}{dt} \leq \frac{27\,C^2}{128\nu^3}\mathcal{E}(t)^3$$

  - corresponding estimate not available ....
  - upper bound on $\mathcal{E}(t)$ blows up in finite time

$$\mathcal{E}(t) \leq \frac{\mathcal{E}(0)}{\sqrt{1 - 4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$$

  - singularity in finite time cannot be ruled out!

- Can we actually find solutions "saturating" a given estimate?

- Lu & Doering (2008) constructed vector fields maximizing $\frac{d\mathcal{E}(t)}{dt}$ instantaneously by solving the problem

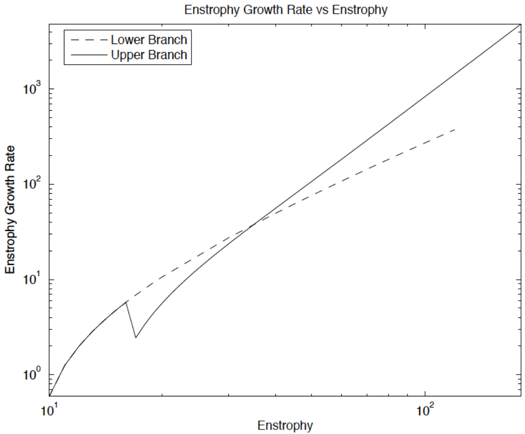
$$\max_{\mathbf{u}\in H^2(\Omega),\ \boldsymbol{\nabla}\cdot\mathbf{u}=0} \frac{d\mathcal{E}(t)}{dt}$$

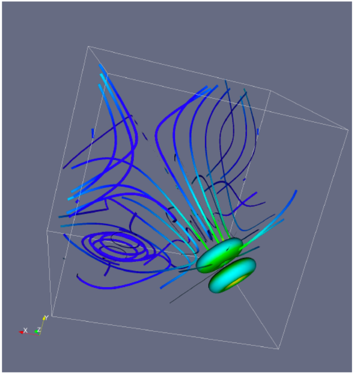
$$\text{subject to } \ \mathcal{E}(t) = \mathcal{E}_0$$

  where

  - $$\frac{d\mathcal{E}(t)}{dt} = -\nu\|\boldsymbol{\Delta}\mathbf{u}\|_2^2 + \int_\Omega \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\cdot\boldsymbol{\Delta}\mathbf{u}\, d\Omega$$
  - $\mathcal{E}_0$ is a parameter

- Numerical solution using a gradient-based descent method

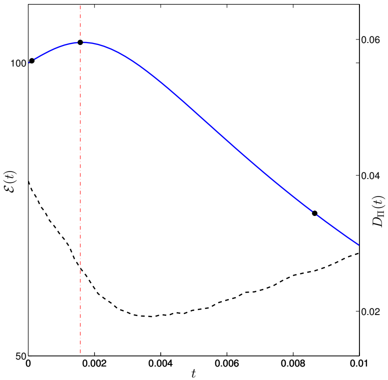




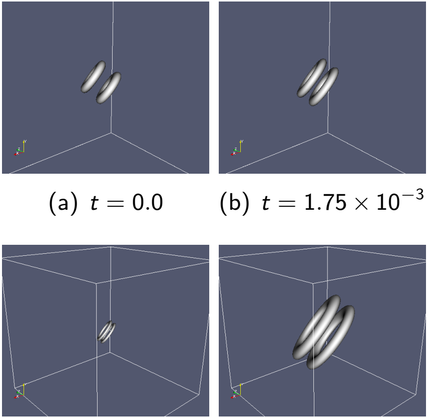
vorticity field (top branch)

$$\left[\frac{d\mathcal{E}(t)}{dt}\right]_{max} = 8.97 \times 10^{-4}\ \mathcal{E}_0^{2.997}$$

The instantaneous estimate $d\mathcal{E}(t)/dt \leq c\mathcal{E}(t)^3$ is sharp, up to prefactor!
(Lu & Doering, 2008)



$\mathcal{E}_0 = 100$



(a) $t = 0.0$ (b) $t = 1.75 \times 10^{-3}$

(c) $t = 8.63 \times 10^{-3}$ (d) $t = 0.198$

The extreme initial rate of growth of enstrophy is rapidly depleted (Ayala & Protas, 2017)

Energy-type Estimates for Related Problems

	Best Estimate	Sharp?
1D Burgers instantaneous	$\frac{d\mathcal{E}(t)}{dt} \le \frac{3}{2}\left(\frac{1}{\pi^2\nu}\right)^{1/3}\mathcal{E}(t)^{5/3}$	Yes Lu & Doering (2008)
1D Burgers finite–time	$\max_{t\in[0,T]}\mathcal{E}(t) \le \left[\mathcal{E}_0^{1/3} + \left(\frac{L}{4}\right)^2\left(\frac{1}{\pi^2\nu}\right)^{4/3}\mathcal{E}_0\right]^3$	No Ayala & P. (2011)
2D Navier–Stokes instantaneous	$\frac{d\mathcal{P}(t)}{dt} \le -\left(\frac{\nu}{\mathcal{E}}\right)\mathcal{P}^2 + C_1\left(\frac{\mathcal{E}}{\nu}\right)\mathcal{P}$ $\frac{d\mathcal{P}(t)}{dt} \le \frac{C_2}{\nu}\mathcal{K}^{1/2}\mathcal{P}^{3/2}$	[Yes] Ayala & P. (2013) Ayala, Doering & Simon (2017)
2D Navier–Stokes finite–time	$\max_{t>0}\mathcal{P}(t) \le \left[\mathcal{P}_0^{1/2} + \frac{C_2}{4\nu^2}\mathcal{K}_0^{1/2}\mathcal{E}_0\right]^2$	[Yes] Ayala & P. (2013)
3D Navier–Stokes instantaneous	$\frac{d\mathcal{E}(t)}{dt} \le \frac{27C^2}{128\nu^3}\mathcal{E}(t)^3$	Yes Lu & Doering (2008)
3D Navier–Stokes finite–time	$\mathcal{E}(t) \le \frac{\mathcal{E}(0)}{\sqrt{1-4\frac{C\mathcal{E}(0)^2}{\nu^3}t}}$	???

- Look for trajectories with a rate of growth $d\mathcal{E}/dt \sim \mathcal{E}^{\alpha}$, $2 < \alpha < 3$, sustained over sufficiently long times
  - maximize the growth of enstrophy over *finite* time T, with $\mathcal{E}_0 > 0$ fixed

$$\max_{\mathbf{u} \in \mathcal{Q}_{\mathcal{E}_0}} \mathcal{E}(\mathbf{u}(T)), \quad \text{where}$$

$$\mathcal{Q}_{\mathcal{E}_0} = \left\{ \mathbf{u} \in H^1(\Omega) : \ \boldsymbol{\nabla} \cdot \mathbf{u} = 0, \ \mathcal{E}(\mathbf{u}) = \mathcal{E}_0 \right\},$$

$$\text{subject to:} \quad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} + \boldsymbol{\nabla} p - \nu \Delta \mathbf{u} = \mathbf{0}, & \text{in } \Omega \times (0, T] \\ \boldsymbol{\nabla} \cdot \mathbf{u} = 0, & \text{in } \Omega \times (0, T] \\ \mathbf{u} = \mathbf{u}_0 & \text{in } \Omega \text{ at } t = 0 \\ \text{Periodic Boundary Condition} & \text{on } \Gamma \times (0, T] \end{cases}$$

  - A formidable, but solvable, PDE optimization problem

- Solution via discretized gradient flow (gradient ascent method)

$$\mathbf{u}^{(n+1)}_{0;\mathcal{E}_0,T} = \mathbb{P}_{\mathcal{Q}_{\mathcal{E}_0}} \left(\mathbf{u}^{(n)}_{0;\mathcal{E}_0,T} + \tau_n \nabla \mathcal{E}_T \left(\mathbf{u}^{(n)}_{0;\mathcal{E}_0,T} \right) \right),$$
$$\mathbf{u}^{(1)}_{0;\mathcal{E}_0,T} = \mathbf{u}^0,$$

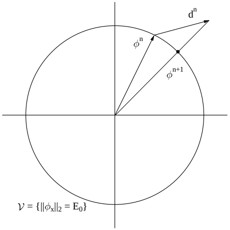
where:

- the gradient $\nabla \mathcal{E}_T$ determined from *adjoint system* via H^1 Sobolev preconditioning

$$\mathcal{L}^* \begin{bmatrix} \mathbf{u}^* \\ p^* \end{bmatrix} := \begin{bmatrix} -\partial_t \mathbf{u}^* - \left[\boldsymbol{\nabla} \mathbf{u}^* + \boldsymbol{\nabla} \mathbf{u}^{*T} \right] \mathbf{u} - \boldsymbol{\nabla} p^* - \nu \boldsymbol{\Delta} \mathbf{u}^* \\ -\boldsymbol{\nabla} \cdot \mathbf{u}^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Delta} \mathbf{u} \\ 0 \end{bmatrix},$$
$$\mathbf{u}^*(T) = \mathbf{0}$$

- step size $\tau^{(n)}$ is found via *arc minimization* and the projection on the constraint manifold $\mathcal{Q}_{\mathcal{E}_0}$ is given by

$$\mathbb{P}_{\mathcal{Q}_{\mathcal{E}_0}}(\mathbf{u}_0) = \sqrt{\frac{\mathcal{E}_0}{\mathcal{E}_T(\mathbf{u}_0)}}\, \mathbf{u}_0$$

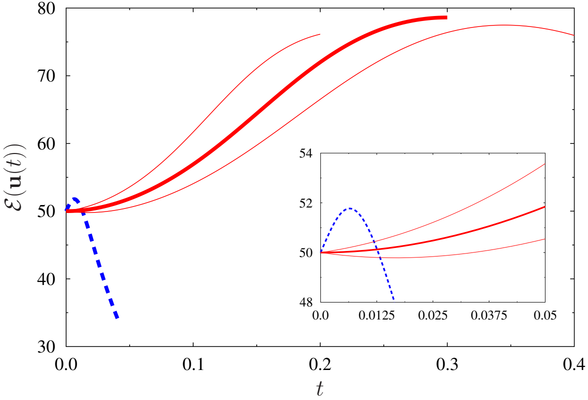


Computational Algorithm

- set $\mathcal{E}_0$ and T
- provide initial guess for the initial data $\mathbf{u}_{0;\mathcal{E}_0,T}$
  1. solve the Navier-Stokes system for $\{\mathbf{u}, p\}$
  2. solve the adjoint Navier-Stokes system for $\{\mathbf{u}^*, p^*\}$
  3. use $\mathbf{u}$ and $\mathbf{u}^*$ to compute $\boldsymbol{\nabla}^{L_2}\mathcal{E}_T$
  4. determine the Sobolev gradient $\boldsymbol{\nabla}^{H^1}\mathcal{E}_T$
  5. update the initial data while enforcing the enstrophy constraint
  $$\mathbf{u}^{(n+1)}_{0;\mathcal{E}_0,T} = \mathbb{P}_{\mathcal{Q}_{\mathcal{E}_0}}\left(\mathbf{u}^{(n)}_{0;\mathcal{E}_0,T} + \tau_n \nabla\mathcal{E}_T\left(\mathbf{u}^{(n)}_{0;\mathcal{E}_0,T}\right)\right)$$
- iterate 1. through 5. until convergence, i.e. until

$$\frac{\mathcal{E}(\mathbf{u}^{(n+1)}_{0;\mathcal{E}_0,T}) - \mathcal{E}(\mathbf{u}^{(n)}_{0;\mathcal{E}_0,T})}{\mathcal{E}(\mathbf{u}^{(n)}_{0;\mathcal{E}_0,T})} < \epsilon$$

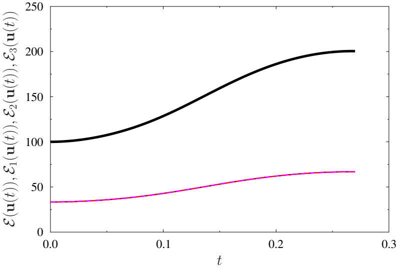
Enstrophy $\mathcal{E}(\mathbf{u}(t))$ in function of time for $\mathcal{E}_0 = 50$



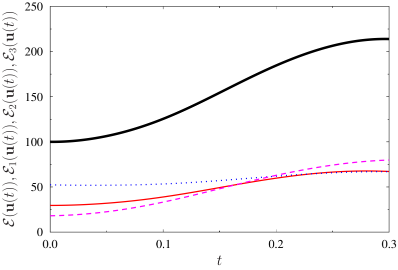
- instantaneously optimal initial data $\mathbf{u}_0 = \tilde{\mathbf{u}}_{\mathcal{E}_0}$
- initial data $\mathbf{u}_0 = \widetilde{\mathbf{u}}_{0;\mathcal{E}_0,T}$ optimized over $[0, T]$, where $T = 0.2, 0.3, 0.4$

Componentwise enstrophy $\mathcal{E}_i(\mathbf{u}(t))$, $i = 1, 2, 3$

Extremal flow evolution with $\mathbf{u}_0 = \widetilde{\mathbf{u}}_{0;\mathcal{E}_0,T}$ for $\mathcal{E}_0 = 100$, $\widetilde{T}_{\mathcal{E}_0} = 0.27$



(symmetric)



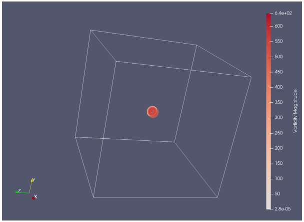
(asymmetric)

— total enstrophy $\mathcal{E}(\mathbf{u}(t)) = \sum_{i=1,2,3} \mathcal{E}_i(\mathbf{u}(t))$

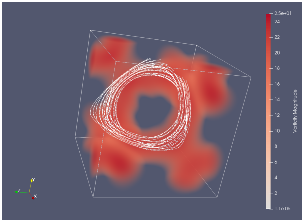
— $\mathcal{E}_1(\mathbf{u}(t))$, ··· $\mathcal{E}_2(\mathbf{u}(t))$, - - - $\mathcal{E}_3(\mathbf{u}(t))$,

where $\mathcal{E}_i(\mathbf{u}(t)) := \int_\Omega |(\boldsymbol{\nabla} \times \mathbf{u}(t)) \cdot \mathbf{e}_i|^2 \, d\mathbf{x}, \quad i = 1, 2, 3,$

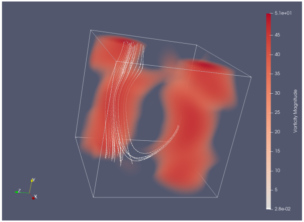
Optimal initial conditions $\widetilde{\mathbf{u}}_{\mathcal{E}_0}$ and $\widetilde{\mathbf{u}}_{0;\mathcal{E}_0,T}$ for $\mathcal{E}_0 = 100$



instantaneous $\widetilde{\mathbf{u}}_{\mathcal{E}_0}$



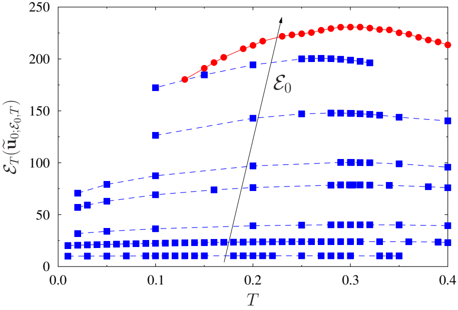
$\widetilde{\mathbf{u}}_{0;\mathcal{E}_0,T}$ with $T = 0.3$ (symmetric)

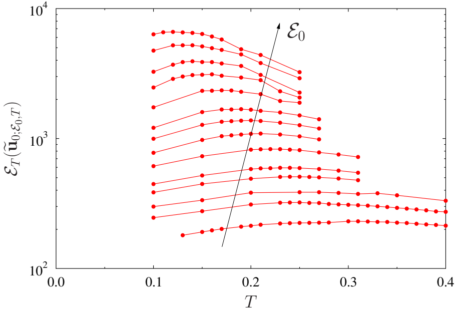


$\widetilde{\mathbf{u}}_{0;\mathcal{E}_0,T}$ with $T = 0.3 = \widetilde{T}_{\mathcal{E}_0}$ (asymmetric)

Finite-time optimal initial conditions $\widetilde{\mathbf{u}}_{0;\mathcal{E}_0,T}$ are much less localized than the instantaneous maximizers $\widetilde{\mathbf{u}}_{\mathcal{E}_0}$!

Maximum enstrophy $\max_{\mathbf{u}_0} \mathcal{E}(T)$ versus T for different $\mathcal{E}_0$



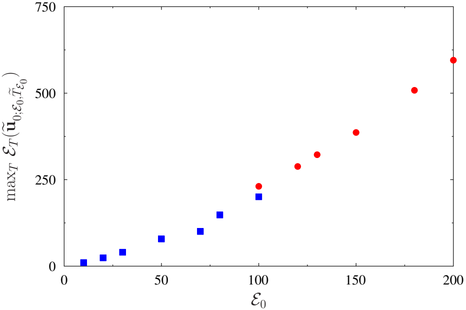


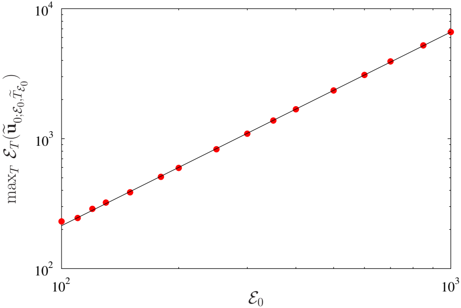
symmetric branch

asymmetric branch

Computational cost of one data point: $\mathcal{O}(10^2)$ hours on $\mathcal{O}(10^2)$ cores

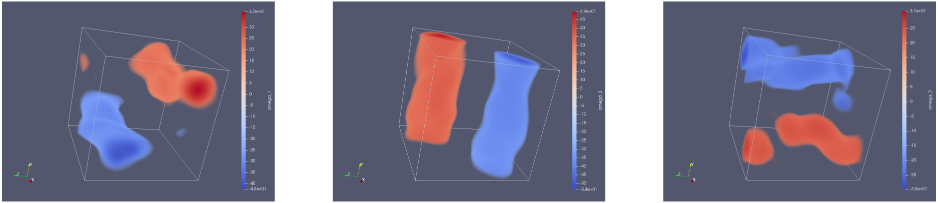
Maximum enstrophy $\max_T \max_{\mathbf{u}_0} \mathcal{E}(T)$ vs. $\mathcal{E}_0$





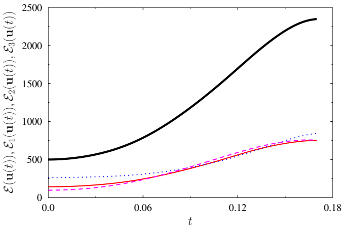
$$\max_T \max_{t\in[0,T]} \mathcal{E}(\widetilde{\mathbf{u}}_{0;\mathcal{E}_0,T}(t)) \sim (0.224 \pm 0.006)\ \mathcal{E}_0^{1.490 \pm 0.004}$$

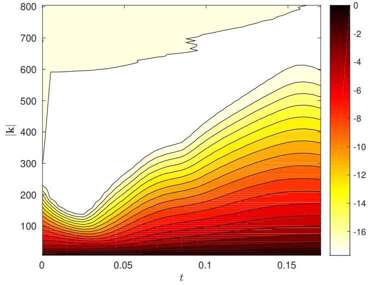
Structure of the optimal initial data $\widetilde{\mathbf{u}}_{0;\mathcal{E}_0,T}$ $(\mathcal{E}_0 = 500,\ T = 0.017)$

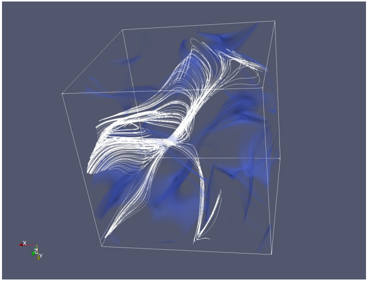


(a) ω_x (b) ω_y (c) ω_z

Evidence for reconnection ($\mathcal{E}_0 = 500$ and $\widetilde{T}_{\mathcal{E}_0} = 0.17$)





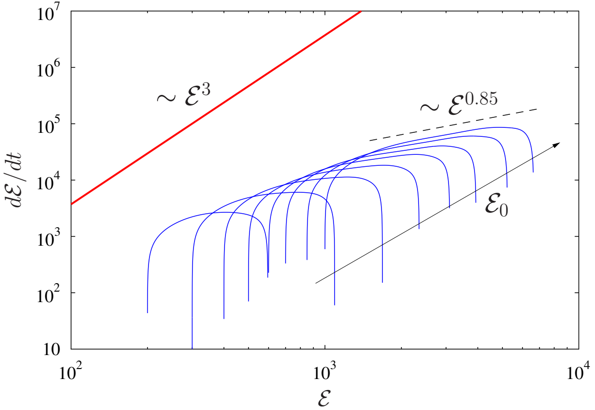


$t = 0.09$

Energy spectrum:

$$e(|\mathbf{k}|, t) := \int_{S_{|\mathbf{k}|}} |\mathbf{k}|^2 \left|[\widehat{\mathbf{u}}(t)]_{\mathbf{k}}\right|^2 \, d\sigma,$$

Maximum Sustained Rate of Enstrophy Growth $\frac{d\mathcal{E}}{dt} \sim C\,\mathcal{E}^{\alpha}$



- extreme trajectories with optimal initial data $\widetilde{\mathbf{u}}_{0;\mathcal{E}_0,T}$
- instantaneous maximizers $\widetilde{\mathbf{u}}_{\mathcal{E}_0}$

Conclusions

- In 3D the maximum growth of enstrophy is *finite* and scales in proportion to $\mathcal{E}_0^{3/2}$ as $\mathcal{E}_0$ becomes large
  - hence, even in such worst-case scenario there is no evidence for formation of singularity in finite time
  - the extreme behavior is realized by a series of reconnection events
  - the dependence of the maximum growth of enstrophy on $\mathcal{E}_0$ is the same as in 1D Burgers flows

- On-going work — probing the Ladyzhenskaya-Prodi-Serrin family of conditional regularity results $\mathbf{u} \in L^p([0,T]; L^q(\Omega))$, $2/p + 3/q \le 1, q > 3$

$$\text{blow-up at } t = t_0 \quad \Longleftrightarrow \quad \lim_{t \to t_0} \int_0^t \|\mathbf{u}(\tau)\|^p_{L^q(\Omega)}\, d\tau \to \infty$$

  - optimization problem (Q, T fixed): $\max_{\|\mathbf{u}_0\|_{L^q} = Q} \int_0^T \|\mathbf{u}(\tau)\|^p_{L^q(\Omega)}\, d\tau$

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